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Research Article

A semi-analytic and numerical approach to the fractional differential equations

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Abstract

A class of linear and nonlinear fractional differential equations (FDEs) in the Caputo sense is considered and studied through two novel techniques called the Homotopy analysis method (HAM). A reliable approach is proposed for solving fractional order nonlinear ordinary differential equations, and the Haar wavelet technique (HWT) is a numerical approach for both integer and noninteger orders. Perturbation techniques are widely applied to gain analytic approximations of nonlinear equations. However, perturbation methods are essentially based on small physical parameters (called

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perturbation quantity), but unfortunately, many nonlinear problems have no such kind of small physical parameters at all. HAM overcomes this, and HWT does not require any parameters. Due to this, we opt for HAM and HWT to study FDEs. We have drawn a semi-analytical solution in terms of a series of polynomials and numerical solutions for FDEs. First, we solve the models by HAM by choosing the preferred control parameter. Second, HWT is considered. Through this technique, the operational matrix of integration is used to convert the given FDEs into a set of algebraic equation systems. Four problems are discussed using both techniques. Obtained results are expressed in graphs and tables. Results on convergence have been discussed in terms of theorems.

AMS subject classifications (2020): Primary 26A33; Secondary 34A08, 65H20, 90C30, 65T60.

Keywords: Homotopy analysis method; Haar wavelet; Convergence; Collocation method.

1 Introduction

Many engineering and scientific areas, including mathematical modeling in chemistry, physics, electrodynamics, and aerodynamics, involve fractional differential equations (FDEs). For instance, fractional derivatives can be used to describe nonlinear seismic oscillation, and they can also be used to solve the flaw in the continuum traffic flow assumption in the fluid-dynamic traffic model. Recent research has shown that FDEs are helpful tools for modeling various physical phenomena [29, 34]. This is due to the accurate representation of physical phenomena requiring the knowledge of both the present and the past, which can also be accomplished through the application of fractional calculus. Additionally, as the model contains memory terms, fractional derivatives offer an effective tool for explaining memory and hereditary qualities of different materials and actions [12]. This memory concept covers the past and how it affects the present and the future. Compared to integer-order, the fractional-order models are much more precise; this phenomenon has drawn the interest of numerous academicians [27, 17].

In the literature, we have yet to find an approachable analytical method for the general form of FDEs. So, we need to switch over to the semi-analytical or numerical methods.

Consider the general FDE of the following form:

$$aD^{\alpha}y(t) + bg(y(t)) + cy(t) = f(t) \tag{1}$$

with initial conditions

$$y(\beta_1) = \alpha_1, \quad y'(\beta_1) = \alpha_2,$$

or boundary conditions

$$y(\beta_2) = \alpha_3, \quad y'(\beta_2) = \alpha_4,$$

where $\alpha \in \mathbb{R}$ and a, b, c, β_1, β_2, α_1, α_2, α_3, and α_4 are real constants, $g(y(t))$ is nonlinear term, and $f(t)$ is continuous function, for all $t \geq 0$.

A method that yields the analytic solution after some iterations is called a semi-analytical method. Here, we considered one of the semi-analytic methods to solve FDEs called the Homotopy analysis method (HAM). The HAM creates a convergent series solution for nonlinear mathematical models by using the idea of the Homotopy in the topology. It is possible to use a Homotopy-Maclaurin series to handle the system's nonlinearities. In 1992, HAM was created for the first time in Shanghai Jiaotong University by Liao Shijun for his Ph.D. thesis [20]. Then, in 1997, it was modified by adding an auxiliary parameter [19] C_0 ($\neq 0$), known as the convergence-control parameter [21]. A nonphysical variable called the convergence-control parameter offers a straightforward approach to confirming and enforcing the convergence of a solution series. In analytical and semi-analytic techniques to nonlinear differential equations, it is unusual for the HAM to demonstrate the convergence naturally. First, unlike other series expansion techniques, the HAM does not rely on either small or large physical factors directly, which allows it to apply to both strongly and weakly nonlinear problems, overcoming some inherent limitations of the standard perturbation methods. Second, the HAM unifies the Adomian decomposition method (ADM), the delta expansion method, the Homotopy perturbation method, and the Lyapunov artificial small parameter approach [18, 30]. Strong convergence of the solution over broader geographical and parameter domains is frequently possible due to the method's enhanced generality. Third, the HAM offers super flexibility in the solution's representation and in the method by which it is expressly achieved. The basis functions of the intended solution and the related auxiliary linear operator of the Homotopy can both be chosen with a significant deal of freedom. Finally, the HAM offers a straightforward method to guarantee the convergence of the solution series, in contrast to the other analytic approximation methods. The Homotopy analysis approach can be used with other nonlinear differential equations methods, including spectral methods [25] and Padé approximants. To enable the linear method to handle nonlinear systems, it may also be supplemented with computational techniques like the boundary element method. The HAM, in contrast to the discrete computational approach of Homotopy continuation, is an analytic approximation method. In contrast, the HAM uses the Homotopy parameter only theoretically to show that a nonlinear system may be split into an infinite set of linear systems that are solved analytically. The HAM was created as an analytical approximation technique for the computer epoch with the intention of "computing with functions instead of numbers." Using the HAM in conjunction with a computer algebra system like Maple or Mathematica, we can quickly obtain analytic approximations of a higher-order strongly nonlinear problem. BVPh is a Mathematica package based on the HAM that has been made available online for solving nonlinear boundary-value problems as a result of the HAM's recent successes in several disciplines [1].

Solutions for the highly complex mathematical model by semi-analytical techniques require much time for each iteration, and sometimes even an infinite series of terms also cannot be possible to put in the analytical format. For such models, numerical techniques are better. In this study, we considered the Haar wavelet technique (HWT) to solve FDEs. When representing data or other functions, wavelets are mathematical functions that meet specific criteria. Since Joseph Fourier realized that sines and cosines could be superposed to describe other functions in the early 1800s, approximation utilizing the superposition of functions has been used. In the 1980s and 1990s, wavelets were created as an alternative to Fourier analysis of signals. Jean Morlet, Baroness Ingrid Daubechies, Alex Grossman, Palle Jorgensen, Yves Meyer, Ronald Coifman, Alfred Haar, and Stephane Mallat were a few key players in this invention. Yet, the scale at which we examine the data has a specific significance in wavelet analysis. Different scales or resolutions of data are processed by wavelet algorithms. Wavelet transforms are extremely helpful for signal analysis, compression, and denoising. Fourier analysis is impoverished at approximating sharp spikes when investigating its solutions; however, we can employ approximation functions that are tidily contained in finite domains appreciations to wavelet analysis. For estimating data with sharp discontinuities, wavelets work well. Compared to other methods [14, 13], the wavelet approach gives better outcomes. Numerical methods based on wavelets are effective for solving FDEs [36]. The Haar wavelets are compactly supported, and orthogonal with multi-resolution analysis [15]. Here we found several methods on FDEs, for instance, the reduced differential transform method [35], Sawi transform combined with the homotopy perturbation method [9], ADM [2], Legendre wavelet operational matrix method [3], fractional reduced differential transform method [37], fractional differential transform method [8] and positive solutions method [4], Adomian decomposition Shehu transform method [10], homotopy technique [28, 7], Adomian decomposition general transform method [5].

The paper is arranged as described as follows. Preliminaries of the Haar wavelets and their operational integration matrix, fractional derivative, and HAM are covered in Section 2. HAM and HWT have been explained in Section 3. Convergence analysis of HAM and HWT is drawn in Section 4. The implementation of test problems is in Section 5. The conclusion is enclosed with concluding remarks in Section 6.

2 Preliminaries

Definition 1 (Haar Wavelet)**.** A wavelet can be expressed as a real valued function $\Psi(t)$ that satisfies the following conditions [33]:

$$\int_{-\infty}^{\infty} \Psi(t)dt = 0 \quad \text{and} \quad \int_{-\infty}^{\infty} |\Psi(t)|^2 dt = 1.$$

This means that $\Psi(t)$ is an oscillatory function with zero mean, and the wavelet function has unit energy. more precisely, wavelets are defined as

$$\Psi_{a,b}(t) = \frac{1}{\sqrt{a}}\Psi(\frac{t-b}{a}), \quad a \neq 0, b \in \mathbb{R}.$$

Here a and b represent, respectively, the dilation and translation. Consider an interval $[A, B] \subset \mathbb{R}$, which is divided into m equal subintervals, each of width $\Delta t = \frac{B-A}{m}$. The i^{th} orthogonal set of Haar functions on the interval $[A, B]$ is defined as

$$h_i(t) = \begin{cases} 1, & \zeta_1(i) \leq t < \zeta_2(i), \\ -1, & \zeta_2(i) \leq t < \zeta_3(i), \\ 0, & \text{otherwise}, \end{cases} \tag{2}$$

where

$$\zeta_1(i) = A + \frac{k-1}{2^j} m\Delta t,$$
$$\zeta_2(i) = A + \frac{k-(\frac{1}{2})}{2^j} m\Delta t,$$
$$\zeta_3(i) = A + \frac{k}{2^j} m\Delta t,$$

for $i = 1, 2, \ldots, m, m = 2^J$ and J is a positive integer, which is called the maximum level of resolution. Here j and k represent the integer decomposition of the index i; that is, $i = k + 2^j - 1$, $0 \leq j < 1$, and $1 \leq k < 2^j + 1$.

