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Research Article

Minimization of sub-topical functions over a simplex

M.H. Daryaei*, and M.A. Yaghoobi

Abstract

This article investigates a particular version of the cutting angle method for finding the global minimizer of sub-topical (increasing and plus sub-homogeneous) functions over a simplex. The algorithm is based on the abstract convexity of sub-topical functions. Furthermore, we discuss the proof of convergence of the algorithm and provide results from numerical experiments.

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*Corresponding author

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Mohammad Hossein Daryaei
Department of Applied Mathematics, Faculty of Mathematics and Computer, Shahid Bahonar University of Kerman, Kerman, Iran. e-mail: daryaei@uk.ac.ir

Mohammad Ali Yaghoobi
Department of Applied Mathematics, Faculty of Mathematics and Computer, Shahid Bahonar University of Kerman, Kerman, Iran. e-mail: yaghoobi@uk.ac.ir

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1 Introduction

The cutting angle method is a powerful technique that can solve a wide range of global optimization problems based on abstract convexity. This method was proposed in 1999 as a method of global Lipschitz optimization and is a deterministic global optimization technique that involves constructing a sequence of lower approximations to an objective function [2]. In fact, this method was developed as a generalization of the cutting plane method of convex minimization. The original optimization problem is replaced by a sequence of relaxed auxiliary problems that are based on the lower approximations. Through this process, the sequence of global minimum solutions of the relaxed problems eventually converges to the global minimum of the original objective function. The articles [1, 3] discussed a method for globally minimizing an increasing positively homogeneous function over the unit simplex $S = \{x = (x_1, \ldots, x_n) \in \mathbb{R}^n_+ : \sum_{i=1}^n x_i = 1\}$. They provided a thorough examination of this method. In addition, Ferrer, Bagirov, and Beliakov [6] employed the cutting angle method to solve the problem of minimizing the difference of two convex functions.

A function $f : \mathbb{R}^n \longrightarrow \mathbb{R}$ is classified as topical if it is increasing in the natural partial ordering of $\mathbb{R}^n$ and plus-homogeneous, meaning that $f(x + \lambda\mathbf{1}) = f(x) + \lambda$ for all $x \in \mathbb{R}^n$ and all $\lambda \in \mathbb{R}$, where $\mathbf{1}$ denotes the vector of the corresponding dimension with all components equal to one. Topical functions have been studied in [7, 8, 10] and have found numerous applications in different areas of applied mathematics, particularly in the modeling of discrete event systems (see [7]).

Functions that are both increasing and plus-sub-homogeneous are referred to as sub-topical functions, which are viewed as a natural extension of topical functions [4]. The class of sub-topical functions is one of the most important classes of abstract convex functions. In [8, 10], the authors showed that every sub-topical function defined on $\mathbb{R}^n$ can be characterized as an abstract convex function with respect to a set L that is called the set of elementary functions. For each $y \in \mathbb{R}^n$ and each $\alpha \in \mathbb{R}$, the function $\varphi_{(y,\alpha)} : \mathbb{R}^n \longrightarrow \mathbb{R}$ is defined by

$$\varphi_{(y,\alpha)}(x) = \sup\{\lambda \in \mathbb{R} : \lambda \leq \alpha,\ \lambda\mathbf{1} \leq x + y\} \quad \text{for all } x \in \mathbb{R}^n,$$

where

$$L = \{\varphi_{(y,\alpha)} : y \in \mathbb{R}^n,\ \alpha \in \mathbb{R}\}.$$

It is worth noting that L is a more general class of functions than the class of min-type functions [3, 9].

Here, we study the problem

$$\min_{x \in S_a} f(x),$$

where $f : \mathbb{R}^n \longrightarrow \mathbb{R}$ is a sub-topical function and S_a is a simplex in $\mathbb{R}^n$; that is, $S_a = \{x = (x_1, \dots, x_n) \in \mathbb{R}^n_+ : \sum_{i=1}^n \frac{x_i}{a_i} = 1\}$, where $a = (a_1, \dots, a_n) \in \mathbb{R}^n_{++}$. We developed a version of the cutting angle method to solve this problem.

The rest of this paper is organized as follows. Section 2 presents definitions and preliminary results on sub-topical functions. In Section 3, a cutting angle method is proposed for finding the global minimizers of sub-topical functions over a simplex. Then, we present an approach to the numerical solution of the subproblem of the algorithm. Finally, Section 4 presents the results of numerical experiments.

2 Preliminaries

Let $\mathbb{R}^n_+$ denote the cone of all n-vectors with nonnegative coordinates. We assume that the usual order relation is defined in this cone as follows. For $x = (x_1, \dots, x_n) \in \mathbb{R}^n$ and $y = (y_1, \dots, y_n) \in \mathbb{R}^n$, $x \le y$ if and only if $x_i \le y_i$ for all i. A function $f : \mathbb{R}^n \longrightarrow \bar{\mathbb{R}} = [-\infty, +\infty]$ is said to be increasing if $f(x) \le f(y)$ for each x, $y \in \mathbb{R}^n$ such that $x \le y$. The function f is called plus sub-homogeneous if $f(x + \lambda\mathbf{1}) \le f(x) + \lambda$ for all $x \in \mathbb{R}^n$ and all $\lambda \ge 0$, where $\mathbf{1} = (1, \dots, 1) \in \mathbb{R}^n$. It is easy to see that f is plus sub-homogeneous if and only if $f(x + \lambda\mathbf{1}) \ge f(x) + \lambda$ for all $x \in \mathbb{R}^n$ and all $\lambda \le 0$. The definitions and results that follow can be found in [9, 10].

Definition 1. A function $f : \mathbb{R}^n \longrightarrow \bar{\mathbb{R}}$ is called sub-topical if it is increasing and plus sub-homogeneous.

Remark 1. A function $f : \mathbb{R}^n \longrightarrow \bar{\mathbb{R}}$ is called topical if it is increasing and $f(x+\lambda\mathbf{1}) = f(x)+\lambda$ for all $x \in \mathbb{R}^n$ and all $\lambda \in \mathbb{R}$. It is clear that any topical function is sub-topical.

Lemma 1 (see [10])**.** Let $f : \mathbb{R}^n \longrightarrow \bar{\mathbb{R}}$ be a sub-topical function. The following statements hold:

(i) If $f(x) = +\infty$ for some $x \in \mathbb{R}^n$, then f is identically equal to $+\infty$.

(ii) If $f(x) = -\infty$ for some $x \in \mathbb{R}^n$, then f is identically equal to $-\infty$.

Lemma 1 implies that a sub-topical function is either finite (i.e., finite-valued at each $x \in \mathbb{R}^n$) or identically $+\infty$ or $-\infty$. Therefore, in the remainder of the paper, we only consider finite-valued sub-topical functions. Here are some simple examples of finite-valued sub-topical functions.

Example 1. Any sub-linear function $f : \mathbb{R}^n \longrightarrow \mathbb{R}$ such that $f(\mathbf{1}) \leq 1$ is plus sub-homogeneous. So, an increasing and sub-linear function is sub-topical.

Example 2. Any function of the form

$$f(x) = \frac{1}{\theta} \ln \big(\sum_{i=1}^{n} e^{\langle a_i, x\rangle} \big), \quad x \in \mathbb{R}^n,$$

where $a_i \in \mathbb{R}^n$, $a_i \geq 0$, $i = 1, 2, \dots, n$, and $\theta \geq \max_{1\leq i\leq n}\langle a_i, \mathbf{1}\rangle$, is sub-topical. Indeed,

$$\begin{aligned}
f(x+\lambda\mathbf{1}) &= \frac{1}{\theta} \ln \big(\sum_{i=1}^{n} e^{\langle a_i, x+\lambda\mathbf{1}\rangle} \big) \\
&= \frac{1}{\theta} \ln \big(\sum_{i=1}^{n} e^{\langle a_i, x\rangle} e^{\lambda\langle a_i, \mathbf{1}\rangle} \big) \\
&\leq \frac{1}{\theta} \ln \big(e^{\lambda\theta} \sum_{i=1}^{n} e^{\langle a_i, x\rangle} \big) \\
&= \frac{1}{\theta} \Big(\ln \big(e^{\lambda\theta}\big) + \ln \big(\sum_{i=1}^{n} e^{\langle a_i, x\rangle} \big) \Big) \\
&= \lambda + \frac{1}{\theta} \ln \big(\sum_{i=1}^{n} e^{\langle a_i, x\rangle} \big) \\
&= \lambda + f(x).
\end{aligned}$$

The set $\mathbf{\Gamma}$ of all sub-topical functions $f : \mathbb{R}^n \longrightarrow \bar{\mathbb{R}}$ has several notable properties, which we will mention below.

(1) We have $\mathbf{\Gamma} + \mathbb{R} = \mathbf{\Gamma}$, that is, if $f \in \mathbf{\Gamma}$ and $\gamma \in \mathbb{R}$, then $f + \gamma \in \mathbf{\Gamma}$.

(2) $\mathbf{\Gamma}$ is a convex set.

(3) Let $f_i \in \mathbf{\Gamma}$ for $1 \leq i \leq K$, and let

$$\hat{f}(x) = \min_{1 \leq i \leq K} f_i(x), \quad \tilde{f}(x) = \max_{1 \leq i \leq K} f_i(x), \quad (x \in \mathbb{R}^n).$$

Then the functions $\hat{f}$ and $\tilde{f}$ belong to $\mathbf{\Gamma}$.

(4) $\mathbf{\Gamma}$ is a complete lattice, that is, if $\{f_\beta\}_{\beta \in B}$ is an arbitrary family of elements of $\mathbf{\Gamma}$ and

$$f(x) = \sup_{\beta \in B} f_\beta(x), \quad (x \in \mathbb{R}^n),$$

then the function f belongs to $\mathbf{\Gamma}$.

Remark 2. If $f : \mathbb{R}^n \longrightarrow \mathbb{R}$ is a sub-topical function, then f is continuous on $\mathbb{R}^n$. Indeed, consider an arbitrary point $x \in \mathbb{R}^n$ and a sequence $\{x_k\} \subseteq \mathbb{R}^n$ that $x_k \longrightarrow x$. Let $\epsilon > 0$ be arbitrary. Then, there exists $k_0 \geq 1$ such that $x - \epsilon\mathbf{1} \leq x_k \leq x + \epsilon\mathbf{1}$ for all $k \geq k_0$. Since f is increasing and plus sub-homogeneous, it follows that

$$f(x) - \epsilon \leq f(x - \epsilon\mathbf{1}) \leq f(x_k) \leq f(x + \epsilon\mathbf{1}) \leq f(x) + \epsilon \quad \text{for all } k \geq k_0.$$

This implies that $f(x_k) \longrightarrow f(x)$, which shows that f is continuous at x.