Equation (2) is valid for $i \geq 2$, and for $i = 1$, we have

$$h_i(t) = \begin{cases} 1, & \text{for} \quad x \in [A, B], \\ 0, & \text{otherwise}. \end{cases} \tag{3}$$

The Haar wavelet operational matrix Q^α for integration of the general order α is given by

$$Q^\alpha H_m(t) = J^\alpha H_m(t) = [J^\alpha h_0(t), J^\alpha h_1(t), J^\alpha h_2(t), \ldots, J^\alpha h_{m-1}(t)],$$
$$Q^\alpha H_m(t) = [Qh_0(t), Qh_1(t), Qh_2(t), \ldots, Qh_{m-1}(t)], \tag{4}$$

where

$$Qh_i(t) = \begin{cases} 0 & A \leq t < \zeta_1(i), \\ \Phi_1 & \zeta_1(i) \leq t < \zeta_2(i), \\ \Phi_2 & \zeta_2(i) \leq t < \zeta_3(i), \\ \Phi_3 & \zeta_2(i) \leq t < B, \end{cases} \tag{5}$$

in which

$$\Phi_1 = \frac{(t-\zeta_1(i))^\alpha}{\Gamma(\alpha+1)},$$
$$\Phi_2 = \frac{(t-\zeta_1(i))^\alpha}{\Gamma(\alpha+1)} - 2\frac{(t-\zeta_2(i))^\alpha}{\Gamma(\alpha+1)},$$
$$\phi_3 = \frac{(t-\zeta_1(i))^\alpha}{\Gamma(\alpha+1)} - 2\frac{(t-\zeta_2(i))^\alpha}{\Gamma(\alpha+1)} + \frac{(t-\zeta_3(i))^\alpha}{\Gamma(\alpha+1)}.$$

Equation (5) is valid for $i \geq 1$. For $i = 0$, we have

$$Qh_0(t) = \begin{cases} \frac{t^\alpha}{\Gamma(\alpha+1)} & t \in [A, B], \\ 0 & \text{otherwise.} \end{cases}$$

For instance, if $\alpha \in \mathbb{R}$, then we have the following cases:

Case 1: For $\alpha = 1, J = 3$

$$Q^1 H_m(t) = \begin{bmatrix} 0.0625 & 0.1875 & 0.3125 & 0.4375 & 0.5625 & 0.6875 & 0.8125 & 0.9375 \\ 0.0625 & 0.1875 & 0.3125 & 0..4375 & 0.4375 & 0.3125 & 0.1875 & 0.0625 \\ 0.0625 & 0.1875 & 0.1875 & 0.0625 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0.0625 & 0.1875 & 0.1875 & 0.0625 \\ 0.0625 & 0.0625 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0.0625 & 0.0625 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0.0625 & 0.0625 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0.0625 & 0.0625 \end{bmatrix}.$$

Case 2: For $\alpha = 2, J = 3$

$Q^2 H_m(t)$

$$= \begin{bmatrix} 0.00195313 & 0.0175781 & 0.0488281 & 0.0957031 & 0.158203 & 0.236328 & 0.330078 & 0.439453 \\ 0.00195313 & 0.0175781 & 0.0488281 & 0.0957031 & 0.154297 & 0.201172 & 0.232422 & 0.248047 \\ 0.00195313 & 0.0175781 & 0.0449219 & 0.0605469 & 0.0625 & 0.0625 & 0.0625 & 0.0625 \\ 0 & 0 & 0 & 0 & 0.00195313 & 0.0175781 & 0.0449219 & 0.0605469 \\ 0.00195313 & 0.0136719 & 0.015625 & 0.015625 & 0.015625 & 0.015625 & 0.015625 & 0.015625 \\ 0 & 0 & 0.00195313 & 0.0136719 & 0.015625 & 0.015625 & 0.015625 & 0.015625 \\ 0 & 0 & 0 & 0 & 0.00195313 & 0.0136719 & 0.015625 & 0.015625 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0.00195313 & 0.0136719 \end{bmatrix}.$$

Case 3: Similarly, we obtain the operational matrix for $\alpha = 1.5, J = 3$

$Q^{1.5}H_m(t)$

$$= \begin{bmatrix} 0.0117539 & 0.0610753 & 0.131413 & 0.217686 & 0.317357 & 0.428818 & 0.550933 & 0.682843 \\ 0.0117539 & 0.0610753 & 0.131413 & 0.217686 & 0.293849 & 0.306667 & 0.288107 & 0.24747 \\ 0.0117539 & 0.0610753 & 0.107905 & 0.0955356 & 0.0662843 & 0.0545208 & 0.047633 & 0.0428933 \\ 0 & 0 & 0 & 0 & 0.0117539 & 0.0610753 & 0.107905 & 0.0955356 \\ 0.117539 & 0.0375674 & 0.0210165 & 0.0159352 & 0.0133974 & 0.0117908 & 0.010654 & 0.00979445 \\ 0 & 0 & 0.117539 & 0.0375674 & 0.0210165 & 0.0159352 & 0.0133974 & 0.0117908 \\ 0 & 0 & 0 & 0 & 0.117539 & 0.0375674 & 0.0210165 & 0.0159352 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0.117539 & 0.0375674 \end{bmatrix}.$$

In the same way, we can generate the operational matrix of Haar wavelets for different values of α as per our requirements.

Definition 2. [32] A real function $f(x)$, for all $x > 0$, is said to be in the space $C_\mu, \mu \in \mathbb{R}$, if there exists a real number $p > \mu$, such that $f(x) = x^p f_1(x)$, where $f_1(x) \in C[0, \infty)$ and it is said to be in the space C_μ^n if and only if $f^{(n)}(x) \in C_\mu, n \in \mathbb{N}$.

Definition 3. [32] The fractional derivative of $f(x)$ in the Caputo sense is defined as

$$D^\alpha f(x) = I^{n-\alpha} D^n f(x) = \frac{1}{\Gamma(n-\alpha)} \int_0^x (x-t)^{x-\alpha-1} f^n(t) dt, \tag{6}$$

for $n - 1 < \alpha \leq n, n \in \mathbb{N}, x > 0, f \in C_{-1}^n$.

Definition 4. The fractional integration of x^n in the Caputo sense is given by

$$I^\alpha x^n = \frac{x^{n+\alpha}\Gamma(1+n)}{\Gamma(1+n+\alpha)}, \tag{7}$$

for $n - 1 < \alpha \leq n, n \in \mathbb{N}, x > 0$.

3 Method of solution

3.1 Homotopy analysis method

Consider the nonlinear FDEs with different physical conditions,

$$\mathscr{N}[y(t)] = 0, \quad t \geq 0, \tag{8}$$

where $\mathscr{N}$ represents the nonlinear differential operator, and $y(t)$ is the function to be determined.

Zeroth order deformation equation:
Let $y_0(t)$ be the initial approximation to the actual solution of (8). Liao constructed zeroth deformation equations taking the auxiliary function $\mathscr{H}(t)$ ($\neq 0$) and auxiliary parameter $\hbar$ ($\neq 0$) as [16],

$$(1-q)\mathscr{L}[\phi(t;q) - y_0(t)] = q\hbar\mathscr{H}(t)\mathscr{N}[\phi(t;q)], \tag{9}$$

where, $\phi(t;q)$ is a unknown function and $\mathscr{L}$ is a Linear operator.

When $q = 0$, (9) becomes, $\phi(t;0) = y_0\ (t)$ and at $q = 1$, (9) becomes, $\phi(t;1) = y(t)$. So as the q varies from 0 to 1, the function $\phi(t;q)$ varies from initial approximation $y_0\ (t)$ to the actual solution $y(t)$. Defining the mth order deformation derivatives,

$$y_m(t) = \frac{1}{m!}\frac{\partial^m \phi(t;q)}{\partial q^m}, \tag{10}$$

expanding $\phi(t;q)$, and using the Taylor series with respect to q, we get

$$\phi(t;q) = y_0(t) + \sum_{m=1}^{\infty} y_m(t) q^m. \tag{11}$$

As we know at $q = 1$ $\phi(t;q)$ becomes the required solution, (11) at $q = 1$ becomes

$$\phi(t;1) = y(t) = y_0(t) + \sum_{m=1}^{\infty} y_m(t). \tag{12}$$

Similarly, mth order deformation equation is given by

$$\mathscr{L}[y_m(t) - \chi_m y_{m-1}(t)] = \hbar\mathscr{H}(t)\mathscr{R}_m(y_{m-1}(t)), \tag{13}$$

where

$$\chi_m = \begin{cases} 0 & \text{if} \quad m \leq 1, \\ 1 & \text{otherwise}, \end{cases} \tag{14}$$

$$\mathscr{R}_m(y_{m-1}(t)) = \frac{1}{(m-1)!} \frac{\partial^{m-1}[\mathscr{N}[\phi(t;q)]]}{\partial q^{m-1}}. \tag{15}$$

Thus $y_1(t)$, $y_2(t)$, $y_3(t)$, $\ldots$ can be obtained from solving (13). The mth order approximation of $y(t)$ [21, 22, 23] is given by

$$y(t) = \sum_{m=0}^{m} y_m(t). \tag{16}$$

Equation (16) is the semi-analytical solution of (8).

NOTE: The above method is the same for solving fractional order differential equations. The inverse of the linear operator will be integration in a nonfractional order differential equation, whereas, for fractional order, it will be a fractional integration.

3.2 Haar wavelet method

Consider the general FDEs of the form

$$aD^{\alpha}y(t) + bg(y(t)) + cy(t) = f(t), \tag{17}$$

with initial conditions

$$y(\beta_1) = \alpha_1, \quad y'(\beta_1) = \alpha_2.$$

The Haar wavelet approximation is given as

$$D^{''}y(t) = \sum_{i=1}^{m} a_i H_m(t), \tag{18}$$

$$D^{'}y(t) = y^{'}(0) + \sum_{i=1}^{m} a_i Q^1 H_m(t),$$

$$D^{'}y(t) = \beta_2 + \sum_{i=1}^{m} a_i Q^1 H_m(t). \tag{19}$$

Integrating the above equation with respect to "t" from 0 to t, we get

$$y(t) = y(0) + \beta_2 t + \sum_{i=1}^{m} a_i Q^2 H_m(t),$$
$$y(t) = \beta_1 + \beta_2 t + \sum_{i=1}^{m} a_i Q^2 H_m(t). \tag{20}$$

Fractionally differentiate the above equation of order α, we obtain

$$D^\alpha y(t) = D^\alpha\{\beta_1 + \beta_2 t\} + \sum_{i=1}^{m} a_i Q^{2-\alpha} H_m(t), \tag{21}$$

where $Q^{2-\alpha}H_m(t)$ is the $(2-\alpha)$th order operational matrix of integration.