Now, we recall some definitions from abstract convexity (see for more details [9, 12]). Consider a set X and a set H of functions $h : X \longrightarrow \bar{\mathbb{R}}$. The function $f : X \longrightarrow \bar{\mathbb{R}}$ is called abstract convex with respect to H (or H-convex) if there exists a subset U of H such that

$$f(x) = \sup_{h \in U} h(x), \quad (x \in X).$$

The set H is called the set of elementary functions. Consider the function $\varphi : \mathbb{R}^n \times \mathbb{R}^n \times \mathbb{R} \longrightarrow \mathbb{R}$ defined by

$$\varphi(x,y,\alpha)=\min_{1\leq i\leq n}\{\alpha,x_i+y_i\},\quad (x,\ y\in\mathbb{R}^n,\ \alpha\in\mathbb{R}).$$

Let $(y,\alpha)\in\mathbb{R}^n\times\mathbb{R}$ be arbitrary. Denote by $\varphi_{(y,\alpha)}$ the function defined on $\mathbb{R}^n$ by the formula $\varphi_{(y,\alpha)}(x)=\varphi(x,y,\alpha)$. Let $X_\varphi=\{\varphi_{(y,\alpha)}\ :\ y\in\mathbb{R}^n,\ \alpha\in\mathbb{R}\}$. Then it is known that any function f defined on $\mathbb{R}^n$ is sub-topical if and only if f is X_φ-convex (see for more details [5, Theorem 3.2]).

Theorem 1. (see [5, Theorem 3.1]) The function $f:\mathbb{R}^n\longrightarrow\bar{\mathbb{R}}$ is sub-topical if and only if it satisfies the following inequality:

$$f(x)\geq\varphi_{(y,\alpha)}(x)+f(-y+\alpha\mathbf{1})-\alpha\quad\text{for all } x,\ y\in\mathbb{R}^n\ \text{ for all }\ \alpha\in\mathbb{R}.\quad(1)$$

Inequality (1) implies the following statement.

Proposition 1. (see [5, Theorem 3.2]) Let $f:\mathbb{R}^n\longrightarrow\mathbb{R}$ be a sub-topical function and let $\bar{x}\in\mathbb{R}^n$. Put $\bar{y}=-\bar{x}+f(\bar{x})\mathbf{1}$ and $\bar{\alpha}=f(\bar{x})$. Then $\varphi_{(\bar{y},\bar{\alpha})}(x)\leq f(x)$ for all $x\in\mathbb{R}^n$ and $\varphi_{(\bar{y},\bar{\alpha})}(\bar{x})=f(\bar{x})$.

Remark 3. Let $\bar{x}\in S_a$, $\bar{y}=-\bar{x}+f(\bar{x})\mathbf{1}$ and let $\bar{\alpha}=f(\bar{x})$. Define the function $\varphi_{\bar{y}}$ as follows:

$$\varphi_{\bar{y}}(x)=\min_{1\leq i\leq n}\{x_i+\bar{y}_i\},\quad (x\in S_a).\quad(2)$$

Then for each $x\in S_a$ we have $\varphi_{(\bar{y},\bar{\alpha})}(x)=\varphi_{\bar{y}}(x)$. Indeed, since $\bar{x},\ x\in S_a$, we get $\sum_{i=1}^n\frac{x_i-\bar{x}_i}{a_i}=0$. This implies that there exists $1\leq i_0\leq n$ such that $x_{i_0}-\bar{x}_{i_0}\leq 0$. Therefore

$$x_{i_0}+\bar{y}_{i_0}=x_{i_0}-\bar{x}_{i_0}+f(\bar{x})\leq f(\bar{x})=\bar{\alpha}.$$

Hence, $\min_{1\leq i\leq n}\{\bar{\alpha},x_i+\bar{y}_i\}=\min_{1\leq i\leq n}\{x_i+\bar{y}_i\}$.

3 Cutting angle method

We now present the cutting angle method for finding a global minimizer of the problem

$$\min_{x\in S_a}f(x),\quad(3)$$

where $f:\mathbb{R}^n\longrightarrow\mathbb{R}$ is a sub-topical function and $S_a=\{x=(x_1,\ldots,x_n)\in\mathbb{R}^n_+\ :\ \sum_{i=1}^n\frac{x_i}{a_i}=1\}$, where $a=(a_1,\ldots,a_n)\in\mathbb{R}^n_{++}$. The ith unit vector is denoted by $\mathbf{e}_i = (0,\dots,0,1,0,\dots,0)$. It is worth noting that, if $x^i = a_i\mathbf{e}_i$, $y^i = -x^i + f(x^i)\mathbf{1}$ and $\alpha_i = f(x^i)$, then $\varphi_{(y^i,\alpha_i)}(x) = \varphi_{y^i}(x) = x_i + y^i_i = x_i + f(x^i) - a_i$ for all $x = (x_1,\dots,x_n) \in S_a$.

Algorithm 2:

Step **0:** (initialization)

a) Take points $x^i = a_i\mathbf{e}_i$ for $i = 1,\dots,n$, and for each $1 \le i \le n$, construct the points $y^i = -x^i + f(x^i)\mathbf{1}$.

b) Define the function $h_n(x) = \max_{1\le i\le n} \varphi_{y^i}(x) = \max_{1\le i\le n}(x_i + y^i_i)$ for all $x = (x_1,\dots,x_n) \in S_a$.

c) Set $k := n$.