Substituting (18), (20), and (21) in (17), we get

$$aD^\alpha\{\beta_1 + \beta_2 t\} + \sum_{i=1}^{m} a_i Q^{2-\alpha} H_m(t) + bg(\beta_1 + \beta_2 t + \sum_{i=1}^{m} a_i Q^2 H_m(t))$$
$$+ c(\beta_1 + \beta_2 t + \sum_{i=1}^{m} a_i Q^2 H_m(t)) = f(t). \tag{22}$$

Collocate this equation by $t_l = A + (l - 0.5)(B - A)/m$, which yields the following equation:

$$aD^\alpha\{\beta_1 + \beta_2 t_l\} + \sum_{i=1}^{m} a_i Q^{2-\alpha} H_m(t_l) + bg(\beta_1 + \beta_2 t_l + \sum_{i=1}^{m} a_i Q^2 H_m(t_l))$$
$$+ c(\beta_1 + \beta_2 t_l + \sum_{i=1}^{m} a_i Q^2 H_m(t_l)) = f(t_l). \tag{23}$$

Solving the system of algebraic system equation (23) by the Newton–Raphson method gives the unknown coefficient values. Substituting these obtained unknown values in (20) will provide the wavelet numerical solution of (17).

4 Convergence analysis

Theorem 1. [21] As long as the series $y_0(t) + \sum_{m=1}^{\infty} y_m(t)$ converges, where $y_m(t)$ is governed by the higher order deformation equation number the χ_m given by (14), it must be the exact solution.

Theorem 2. [26] Let $\phi_0, \phi_1, \phi_2, \ldots$ be the solution components of a given equation. The series solution $\sum_{k=0}^{\infty} \phi_k(t)$ converges if there exists $0 < \gamma < 1$ such that $||\phi_{k+1}|| \leq \gamma\|\phi_k\|$, for all $k \geq k_0$ for some $k_0 \in \mathbb{N}$.

Theorem 3. [26] Assume that the series solution $\sum_{k=0}^{\infty} \phi_k(t)$ is convergent to the solution $y(t)$. If the truncation series $\sum_{k=0}^{m} \phi_k(t)$ is used as an approximation to the solution $y(t)$, then the maximum absolute truncation error is estimated as, $\|y(t) - \sum_{k=0}^{m} \phi_k(t)\| \leq \frac{1}{1-\gamma}\gamma^{m+1}\|\phi_0(t)\|$.

Theorem 4. [6] Suppose that the functions $D_*^{\alpha} u_k(t)$ obtained by using Haar wavelets are the approximation of $D_*^{\alpha} u(t)$. Then we have an exact upper bound as follows:

$$\|D_*^{\alpha} u(t) - D_*^{\alpha} u_k(t)\|_E \leq \frac{M}{\Gamma(m-\alpha).(m-\alpha)} \frac{1}{[1-2^{2(\alpha-m)}]^{\frac{1}{2}}} \frac{1}{k^{m-\alpha}},$$

where $\|u(t)\|_E = (\int_0^1 u^2(t)dt)^{\frac{1}{2}}$.

5 Applications

Example 1. Consider the linear fractional model [24],

$$D^{\alpha} y(t) + y(t) = f(t), \quad t \geq 0, \quad and \quad 1 \leq \alpha \leq 2, \tag{24}$$

with the initial conditions

$$y^{(k)}(0) = 0, \quad k = 0, 1, \tag{25}$$

where $f(t) = te^{-t}$. The exact solution is $y(t) = -\frac{1}{2}e^{-t}\left(-1 - t + e^t \cos(t)\right)$.

Method of implementation: Applying the HAM method to solve (24) and concerning (9), the zeroth order deformation is given by

$$(1-q)\mathscr{L}[\phi(t;q) - y_0(t)] = q\hbar\mathscr{H}(t)[D^{\alpha}\phi(t;q) + \phi(t;q) - f(t)]. \tag{26}$$

According to the condition (25), we choose the initial approximation as $y_0(t) = t^2$ and take the linear operator as $L = D^{\alpha}$ with $L(C_1 + C_2 t) = 0$, where C_1 and C_2 are integral constants with $\mathscr{H}(t) = 1$. Therefore, mth order deformation is given by

$$D^{\alpha}[y_m(t) - \chi_m y_{m-1}(t)] = \hbar \mathscr{R}_m(y_{m-1}(t)), \tag{27}$$

where

$$\mathscr{R}_m(y_{m-1}(t)) = \frac{1}{(m-1)!} \frac{\partial^{m-1}[D^{\alpha}\phi(t;q) + \phi(t;q) - f(t)]}{\partial q^{m-1}}, \tag{28}$$

subject to

$$y_m(0) = 0, \qquad y'_m(0) = 0. \tag{29}$$

Applying I^{α}, the inverse of D^{α} on either sides of (27), we get

$$y_m(t) = \chi_m y_{m-1}(t) + \hbar I^{\alpha}[\mathscr{R}_m(y_{m-1}(t))] + C_1 + C_2 t, \quad m \geq 1,$$

where C_1 and C_2 are calculated by using (29).

The HAM series solution upto first seven terms when $\alpha = 1.5$ with $\hbar = -1$ is given by

$$\begin{aligned} y(t) = {} & 0.306625t^{2.5} - 0.178204t^{3.5} - 0.0424592t^{4} + 0.0607691t^{4.5} + 0.0172735t^{5} \\ & - 0.0116399t^{5.5} - 0.00441781t^{6} + 0.00192798t^{6.5} + 0.000664872t^{7} \\ & - 0.000284779t^{7.5} - 0.0000894778t^{8} + 0.0000471467t^{8.5} + 0.0000110138t^{9} \\ & - 6.78818 \times 10^{-6}t^{9.5} - 1.54989 \times 10^{-6}t^{10} + 8.99097 \times 10^{-7}t^{10.5} \\ & + 2.42827 \times 10^{-7}t^{11} - 1.19467 \times 10^{-7}t^{11.5} - 2.23355 \times 10^{-8}t^{12} \\ & + 2.2627 \times 10^{-8}t^{12.5} + 2.62537 \times 10^{-9}t^{13} + 3.25559 \times 10^{-10}t^{13.5} \\ & - 4.43969 \times 10^{-10}t^{14} - 3.57732 \times 10^{-11}t^{14.5} - 5.74907 \times 10^{-12}t^{15} + \cdots . \end{aligned}$$

The HAM series solution upto first seven terms when $\alpha = 2$ with $\hbar = -1$ is given by

$$\begin{aligned} y(t) = {} & \frac{t^3}{6} - \frac{t^4}{12} + \frac{t^5}{60} - \frac{t^6}{360} + \frac{t^7}{1680} - \frac{t^8}{10080} + \frac{t^9}{90720} - \frac{t^{10}}{907200} + \frac{t^{11}}{7983360} \\ & - \frac{t^{12}}{79833600} + \frac{23t^{13}}{1037836800} + \frac{t^{14}}{10897286400} - \frac{139t^{15}}{1307674368000} - \cdots . \end{aligned}$$

Also, the above problem was solved through HWT and ND solvers. The results obtained are explained through the tables and graphs. Figure 1 shows the geometrical comparison of exact, HAM, and HTM solutions. Figure 2 reflects the graphical presentation of the HAM solution with different α values. The error analysis of HAM, HWT, and ND solver solutions with the exact solution is shown in Figure 3. Numerical values for integer values and noninteger values of α have been given in Tables 1 and 2, respectively.

Table 1: Comparison between HAM and HWT solutions for different fractional values of α in Example 1

t	$\alpha = 1.2$		$\alpha = 1.5$		$\alpha = 1.7$	
	HAM	HWT	HAM	HWT	HAM	HWT
0.0	0	0	0	0	0	0
0.1	0.002456	0.002359	0.000911	0.000872	0.000457	0.00047
0.2	0.010388	0.010175	0.004826	0.000872	0.002802	0.002739
0.3	0.023282	0.0228757	0.012430	0.012118	0.007889	0.007729
0.4	0.040204	0.039558	0.023801	0.023309	0.016138	0.015862
0.5	0.060161	0.059252	0.038723	0.0380337	0.027696	0.027294
0.6	0.082198	0.081026	0.056803	0.0559103	0.042517	0.041987
0.7	0.105453	0.104031	0.077547	0.076456	0.060412	0.059758
0.8	0.129169	0.127521	0.100408	0.099131	0.081087	0.080318
0.9	0.152700	0.150860	0.124818	0.0123377	0.104172	0.103302

Table 2: Comparison of solutions obtained from ND Solver, HAM, HWT and their absolute errors (AE) with exact solution for integer order $\alpha = 2$ of Example 1

t	Exact	ND Solver	HAM	HWT	ND Solver Error	HAM Error	HWT Error
0.0	0	0	0	0	0	0	0
0.1	0.000159	0.000159	0.000159	0.000189	4.72×10^{-9}	4.7×10^{-9}	3.09×10^{-5}
0.2	0.001205	0.001205	0.001205	0.001264	1.61×10^{-8}	1.61×10^{-8}	5.88×10^{-5}
0.3	0.003863	0.003864	0.003864	0.003947	6.12×10^{-9}	6.1×10^{-9}	8.32×10^{-5}
0.4	0.008694	0.008694	0.008694	0.008798	6.28×10^{-9}	6.3×10^{-9}	1.04×10^{-4}
0.5	0.016107	0.016107	0.016107	0.016229	7.26×10^{-9}	7.3×10^{-9}	1.22×10^{-4}
0.6	0.026382	0.026382	0.026382	0.026518	8.18×10^{-9}	8.2×10^{-9}	1.37×10^{-4}
0.7	0.039676	0.039677	0.039676	0.039825	1.30×10^{-8}	1.28×10^{-8}	1.49×10^{-4}
0.8	0.056043	0.056043	0.056043	0.056200	1.54×10^{-8}	1.42×10^{-8}	1.57×10^{-4}
0.9	0.075436	0.075436	0.075436	0.075599	1.70×10^{-8}	1.16×10^{-8}	1.63×10^{-4}

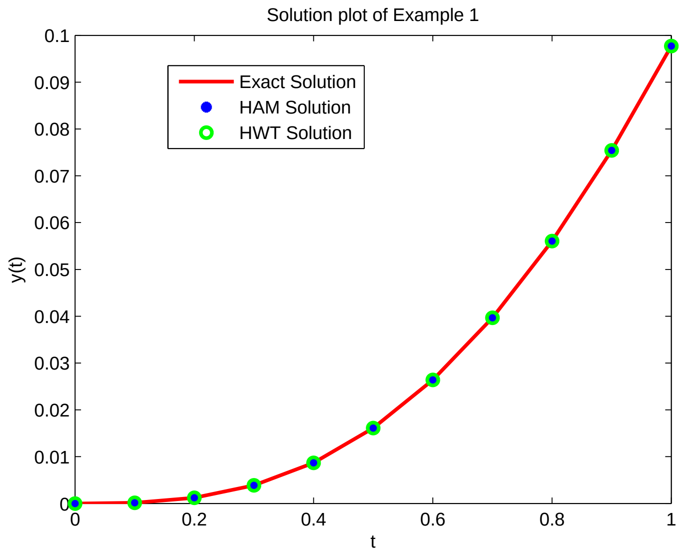

Figure 1: Comparison of the exact solution with HAM and HWT solutions, for Example 1.