Step **1:** Solve the following subproblem:

$$\min_{x\in S_a} h_k(x). \tag{4}$$

Step **2:** Let y^* be an optimal solution of Problem (4). Set $k := k+1$ and $x^k = y^*$.

Step **3:** Compute $y^k = -x^k + f(x^k)\mathbf{1}$. Define the function

$$h_k(x) = \max(h_{k-1}(x),\ \varphi_{y^k}(x)) = \max_{1\le j\le k}\min_{1\le i\le n}(x_i + y^j_i)$$

for all $x = (x_1,\dots,x_n) \in S_a$. Go to Step **1**.

In Algorithm 2, the function h_n is the first lower approximation of the objective function f over S_a (note that the objective function f is from $\mathbb{R}^n$ to $\mathbb{R}$). This algorithm produces the sequence $\{h_k\}_{k\ge n}$ of lower approximations of f such that

$$h_k(x) \le h_{k+1}(x) \quad \text{for all}\, k \ge n \ \text{ for all }\ x \in S_a.$$

Note that Proposition 1 implies that $h_k(x) \le f(x)$ for all $x \in S_a$ and all $k \ge n$.

Algorithm 2 is indeed a version of the cutting angle method, which is a powerful technique for global optimization of sub-topical functions over a simplex. By a number of iterations of this algorithm, it produces x^k as an approximation of the global minimizer of f over S_a.

Let $\lambda_k = \min_{x\in S_a} h_k(x)$, and let $f(x^*) = \min_{x\in S_a} f(x)$. It is clear that $\lambda_k \leq f(x^*)$ for all $k \geq n$.

Lemma 2. In Algorithm 2, let $k \geq n$. Then $y_i^i \leq y_i^k$ for all $1 \leq i \leq n$.

Proof. By Proposition 1 we have $\varphi_{y^i}(x^k) \leq f(x^k)$ for all $1 \leq i \leq n$. So

$$x_i^k + f(x^i) - a_i = x_i^k + y_i^i \leq f(x^k), \quad i = 1, \dots, n.$$

This implies that $y_i^i \leq y_i^k$ for all $1 \leq i \leq n$. □

Proposition 2. In Algorithm 2, let $k \geq n+1$. If $y_j^j = y_j^k$ for some $1 \leq j \leq n$, then x^k is a global minimizer of the Problem (3).

Proof. We have

$$\begin{aligned}\lambda_{k-1} &= h_{k-1}(x^k) \\ &= \max_{1\leq i\leq k-1} \varphi_{y^i}(x^k) \\ &\geq \varphi_{y^j}(x^k) \\ &= y_j^j + x_j^k \\ &= y_j^k + x_j^k \\ &\geq \varphi_{y^k}(x^k) \\ &= f(x^k) \\ &\geq f(x^*) \\ &\geq \lambda_{k-1}.\end{aligned}$$

Hence $\lambda_{k-1} = f(x^k) = f(x^*)$. This completes the proof. □

Proposition 3. In Algorithm 2, let $k \geq n+2$. If $y^k \leq y^j$ for some $n+1 \leq j \leq k-1$, then x^k is a global minimizer of the Problem (3).

Proof. We have $\lambda_{k-1} = h_{k-1}(x^k) \geq \varphi_{y^j}(x^k) \geq \varphi_{y^k}(x^k) = f(x^k) \geq f(x^*) \geq \lambda_{k-1}$. Therefore $\lambda_{k-1} = f(x^k) = f(x^*)$, and so x^k is a global minimizer of the function f over S_a. □

If Algorithm 2 does not terminate after a finite iteration, then we show that the sequence $\{x^k\}$ generated by it converges to a global minimizer of the sub-topical function f on S_a. To this end, we first state and prove some results. Note that the sequence of functions $\{f_k\}_{k\geq 1}$, defined on the set $X \subseteq \mathbb{R}^n$, is called equicontinuous on X if for every $\epsilon > 0$ there exists $\delta > 0$ such that $|f_k(x) - f_k(t)| < \epsilon$ whenever $\|x - t\| < \delta$, $x \in X$, $t \in X$, and $k \geq 1$.

Lemma 3. The sequence $\{\varphi_{y^k}\}_{k\geq 1}$ generated by Algorithm 2 is equicontinuous on S_a.

Proof. Let $\epsilon > 0$ be given and let $k \geq 1$. Put $\delta = \epsilon$, and let $x = (x_1, \dots, x_n)$, $t = (t_1, \dots, t_n) \in S_a$ such that $\|x - t\| < \delta$. Therefore

$$|x_i - t_i| < \epsilon, \quad i = 1, \dots, n. \tag{5}$$

Now, assume that $\min_{1\leq i\leq n}(x_i - x_i^k) = x_{i_k} - x_{i_k}^k$ and $\min_{1\leq i\leq n}(t_i - x_i^k) = t_{i'_k} - x_{i'_k}^k$ for some $i_k,\ i'_k \in \{1, \dots, n\}$. We conclude that

$$x_{i_k} - t_{i_k} \leq \varphi_{y^k}(x) - \varphi_{y^k}(t) \leq x_{i'_k} - t_{i'_k}.$$

It follows from (5) that $|\varphi_{y^k}(x) - \varphi_{y^k}(t)| < \epsilon$. Hence the sequence $\{\varphi_{y^k}\}$ is equicontinuous on S_a. □

Lemma 4. The sequence $\{h_k\}_{k\geq n}$ generated by Algorithm 2 is equicontinuous on S_a.

Proof. Let $\epsilon > 0$ be given. Put $\delta = \epsilon$, and let $x = (x_1, \dots, x_n)$, $t = (t_1, \dots, t_n) \in S_a$ such that $\|x - t\| < \delta$. By Lemma 3, we have

$$|\varphi_{y^k}(x) - \varphi_{y^k}(t)| < \epsilon, \quad k = 1, 2, \dots. \tag{6}$$

Let $k \geq n$. Now, assume that $h_k(x) = \max_{1\leq i\leq k} \varphi_{y^i}(x) = \varphi_{y^{i_0}}(x)$ and $h_k(t) = \max_{1\leq i\leq k} \varphi_{y^i}(t) = \varphi_{y^{i_1}}(t)$ for some $i_0,\ i_1 \in \{1, \dots, k\}$. We conclude that

$$\varphi_{y^{i_1}}(x) - \varphi_{y^{i_1}}(t) \leq h_k(x) - h_k(t) = \varphi_{y^{i_0}}(x) - \varphi_{y^{i_1}}(t) \leq \varphi_{y^{i_0}}(x) - \varphi_{y^{i_0}}(t).$$

It follows from (6) that $|h_k(x) - h_k(t)| < \epsilon$. Hence the sequence $\{h_k\}_{k\geq n}$ is equicontinuous on S_a. □

Remark 4. By the definition of the function h_k, we conclude that the sequence $\{h_k\}$ in Algorithm 2 is also uniformly bounded on S_a. Indeed, we have that the sequence $\{h_k\}$ is increasing and $h_k(x) \leq f(x)$ for all $x \in S_a$ and all $k \geq n$. On the other hand, any sub-topical function is continuous and S_a is a compact set. So we get that the sequence $\{h_k\}$ is uniformly bounded on S_a.

Proposition 4. Let $\{x^k\}$ be the generated sequence by Algorithm 2 and let x^* be a limit point of this sequence. Then x^* is a global minimum point of the function $\varphi(x) = \sup_{k\geq n} h_k(x)$ over S_a.

Proof. Without loss of generality, assume that $x^k \longrightarrow x^*$. Put $\lambda_k = h_k(x^{k+1})$. We have that the sequence $\{\lambda_k\}$ is an increasing sequence and $\lambda_k \leq \varphi(x)$ for all $x \in S_a$. Thus

$$\lim_{k\longrightarrow\infty} \lambda_k \leq \inf_{x\in S_a} \varphi(x) \leq \varphi(x^*). \tag{7}$$

Now, we show that $\lim_{k\longrightarrow\infty} \lambda_k = \varphi(x^*)$. Let $\epsilon > 0$ be given. Since $\{h_k\}$ is equicontinuous on S_a and $x^k \longrightarrow x^*$, there exists $N_1 > 0$ such that for each $m \geq N_1$ we have

$$|h_k(x^m) - h_k(x^*)| < \frac{\epsilon}{2} \quad \text{for all } \ k \geq n. \tag{8}$$

On the other hand, It is clear that $h_k(x^*) \longrightarrow \varphi(x^*)$. So there exists $N_2 > n$ such that

$$|h_k(x^*) - \varphi(x^*)| < \frac{\epsilon}{2} \quad \text{for all } \ k \geq N_2. \tag{9}$$

Let $N = \max\{N_1, N_2\}$. It follows from (8) and (9) that

$$|\lambda_k - \varphi(x^*)| \leq |\lambda_k - h_k(x^*)| + |h_k(x^*) - \varphi(x^*)| < \frac{\epsilon}{2} + \frac{\epsilon}{2} = \epsilon$$

for all $k \geq N$. This implies that $\lim_{k\longrightarrow\infty} \lambda_k = \varphi(x^*)$, and by (7) we conclude that $\varphi(x^*) = \inf_{x\in S_a} \varphi(x)$. This completes the proof. □

We will now demonstrate the convergence of Algorithm 2.

Theorem 2. Let $\{x^k\}$ be the generated sequence by Algorithm 2 and let x^* be a limit point of this sequence. Then x^* is a global optimal solution of the Problem (3).

Proof. Since $\{h_k\}$ is uniformly bounded and equicontinuous on the compact set S_a, then by [11, Theorem 7.25], this sequence has a subsequence that is uniformly convergent on S_a. On the other hand, the sequence $\{h_k\}$ is increasing, so $\{h_k\}$ converges uniformly to $\varphi(x) = \sup_{k\geq n} h_k(x)$ on S_a. It is clear that $\varphi(x) \leq f(x)$ for all $x \in S_a$ and $\varphi(x^k) = f(x^k)$ for all $k \geq n$ (indeed, by Proposition 1, for each $k \geq n$, we have $\varphi_{y^k}(x^k) = f(x^k)$ and thus $h_k(x^k) = f(x^k)$ and $\varphi(x^k) = f(x^k)$). Now, let $x^* \in S_a$ be a limit point of the sequence $\{x^k\}$ (the simplex S_a is a compact set, so $\{x^k\}$ has always a limit point $x^* \in S_a$). We have $\varphi(x) - \varphi(x^k) \leq f(x) - f(x^k)$ for all $x \in S_a$ and all $k \geq n$. This implies that

$$\varphi(x) - \varphi(x^*) \leq f(x) - f(x^*) \quad \text{for all } \ x \in S_a. \tag{10}$$