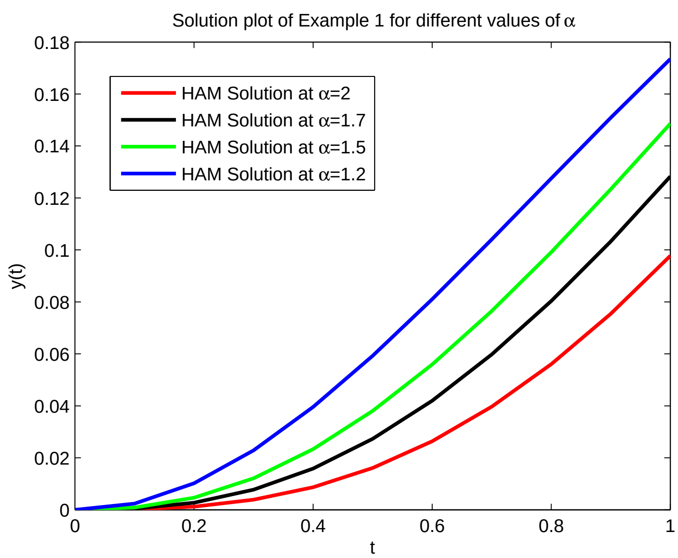

Figure 2: Graphical interpretation of the HAM solutions at different α values for Example 1.

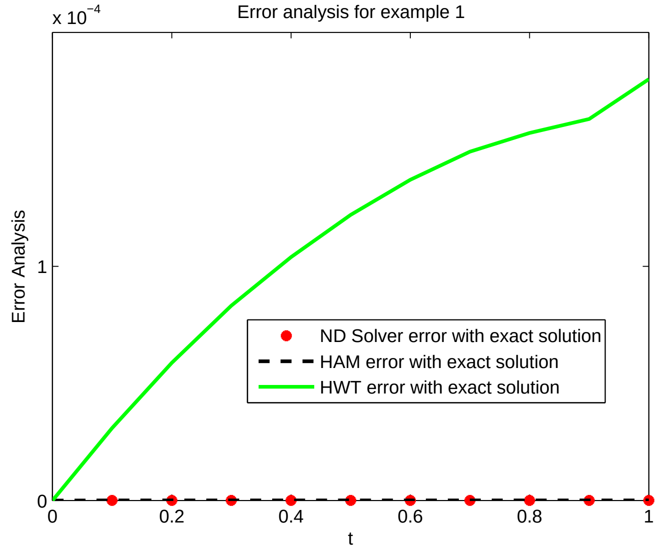

Figure 3: Graphical interpretation of the error analysis, for Example 1.

Example 2. Consider the nonlinear fractional model [24],

$$D^{\alpha}y(t)+y^{2}(t)=f(t),\quad 1\leq\alpha\leq 2, \tag{30}$$

subject to the conditions

$$y(0)=0,\quad y'(0)=0, \tag{31}$$

where $f(t)=\frac{\Gamma(6)}{\Gamma(6-\alpha)}t^{5-\alpha}-\frac{3\Gamma(5)}{\Gamma(5-\alpha)}t^{4-\alpha}+\frac{2\Gamma(4)}{\Gamma(4-\alpha)}t^{3-\alpha}+(t^{5}-3t^{4}+2t^{3})^{2}$. The exact solution is [24] $y(t)=t^{5}-3t^{4}+2t^{3}$.

Method of implementation: Applying HAM method to solve (30) and concerning (9), the zeroth order deformation is given by

$$(1-q)\mathscr{L}[\phi(t;q)-y_{0}(t)]=q\hbar\mathscr{H}(t)[D^{\alpha}\phi(t;q)+\phi^{2}(t;q)-f(t)]. \tag{32}$$

According to the condition (31), choosing the initial approximation $y_0\ (t)=\frac{t^{\alpha}}{\Gamma(\alpha+1)}$ and the linear operator as $\mathscr{L}=D^{\alpha}$ with $\mathscr{L}(C_1+C_2t)=0$, where C_1 and C_2 are integral constants with $\mathscr{H}(t)=1$, therefore, the mth order deformation is given by

$$D^{\alpha}[y_m(t) - \chi_m y_{m-1}(t)] = \hbar \mathscr{R}_m(y_{m-1}(t)), \tag{33}$$

where

$$\mathscr{R}_m(y_{m-1}(t)) = \frac{1}{(m-1)!} \frac{\partial^{m-1}[D^{\alpha}\phi(t;q) + \phi^2(t;q) - f(t)]}{\partial q^{m-1}}, \tag{34}$$

subject to

$$y_m(0) = 0, \qquad y'_m(0) = 0. \tag{35}$$

Applying I^{α}, the inverse of D^{α} on either sides of (33), we get

$$y_m(t) = \chi_m y_{m-1}(t) + \hbar I^{\alpha}[\mathscr{R}_m(y_{m-1}(t))] + C_1 + C_2 t, \quad m \geq 1,$$

where C_1 and C_2 are calculated by using (35).

The HAM series upto first six terms when $\alpha = 1.5$ and taking $\hbar = -1$ is given by

$$\begin{aligned} y(t) = {} & 2t^3 - 3t^4 - 5.55112 \times 10^{-17} t^{4.5} + 1t^5 + 1.11022 \times 10^{-16} t^{8.5} - 1.66533 \\ & \times 10^{-16} t^{9.5} - 1.38778 \times 10^{-17} t^{10} - 0.00107409 t^{10.5} + 3.46945 \times 10^{-18} t^{11} \\ & + 1.38778 \times 10^{-17} t^{11.5} + 0.00410658 t^{12} - 0.00447957 t^{13} - 0.00465867 t^{13.5} \\ & + 0.00114198 t^{14} + 0.010066 t^{14.5} + 0.000170673 t^{15} - 0.00817004 t^{15.5} \\ & - 0.000182659 t^{16} + 0.0049056 t^{16.5} + 0.0000458036 t^{17} - 0.00927567 t^{17.5} \\ & - 0.000316896 t^{18} + 0.0173363 t^{18.5} + 0.00101742 t^{19} - 0.0193873 t^{19.5} \\ & - 0.00135484 t^{20} + 0.0135402 t^{20.5} + 0.00107153 t^{21} - 0.00602444 t^{21.5} \\ & - 0.00102235 t^{22} + 0.00165852 t^{22.5} + 0.00169885 t^{23}. - 0.000236628 t^{23.5} \\ & - 0.00240855 t^{24} - 0.0000267944 t^{24.5} + 0.002328 t^{25} + 0.000048839 t^{25.5} - \cdots . \end{aligned}$$

The HAM series upto first six terms when $\alpha = 2$ and taking $\hbar = -1$, is given by

$$\begin{aligned} y(t) = {} & 2t^3 - 3t^4 + t^5 - \frac{t^{14}}{374400} + \frac{79t^{15}}{4365900} - \frac{12469t^{16}}{232848000} + \frac{19801t^{17}}{257297040} \\ & - \frac{2940419t^{18}}{198486288000} - \frac{6625093t^{19}}{43135092000} + \frac{311661149t^{20}}{985944960000} - \frac{3268452721t^{21}}{9725828112000} \\ & + \frac{45173665739t^{22}}{203779255680000} - \frac{103894399937t^{23}}{1113144184152000} + \frac{455437961213t^{24}}{20778691437504000} \\ & + \frac{18932926013t^{25}}{5081745188520000} - \cdots . \end{aligned}$$

Additionally, the HWT is used to resolve the above problem. Graphs and tables are used to explain the results that were obtained. The geometrical comparison of the exact HAM and HTM solutions is shown in Figure 4. The graphical depiction of the HAM solution with various values of α is shown in Figure 5. Figure 6 displays the error analysis of the HAM and HWT answers with the precise solution. Tables 3 and 4 provide numerical values for the integer and noninteger values of α accordingly, including the results available in the literature.

Table 3: Comparison between HAM and HWT solutions for different fractional α values in Example 2

t	$\alpha = 1.2$		$\alpha = 1.5$		$\alpha = 1.7$	
	HAM	HWT	HAM	HWT	HAM	HWT
0.0	0	0	0	0	0	
0.1	0.001710	0.001799	0.001710	0.001790	0.001710	0.001799
0.2	0.011520	0.011555	0.011520	0.011565	0.011520	0.011511
0.3	0.032129	0.032988	0.032130	0.032153	0.032130	0.032125
0.4	0.061432	0.061457	0.061440	0.061477	0.061440	0.061410
0.5	0.093701	0.093365	0.093750	0.093799	0.093750	0.093702
0.6	0.120744	0.120875	0.120958	0.120921	0.120960	0.120999
0.7	0.132991	0.132822	0.133760	0.133796	0.133770	0.133765
0.8	0.120401	0.120897	0.122840	0.122856	0.122878	0.122825
0.9	0.072946	0.072365	0.080041	0.080035	0.080182	0.080147
1.0	-0.019915	-0.019896	-0.000521	-0.000988	-0.000034	-0.000988

Table 4: Comparison of solutions obtained from HAM, HOFLMSM, HWT, and their AE with exact solution for integer order $\alpha = 2$ of Example 2

t	Exact	HAM	HOFLMSM [24]	HWT	HAM Error	HOFSMLM Error [24]	HWT Error
0	0	0	0	0	0	0	0
0.1	0.001710	0.001710	-	0.002044	0	-	3.34×10^{-4}
0.2	0.011520	0.0115200	0.011517	0.012099	0	3.222×10^{-6}	5.79×10^{-4}
0.3	0.032130	0.032130	-	0.032866	2.1×10^{-9}	-	7.35×10^{-4}
0.4	0.061440	0.061440	0.061411	0.062269	3.66×10^{-8}	2.920×10^{-5}	8.29×10^{-4}
0.5	0.093750	0.093750	-	0.094641	3.268×10^{-7}	-	8.91×10^{-4}
0.6	0.120960	0.120958	0.120920	0.12187	1.991×10^{-6}	3.972×10^{-5}	9.26×10^{-4}
0.7	0.13377	0.133760	-	0.134730	9.592×10^{-6}	-	9.60×10^{-4}
0.8	0.122880	0.122840	0.122845	0.012389	3.979×10^{-5}	3.49×10^{-5}	1.01×10^{-3}
0.9	0.08019	0.080041	-	0.081295	1.494×10^{-4}	-	1.10×10^{-3}
1.0	0	-0.000521	-0.000015	-0.001106	5.212×10^{-4}	1.545×10^{-5}	1.10×10^{-3}

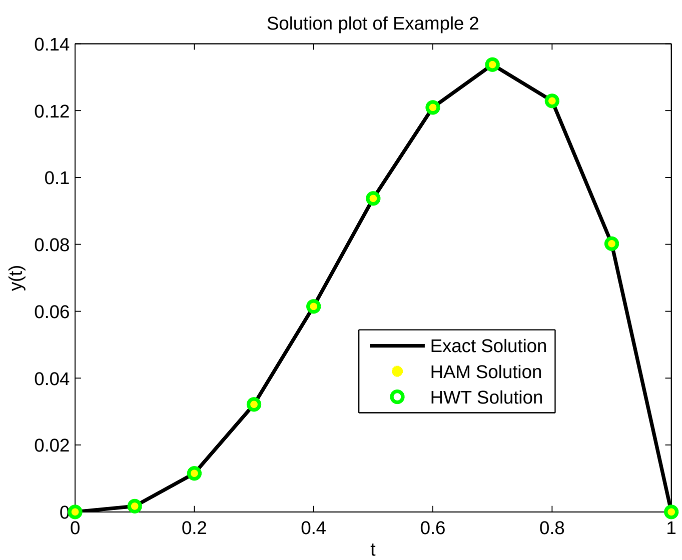

Figure 4: Comparison of the exact solution with HAM and HWT solutions, for Example 2.

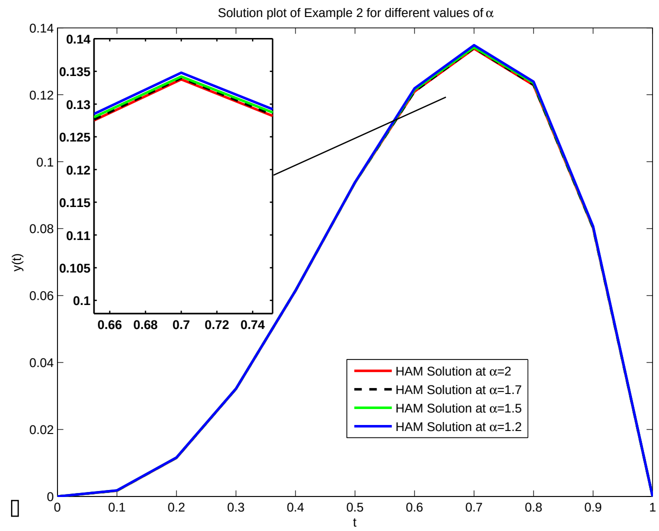

Figure 5: Graphical interpretation of the HAM solutions at different α values for Example 2.

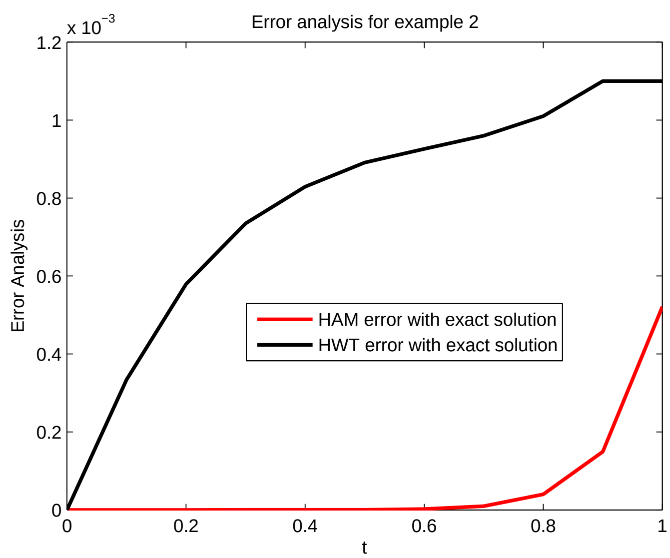

Figure 6: Graphical interpretation of the error analysis, for Example 2.

Example 3. Consider the nonlinear fraction model [31],

$$D^{\alpha}y(t) - t\sinh(y(t)) - 1 = 0, \quad 2 \leq \alpha \leq 3, \tag{36}$$

subject to the conditions

$$y(0) = 0, \quad y(0.25) = 1, \quad y(1) = 0. \tag{37}$$

Method of implementation: Applying HAM method to solve (36) and concerning (9), the zeroth order deformation is given by

$$(1-q)\mathscr{L}[\phi(t;q) - y_0(t)] = q\hbar\mathscr{H}(t)[D^{\alpha}\phi(t;q) - \sinh(\phi(t;q)) - 1]. \tag{38}$$

According to the condition (37) and choosing the initial approximation $y_0(t) = \frac{t^{\alpha}}{\Gamma(\alpha+1)}$, we take the linear operator $\mathscr{L} = D^{\alpha}$ with $\mathscr{L}(C_1 + C_2t + C_3t^2) = 0$, where C_1, C_2, and C_3 are integral constants with $\mathscr{H}(t) = 1$. Therefore, the mth order deformation is given by

$$D^{\alpha}[y_m(t) - \chi_m y_{m-1}(t)] = \hbar\mathscr{R}_m(y_{m-1}(t)), \tag{39}$$

where

$$\mathscr{R}_m(y_{m-1}(t)) = \frac{1}{(m-1)!}\frac{\partial^{m-1}[D^{\alpha}\phi(t;q) - \sinh(\phi(t;q)) - 1]}{\partial q^{m-1}}, \tag{40}$$

subject to

$$y_m(0) = 0, \quad y_m(0.25) = 0, \quad y_m(1) = 0. \tag{41}$$

Applying I^{α}, the inverse of D^{α} on either sides of (39), we get

$$y_m(t) = \chi_m y_{m-1}(t) + \hbar I^{\alpha}[\mathscr{R}_m(y_{m-1}(t))] + C_1 + C_2t + C_3t^2, \quad m \geq 1,$$

where C_1, C_2, and C_3 are calculated by using (41).

The HAM series upto first six terms when $\alpha = 2.5$ and taking $\hbar = -1$ is given by

$$\begin{aligned} y(t) = {} & 5.42172t - 5.85394t^2 + 0.300901t^{2.5} + 0.13254t^{3.5} + 0.00333693t^{4.5} \\ & - 0.0107536t^{5.5} + 0.00486111t^6 + 6.71025 \times 10^{-7}t^{6.5} + 0.00133921t^7 \\ & - 8.70736 \times 10^{-6}t^{7.5} + 0.0000293912t^8 + 0.0000396298t^{8.5} \\ & - 0.000104465t^9 - 0.000025573t^{9.5} + 0.000110933t^{10} - 0.000061787t^{10.5} \end{aligned}$$

$$+ 0.0000292694t^{11} - 0.0000240961t^{11.5} + 9.8944 \times 10^{-6}t^{12} - 8.36776 \times 10^{-7}t^{12.5} - 7.54444 \times 10^{-7}t^{13} + 2.01592 \times 10^{-6}t^{13.5} - 1.14756 \times 10^{-6}t^{14} - 1.27511 \times 10^{-7}t^{14.5} + 3.67453 \times 10^{-7}t^{15} - 1.41232 \times 10^{-7}t^{15.5} + 1.8676 \times 10^{-8}t^{16} - 6.94883 \times 10^{-9}t^{16.5} + 1.11916 \times 10^{-8}t^{17} + 1.86632 \times 10^{-9}t^{17.5} - 1.1398 \times 10^{-8}t^{18} + 7.4029 \times 10^{-9}t^{18.5} - 7.60145 \times 10^{-10}t^{19} - 1.21074 \times 10^{-9}t^{19.5} + 1.38057 \times 10^{-10}t^{20} + 4.59674 \times 10^{-10}t^{20.5} + 2.71044 \times 10^{-10}t^{21} - 9.18445 \times 10^{-10}t^{21.5} + 7.2516 \times 10^{-10}t^{22} - 2.61436 \times 10^{-10}t^{22.5} + 3.73559 \times 10^{-11}t^{23} - 4.92754 \times 10^{-13}t^{23.5} + 1.0376 \times 10^{-12}t^{24} - 6.09013 \times 10^{-13}t^{24.5} + 1.22102 \times 10^{-13}t^{25}.$$