By Proposition 4, x^* is a minimizer of φ over S_a. So $\varphi(x) - \varphi(x^*) \geq 0$ for all $x \in S_a$. Therefore, by (10), we conclude that $f(x) - f(x^*) \geq 0$ for all $x \in S_a$. Hence x^* is a global minimizer of the function f over the set S_a. This completes the proof. □

Step 1 of Algorithm 2 involves finding the global minimum of the function h_k on the set S_a, which is the most challenging part of the algorithm. The problem in Step 1 can be formulated as follows:

$$\begin{aligned} &\min\ h_k(x) \\ &\text{subject to } x \in S_a, \end{aligned} \tag{11}$$

where

$$h_k(x) = \max_{1\leq j\leq k} \min_{1\leq i\leq n} (x_i + y_i^j),$$

$k \geq n$, $x = (x_1, \ldots, x_n) \in S_a$, and $y^j = -x^j + f(x^j)\mathbf{1}$.

We will now discuss an approach for solving the subproblem. We demonstrate that Problem (11) can be transformed into a mixed integer linear programming problem with $0-1$ variables. This technique is widely used and involves the introduction of a large positive parameter M. If we set $t = \max_{1\leq j\leq k} \min_{1\leq i\leq n}(x_i + y_i^j)$, then it is clear that Problem (11) is equivalent to the following problem:

$$\begin{aligned} &\min\ z = t \\ &\text{subject to } \min_{1\le i\le n}(x_i + y_i^j) \le t, \quad 1 \le j \le k, \\ &\qquad x \in S_a. \end{aligned} \tag{12}$$

Each constraint $\min_{1\le i\le n}(x_i + y_i^j) \le t$ can be expressed as a set of $n+1$ linear constraints of the following form

$$\begin{aligned} &x_i + y_i^j - M u_i^j \le t, \quad 1 \le i \le n, \\ &\sum_{i=1}^{n} u_i^j \le n-1, \end{aligned}$$

where $u_i^j \in \{0,1\}$ for all $i,\ j$.
Hence, we conclude that Problem (12) is equivalent to the following mixed integer linear programming problem:

$$\begin{aligned} &\min\ z = t \\ &\text{subject to } x_i - t - M u_i^j \le -y_i^j, \quad 1 \le i \le n, \quad 1 \le j \le k, \\ &\qquad \textstyle\sum_{i=1}^{n} u_i^j \le n-1, \quad 1 \le j \le k, \\ &\qquad \textstyle\sum_{i=1}^{n} \frac{x_i}{a_i} = 1, \\ &\qquad x_i \ge 0, \quad u_i^j \in \{0,1\}, \quad 1 \le i \le n, \quad 1 \le j \le k. \end{aligned} \tag{13}$$

Despite the fact that this technique increases the number of constraints and variables with each iteration, it also has the advantage of allowing the use of software packages designed for solving mixed integer linear programming problems, such as MATLAB, LINGO, and so on, to solve the subproblem.

Remark 5. There is a natural question that, in Problem (13), how big should M be. Let $|f(x)| \le C$ for all $x \in S_a$, and let $A = \max_{1\le i\le n}(a_i)$. Then it is clear that $|x_i + y_i^j| \le A + C$ for all i and j. This implies that, it is enough to assume that $M \ge 2A + 2C$.

At the end of this section, we present the complete version of the cutting angle method for finding the global optimal solution of Problem (3).

Algorithm 3: Cutting angle method

Initialization:

Set $x^i = a_i\mathbf{e}_i$ for $i = 1, \ldots, n$, and for each $1 \le i \le n$, compute the points $y^i = -x^i + f(x^i)\mathbf{1}$. Find the optimal solution y^* of Problem (13) according to $k = n$. Set $k := n + 1$ and $F := \emptyset$.

Iteration $k - n$**:**

1) Set $x^k = y^*$, $F := F \cup \{f(x^k)\}$ and compute the point $y^k = -x^k + f(x^k)\mathbf{1}$.

- i) Let $f(\hat{x}^k) = \min F$.
- ii) If there exists $1 \le j \le n$ such that $y^j_j = y^k_j$, then terminate the algorithm; x^k is a global optimal solution of Problem (3).
- iii) For $k \ge n + 2$, if there exists $n + 1 \le j \le k - 1$ such that $y^k \le y^j$, then terminate the algorithm; x^k is a global optimal solution of Problem (3).

2) Find the optimal solution y^* of Problem (13). Set $k := k + 1$ and go to iteration $k - n$.

Remark 6. It is clear that, if Algorithm 3 does not terminate after a finite iteration, then the sequence $\{\hat{x}^k\}$ tends to a global optimal solution of Problem (3).

4 Numerical results

In this section, we will provide three numerical examples to evaluate the effectiveness of the proposed cutting angle method. We have implemented the method using MATLAB and carried out the computations on a personal computer with the following specifications: Microprocessor: Intel® Core™ i5-7200U (2.5 GHz, up to 3.1 GHz, 3 MB cache, 2 cores); Memory: 8 GB DDR4-2133.

In each of the examples, we have selected an objective function that is sub-topical. We have then summarized the numerical results in tables 1,2, and 3, which allow us to easily evaluate the performance of the algorithm.