The HAM series upto first six terms when $\alpha = 3$ and taking $h = -1$ is given by

$$y(t) = 5.39493t - 5.62542t^2 + \frac{t^3}{6} + 0.0642361t^4 + 0.00101757t^5 - 0.00241502t^6 + 0.000793762t^7 + 0.000184995t^8 + 5.78057 \times 10^{-6}t^9 - 0.0000101977t^{10} + 6.76308 \times 10^{-6}t^{11} - 1.72474 \times 10^{-6}t^{12} - 3.70131 \times 10^{-7}t^{13} + 2.20798 \times 10^{-7}t^{14} + 4.34937 \times 10^{-8}t^{15} - 2.3555 \times 10^{-8}t^{16} + 7.1741 \times 10^{-9}t^{17} - 1.46382 \times 10^{-9}t^{18} + 2.83531 \times 10^{-10}t^{19} - 5.93013 \times 10^{-11}t^{20} + 2.72701 \times 10^{-11}t^{21} - 1.25142 \times 10^{-11}t^{22} + 5.48224 \times 10^{-12}t^{23} - 2.21553 \times 10^{-12}t^{24} + 7.74129 \times 10^{-13}t^{25} - 1.78755 \times 10^{-13}t^{26} + 1.97752 \times 10^{-14}t^{27} - 1.1748 \times 10^{-16}t^{28} + 1.46076 \times 10^{-17}t^{29}.$$

The above problem is also solved using the HWT and ND solver. The graphs and tables provide an explanation of the results. Figure 7 compares ND solver, HAM, and HTM solutions geometrically. The HAM solution is graphically depicted with various values of α in Figure 8. Figure 9 depicts the error analysis of the solutions from the HAM, HWT, and ND solvers. Tables 5 and 6 contain numerical values for the fractional and nonfractional values of α, respectively.

Table 5: Comparison between HAM and HWT solutions for different fractional values of α in Example 3

t	$\alpha = 2.2$		$\alpha = 2.5$		$\alpha = 2.7$	
	HAM	HWT	HAM	HWT	HAM	HWT
0.1	0.4844060642	0.484351	0.4838689878	0.484351	0.4836640338	0.483640
0.2	0.8558871839	0.855887	0.8555770932	0.855887	0.8554919337	0.855491
0.3	1.1166930600	1.116703	1.1170945299	1.117105	1.1171714074	1.117176
0.4	1.2687652290	1.268765	1.2701278494	1.270128	1.2702944408	1.270294
0.5	1.3139530304	1.313949	1.3162987718	1.316293	1.3164440681	1.316442
0.6	1.2540670349	1.254067	1.2571981039	1.257198	1.2572226052	1.257222
0.7	1.0909051719	1.090910	1.0944110459	1.094417	1.0942652940	1.094268
0.8	0.8262733802	0.826273	0.8295325852	0.829533	0.8292490252	0.829249
0.9	0.4620075965	0.461998	0.4641792631	0.464165	0.4638997104	0.463893
1.0	0	0	0	0	0	0

Table 6: Comparison of solutions obtained from HAM, HWT, and their AE with ND Solver solution for integer order $\alpha = 3$ of Example 3

t	ND Solver	HAM	HWT	HAM Error	HWT Error
0.0	0	0	0	0	0
0.1	0.4850443102	0.4834121287	0.483293	1.63×10^{-3}	1.75×10^{-3}
0.2	0.8564165257	0.8554060228	0.855347	1.01×10^{-3}	1.06×10^{-3}
0.3	1.1158509874	1.1172133089	1.117272	1.36×10^{-3}	1.42×10^{-3}
0.4	1.2655112703	1.2702191975	1.270362	4.70×10^{-3}	4.85×10^{-3}
0.5	1.3078943289	1.3159609254	1.316153	8.06×10^{-3}	8.25×10^{-3}
0.6	1.2456730868	1.2561260143	1.256409	1.04×10^{-2}	1.07×10^{-2}
0.7	1.0815016154	1.0925510067	1.093045	1.10×10^{-2}	1.15×10^{-2}
0.8	0.8178293569	0.8272214124	0.828006	9.39×10^{-3}	1.10×10^{-2}
0.9	0.4567647525	0.4622736694	0.463128	5.509×10^{-3}	6.3×10^{-3}
1.0	0	0	0	0	0

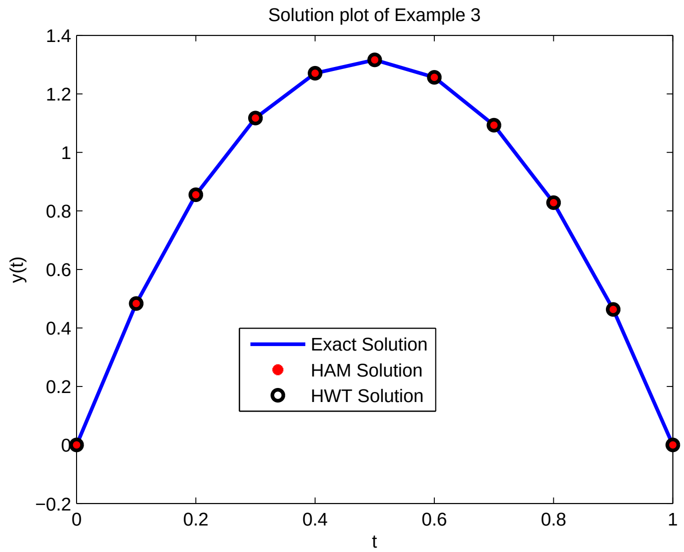

Figure 7: Comparison of the exact solution with HAM and HWT solutions, for Example 3.

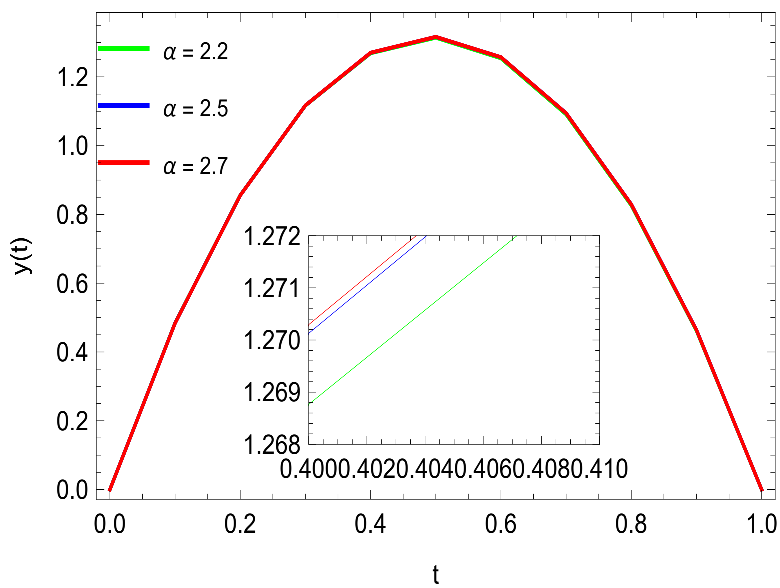

Figure 8: Graphical interpretation of the HAM solutions at different α values for Example 3.

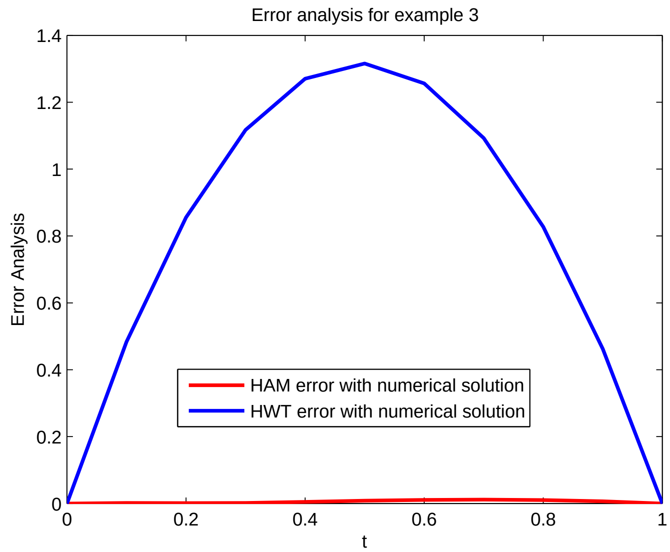

Figure 9: Graphical interpretation of the error analysis, for Example 3.

Example 4. Consider the nonlinear fractional model,

$$D^{\alpha}y(t) - \sin(y(t)) = 0, \quad 1 \leq \alpha \leq 2, \tag{42}$$

subject to the conditions

$$y(0) = \pi, \quad y'(0) = -2. \tag{43}$$

Method of implementation: Applying HAM to solve (42) and concerning (9), the zeroth order deformation is given by

$$(1-q)\mathscr{L}[\phi(t;q) - y_0(t)] = q\hbar\mathscr{H}(t)[D^{\alpha}\phi(t;q) - \sin(\phi(t;q))]. \tag{44}$$

According to the condition (43) and choosing initial approximation $y_0(t) = \pi - 2t$, we take the linear operator as $\mathscr{L} = D^{\alpha}$ with $\mathscr{L}(C_1 + C_2 t) = 0$, where C_1 and C_2 are integral constants with $\mathscr{H}(t) = 1$. Therefore, mth order deformation is given by

$$D^{\alpha}[y_m(t) - \chi_m y_{m-1}(t)] = \hbar\mathscr{R}_m(y_{m-1}(t)), \tag{45}$$

where

$$\mathscr{R}_m(y_{m-1}(t)) = \frac{1}{(m-1)!}\frac{\partial^{m-1}[D^\alpha\phi(t;q) - \sin(\phi(t;q))]}{\partial q^{m-1}} \tag{46}$$

subject to

$$y_m(0) = 0, \qquad y_m'(0) = 0. \tag{47}$$

Applying I^α, the inverse of D^α on either sides of (45), we get

$$y_m(t) = \chi_m y_{m-1}(t) + \hbar I^\alpha[\mathscr{R}_m(y_{m-1}(t))] + C_1 + C_2 t, \quad m \geq 1,$$

where C_1 and C_2 are calculated using (47).

The HAM series upto first six terms when $\alpha = 2$ and taking $h = -1$ is given by

$$\begin{aligned} y(t) = \frac{1}{16986931200}&(100t(72(642315 - 7616t^2)\cos(2t) + 52736\cos(6t) \\ &+ 405\cos(8t) + 48(9(5381 - 40t^2)\cos(4t) + 4t(45691 - 48t^2 + 4293\cos(2t) \\ &+ 50\cos(4t))\sin(2t))) - 5(-3397386240\pi + 4168250428t \\ &+ 1152t^3(-835 + 16t^2) + 1660356360\sin(2t) + 66396240\sin(4t) \\ &+ 2073250\sin(6t) + 33885\sin(8t)) - 1026\sin(10t)). \end{aligned}$$

Additionally, the above problem was solved using the HWT and ND solver. Through the use of graphs and tables, outcomes are explained. Exact HAM and HTM solutions are geometrically compared in Figure 10. Figure 11 shows a graphic representation of the HAM solution with various values for α. Figure 12 displays the exact solution and error analysis of the HAM, HWT, and ND solver solutions. In Tables 7 and 8, respectively, numerical values for integer and noninteger values of α are provided.

Table 7: Comparison between HAM and HWT solutions for different fractional values of α in Example 4

t	$\alpha = 1.2$		$\alpha = 1.5$		$\alpha = 1.7$	
	HAM	HWT	HAM	HWT	HAM	HWT
0.1	2.946783	2.94657	2.943491	2.94343	2.942547	2.94256
0.2	2.765230	2.76433	2.752249	2.75192	2.747753	2.74762
0.3	2.598414	2.59546	2.570587	2.56965	2.559792	2.55934
0.4	2.446371	2.43943	2.400060	2.39788	2.380531	2.37946
0.5	2.308211	2.29522	2.241409	2.23711	2.211257	2.20912
0.6	2.182297	2.16154	2.094669	2.08729	2.052699	2.04892
0.7	2.066382	2.03708	1.959240	1.94802	1.905048	1.89911
0.8	1.957728	1.92051	1.833943	1.81866	1.767964	1.75956
0.9	1.853242	1.8106	1.7171035	1.69842	1.640616	1.62994
1.0	1.749611	1.70617	1.606626	1.58642	1.521720	1.50973

Table 8: Comparison of solutions obtained from HAM, HWT, and their AE with ND Solver solution for integer order $\alpha = 2$ of Example 4

t	ND Solver	HAM	HWT	HAM Error	HWT Error
0.1	2.941925	2.941925	2.94216	8.8×10^{-9}	2.35×10^{-4}
0.2	2.744233	2.744233	2.74474	1.57×10^{-8}	5.17×10^{-4}
0.3	2.550495	2.550395	2.55114	5.4×10^{-9}	7.44×10^{-4}
0.4	2.362110	2.362110	2.36307	3.5×10^{-9}	9.60×10^{-4}
0.5	2.180831	2.180831	2.18198	4.87×10^{-8}	1.14×10^{-3}
0.6	2.007722	2.007721	2.00903	2.033×10^{-7}	1.30×10^{-3}
0.7	1.843649	1.843648	1.84509	7.507×10^{-7}	1.44×10^{-3}
0.8	1.689183	1.689181	1.69073	2.3361×10^{-6}	1.55×10^{-3}
0.9	1.544628	1.544622	4.54625	5.8886×10^{-6}	1.62×10^{-3}
1.0	1.410054	1.410042	1.41194	1.16915×10^{-5}	1.89×10^{-3}

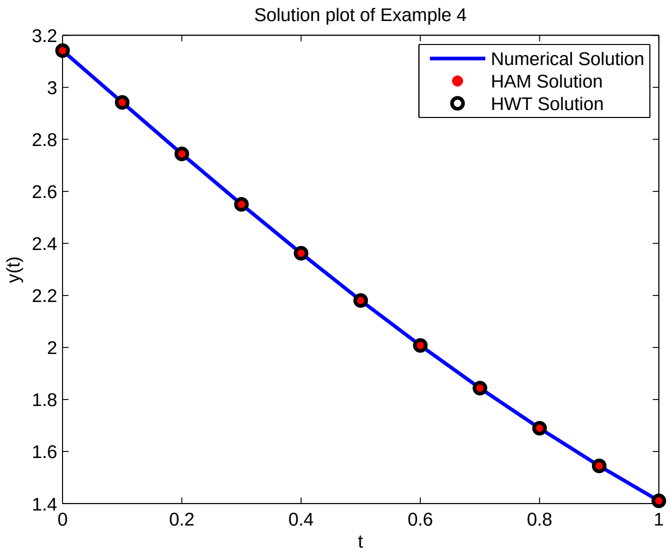

Figure 10: Comparison of the exact solution with HAM and HWT solutions, for Example 4.

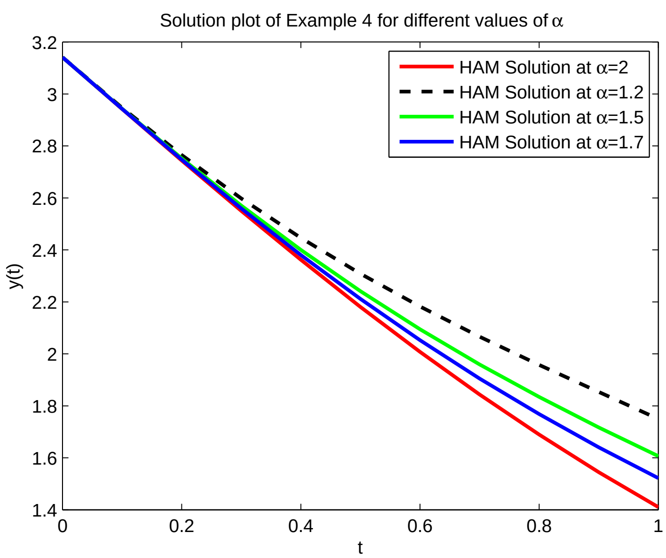

Figure 11: Graphical interpretation of the HAM solutions at different α values for Example 4.

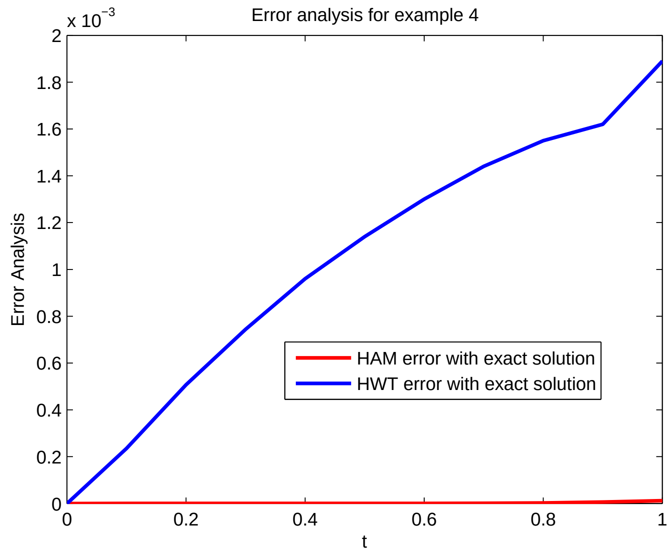

Figure 12: Graphical interpretation of the error analysis, for Example 4.

Example 5. Consider the one-dimensional nonlinear homogeneous time fractional Korteweg–de Vries (KdV) equation [11],

$$D_t^{\alpha}[v(x,t)] + 6vv_x + v_{xxx} = 0, \quad 0 \leq \alpha \leq 1, \tag{48}$$

with the initial condition

$$v(x,0) = \frac{1}{2} sech^2 \left(\frac{x}{2}\right). \tag{49}$$

The exact solution [11] is $v(x,t) = \frac{1}{2} sech^2 \left(\frac{x+t}{2}\right)$.

Method of implementation: Applying HAM method to solve (48) and concerning (9), the zeroth order deformation is given by

$$\begin{aligned}(1-q)&\mathscr{L}[\phi(x,t;q) - v_0(x,t)] \\ &= q\hbar\mathscr{H}(t)[D_t^{\alpha}\phi(x,t;q) + 6\phi(x,t;q)\phi(x,t;q)_x + \phi(x,t;q)_{xxx}].\end{aligned} \tag{50}$$

According to the condition (49), choosing the initial approximation $v_0(x,t) = \frac{1}{2} sech^2\left(\frac{x}{2}\right)$, we take the linear operator $\mathscr{L} = D_t^{\alpha}$ with $\mathscr{L}(C_1) = 0$, where C_1 is a integral constant with $\mathscr{H}(t) = 1$. Therefore, mth order deformation is given by

$$D_t^{\alpha}[v_m(x,t) - \chi_m v_{m-1}(x,t)] = \hbar \mathscr{R}_m(v_{m-1}(x,t)), \tag{51}$$

where

$$\mathscr{R}_m(v_{m-1}(x,t)) = \frac{1}{(m-1)!}\frac{\partial^{m-1}[D_t^{\alpha}\phi(x,t;q) + 6\phi(x,t;q)\phi(x,t;q)_x + \phi(x,t;q)_{xxx}]}{\partial q^{m-1}}, \tag{52}$$

subject to

$$v_m(x,0) = 0, \tag{53}$$

Applying I_t^{α}, the inverse of D_t^{α} on either sides of (51), we get

$$v_m(x,t) = \chi_m v_{m-1}(xt) + \hbar I_t^{\alpha}[\mathscr{R}_m(v_{m-1}(x,t))] + C_1, \quad m \geq 1,$$

where C_1 is calculated by using (53).