Example 3. Consider

$$\begin{aligned} &\min\ f(x_1,x_2)\\ &\text{s.t. } x_1+x_2=1, \\ &\quad x_1,x_2\geq 0, \end{aligned} \tag{14}$$

where

$$f(x_1,x_2)=\frac{1}{5}\ln(e^{3x_1}+e^{5x_2}).$$

The graph of the objective function f and functions h_i $(i=2,3,4,5)$ over the unit simplex $S_{\mathbf{1}}=\{(x_1,x_2):x_1+x_2=1,\ x_1,x_2\geq 0\}$ are shown in Figure 1. The optimal solution and the optimal value of Problem (14) are $x^*=(0.688853,0.311147)$ and $f^*=0.507312$, respectively.

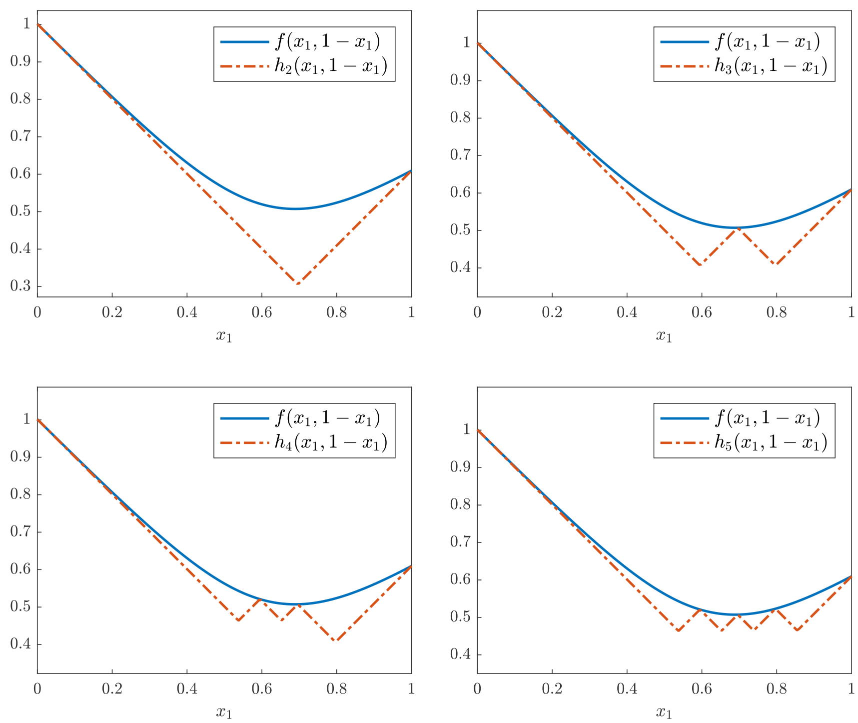

Figure 1: Graph of the objective function f of Problem (14) and the functions h_i $(i=2,3,4,5)$ over the unit simplex $S_{\mathbf{1}}=\{(x_1,x_2):x_1+x_2=1,\ x_1,x_2\geq 0\}$.

Table 1: Obtained results by the cutting angle method for Example 3

Iter	Optimal point found $\hat{x}^k$	Optimal value found $f(\hat{x}^k)$	Time (sec)
1	$(0.695813, 0.304187)$	0.507385	0.01
13	$(0.685517, 0.314483)$	0.507329	0.19
20	$(0.690637, 0.309363)$	0.507317	0.34
34	$(0.688083, 0.311917)$	0.507314	0.73
64	$(0.688721, 0.311279)$	0.507312	2.71

Example 4. Consider

$$\begin{aligned} &\min\ f(x_1,x_2,x_3)\\ &\text{s.t. } x_1+\tfrac{1}{2}x_2+\tfrac{1}{3}x_3=1,\\ &\qquad x_1,x_2,x_3\geq 0, \end{aligned} \tag{15}$$

where

$$\begin{aligned} f(x_1,x_2,x_3) = 0.1\max\{&0.2x_1+0.3x_2+0.5x_3, 0.1x_1+0.7x_2+0.1x_3,\\ &0.4x_1+0.38x_2+0.2x_3\} + 0.025\ln(e^{9x_1}+e^{5x_2}+e^{12x_3}). \end{aligned}$$

Note that the function f is not differentiable on the simplex $S_a = \{(x_1,x_2,x_3) : x_1+\frac{1}{2}x_2+\frac{1}{3}x_3=1,\ x_1,x_2,x_3\geq 0\}$. The graph of the objective function f over the simplex S_a is shown in Figure 2. The optimal solution and the optimal value of Problem (15) are $x^*=(0.5658, 0.6699, 0.2978)$ and $f^*=0.1911$, respectively.