The HAM series upto first six terms when $\alpha = 1$ and taking $\hbar = -1$ is given by

$$v(x,t) = \frac{2}{\left(e^{\frac{-x}{2}} + e^{\frac{x}{2}}\right)^2} + \frac{2e^x t}{(1+e^x)^4} - \frac{2e^{3x}t}{(1+e^x)^4} + \frac{4e^x(-1+e^x)t}{(1+e^x)^3} + \frac{e^x t^2}{(1+e^x)^4} - \frac{4e^{2x}t^2}{(1+e^x)^4} + \frac{e^{3x}t^2}{(1+e^x)^4} + \cdots .$$

The HAM series upto first six terms when $\alpha = 0.4$ and taking $h = -1$ is given by

$$y(t) = \frac{2}{\left(e^{\frac{-x}{2}} + e^{\frac{x}{2}}\right)^2} + \frac{2.25412e^x t^{0.4}}{(1+e^x)^6} + \frac{4.50824e^{2x}t^{0.4}}{(1+e^x)^6} - \frac{4.50824e^{4x}t^{0.4}}{(1+e^x)^6} - \frac{2.25412e^{5x}t^{0.4}}{(1+e^x)^6} + \frac{4.50824e^x(e^x-1)t^{0.4}}{(1+e^x)^3} + \frac{2.25412e^x t^{0.8}}{(1+e^x)^6} - \frac{4.29469e^{2x}t^{0.8}}{(1+e^x)^6} + \cdots .$$

The graphs and tables provide an explanation of the results. Figure 13 compares Exact, HAM, and FRPSM solutions geometrically. The HAM solution is graphically depicted with various values of α in Figure 14. Figure 15 depicts the error analysis of the solutions from the HAM, FRSPM, and Exact.

Tables 9 and 10 contain numerical values for the fractional and nonfractional values of α, respectively.

Table 9: Comparison of solutions obtained from exact solutions and HAM and AE of HAM, FRSM, ADM solutions with exact solution for integer order $\alpha = 1$ at $x = 0$ of Example 5

t	Exact	HAM	HAM Error	FRPSM Error[11]	ADM Error[11]
0	0.500000	0.500000	0.	-	-
0.1	0.498752	0.498752	5.551×10^{-17}	2.900×10^{-9}	2.080×10^{-6}
0.2	0.495033	0.495033	5.551×10^{-17}	1.879×10^{-7}	3.31×10^{-5}
0.3	0.488917	0.488917	1.110×10^{-16}	2.216×10^{-6}	3.054×10^{-4}
0.4	0.480521	0.480521	5.551×10^{-17}	1.184×10^{-5}	2.763×10^{-4}
0.5	0.470007	0.470007	5.551×10^{-17}	4.465×10^{-5}	2.500×10^{-3}
0.6	0.457568	0.457568	2.220×10^{-16}	-	-
0.7	0.443426	0.443426	6.106×10^{-16}	-	-
0.8	0.427819	0.427819	3.608×10^{-15}	-	-
0.9	0.411001	0.411001	7.622×10^{-14}	-	-
1.0	0.393224	0.393224	1.171×10^{-12}	-	-

Table 10: Comparison between HAM and FRPSM solutions for different fractional values of α at $x = 2$ in Example 5

t	$\alpha = 0.2$		$\alpha = 0.4$		$\alpha = 0.6$		$\alpha = 0.8$		$\alpha = 1$	
	HAM	FRPSM[11]	HAM	FRPSM[11]	HAM	FRPSM[11]	HAM	FRPSM[11]	HAM	FRPSM[11]
0	0.209987	-	0.209987	-	0.209987	-	0.209987	-	0.209987	-
0.1	0.354751	0.354750978	0.294966	0.29496648	0.259396	0.259395686	0.238566	0.238566145	0.226368	0.226368188
0.2	0.382232	0.382232355	0.327693	0.327692525	0.288355	0.28835476	0.261508	0.261507702	0.243526	0.243526238
0.3	0.400997	0.400996543	0.353185	0.353185341	0.31353	0.313530228	0.283442	0.283441821	0.261461	0.261461322
0.4	0.415703	0.415702913	0.375006	0.375006055	0.336761	0.336761221	0.30503	0.305029944	0.280173	0.280173439
0.5	0.427989	0.427988593	0.394504	0.394504163	0.35877	0.358769892	0.326536	0.326536438	0.299663	0.299662589
0.6	0.438641	-	0.412363	-	0.379928	-	0.348096	-	0.319929	-
0.7	0.448105	-	0.428985	-	0.400457	-	0.369786	-	0.340972	-
0.8	0.456661	-	0.44463	-	0.420499	-	0.391655	-	0.362792	-
0.9	0.464496	-	0.459478	-	0.440155	-	0.413735	-	0.385390	-
1.0	0.471742	-	0.47366	-	0.459495	-	0.436046	-	0.408764	-

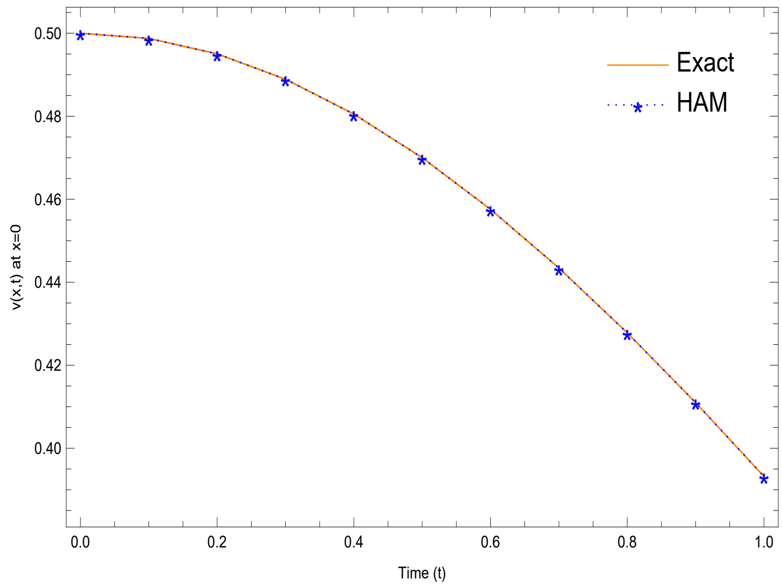

Figure 13: Comparison of the exact solution with HAM and HWT solutions, for Example 5.

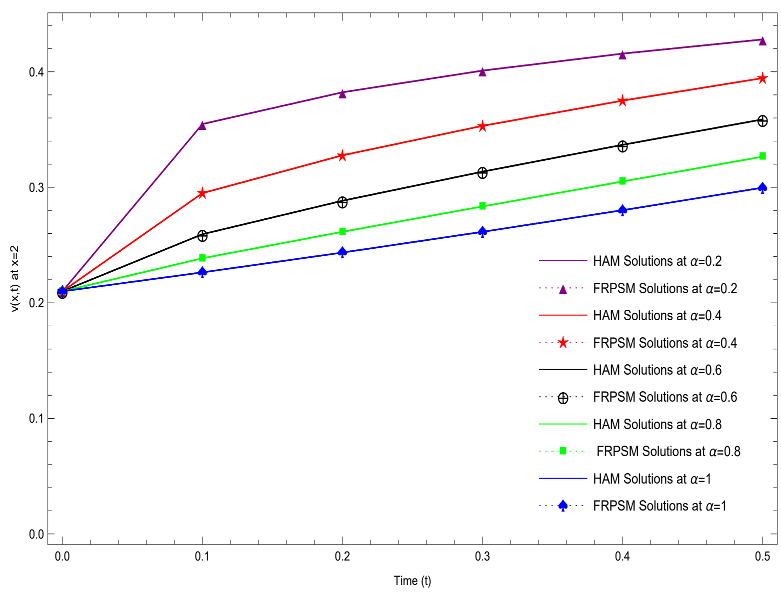

Figure 14: Graphical interpretation of the HAM solutions at different α values for Example 5.

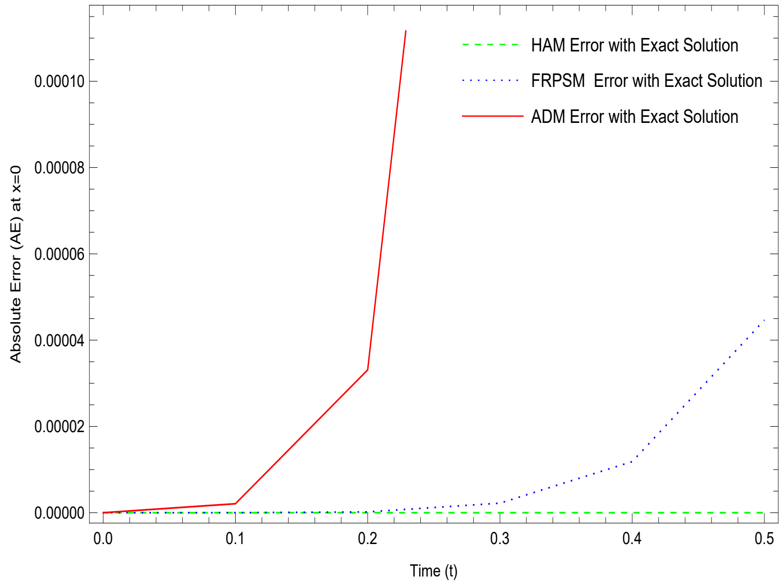

Figure 15: Graphical interpretation of the error analysis, for Example 5.

6 Conclusion

In this study, we discussed the linear and nonlinear fractional models of different orders through two different methods: the HAM (Homotopy analysis method) and the HWT (Haar wavelet technique). Here, we considered five problems to justify the two different methods. HAM is the semi-analytical method that yields the analytical solution of a given model after more deformations. The HWT is a numerical technique that solves the models numerically with the help of the Mathematica software. The obtained solutions were numerically tabulated in Tables 1–10. Figures 1–14 show the performance of the methods. Here, HAM consumes more time to yield solutions for the different models, but HWT yields the numerical results in less time. This study reveals that HAM provides solutions with high accuracy compared to HWT. From the tables and figures, we conclude that HAM is better than HWM, FRPSM [11], and HOFLMSM [24].

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