Table 2: Obtained results by the cutting angle method for Example 4

Iter	Optimal point found $\hat{x}^k$	Optimal value found $f(\hat{x}^k)$	Time (sec)
1	$(0.0859, 0.9609, 1.3009)$	0.4859	0.02
2	$(0.2923, 1.1673, 0.3723)$	0.2407	0.04
4	$(0.4787, 0.6702, 0.5587)$	0.2282	0.11
5	$(0.5668, 0.7583, 0.1623)$	0.1947	0.15
27	$(0.5604, 0.6496, 0.3444)$	0.1919	4.86
33	$(0.5796, 0.6689, 0.25784)$	0.1916	7.64

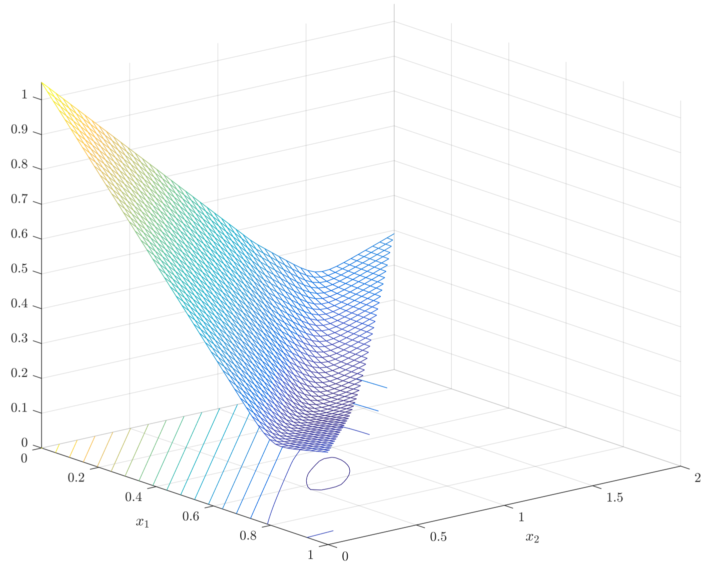

Figure 2: Graph of the objective function f of Problem (15) over the simplex $S_a = \{(x_1, x_2, x_3) : x_1 + \frac{1}{2}x_2 + \frac{1}{3}x_3 = 1,\ x_1, x_2, x_3 \geq 0\}$.

Now, we present an example that the global optimal solution is a boundary point of the feasible set of the problem.

Example 5. Consider

$$\begin{aligned} &\min\ f(x_1, x_2, x_3) \\ &\text{s.t. } x_1 + x_2 + x_3 = 1, \\ &\qquad x_1, x_2, x_3 \geq 0, \end{aligned} \tag{16}$$

where

$$\begin{aligned} f(x_1, x_2, x_3) = {} & 0.1 \max\{0.2x_1 + 0.3x_2 + 0.5x_3, 0.1x_1 + 0.7x_2 + 0.1x_3\} \\ & + 0.4 \min\{0.2x_1 + 0.3x_2 + 0.5x_3, 0.1x_1 + 0.7x_2 + 0.1x_3\} \\ & + \frac{1}{60} \ln(e^{9x_1} + e^{5x_2} + e^{12x_3}). \end{aligned}$$

The graph of the objective function f over the simplex S_a is shown in Figure 3. The optimal solution and the optimal value of Problem (16) are $x^* = (0.6025, 0, 0.3975)$ and $f^* = 0.1693$, respectively (note that x^* is a boundary point of the simplex S_a).

Table 3: Obtained results by the cutting angle method for Example 5

Iter	Optimal point found $\hat{x}^k$	Optimal value found $f(\hat{x}^k)$	Time (sec)
1	$(0.3812, 0.3176, 0.3012)$	0.2199	0.03
3	$(0.6923, 0, 0.3077)$	0.1744	0.18
22	$(0.6136, 0, 0.3864)$	0.1694	3.95

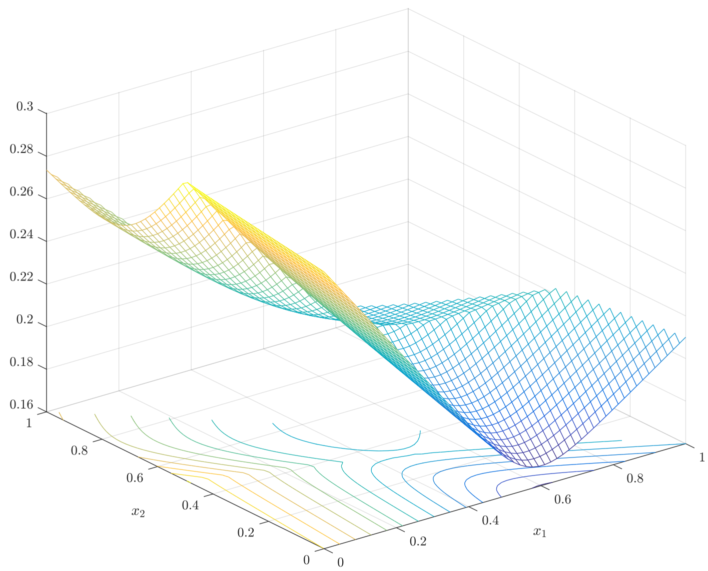

Figure 3: Graph of the objective function f of Problem (16) over the simplex $S_{\mathbf{1}} = \{(x_1, x_2, x_3) : x_1 + x_2 + x_3 = 1,\ x_1, x_2, x_3 \geq 0\}$.

5 Conclusion

This paper has proposed the cutting angle method as a means of finding the global minimum solution of a sub-topical function over a simplex. The algorithm utilized the abstract convexity of sub-topical functions to estimate the optimal value by solving mixed integer linear programming problems. The efficiency of this method has been demonstrated through numerical experiments.

Overall, the proposed cutting angle method provided a promising approach for solving sub-topical optimization problems with a simplex constraint. Future research can explore the extension of this algorithm to handle more complex sub-topical optimization problems with additional constraints.

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