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Research Article

A study on the convergence and error bound of solutions to 2D mixed Volterra–Fredholm integral and integro-differential equations via high-order collocation method

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Abstract

The integral equation is transformed into systems of algebraic equations using standard collocation points, and then the algebraic equations are solved using matrix inversion. Their solutions are substituted into the approximate equation to give the numerical results. We establish the analysis of

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the developed method, which shows that the solution is unique, convergent, and error bound. To illustrate the effectiveness, ease of use, and dependability of the approach, illustrative examples are provided. It demonstrates that the method outperforms other methods.

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Keywords: Volterra integral equation; Fredholm integral equation; Mixed Volterra–Fredholm integral equation; Collocation; Two-dimensional integral.

1 Introduction

Due to certain scientists' inability to solve differential equations, integral equations first appeared in writing in the middle of the seventeenth century. The numerous applications of integral equations can be found in the fields of elasticity, plasticity, heat and mass transfer, fluid dynamics, filtration theory, electrostatics, electrodynamics, bio-mechanics, game theory, control, queuing theory, electrical engineering, economics, and medicine, among other scientific disciplines. In many branches of natural science, exact (closed-form) solutions to integral equations are essential to comprehending the qualitative aspects of numerous processes and occurrences [13].

The integral equations provide a significant tool for describing diverse processes and for solving several sorts of boundary value issues relating to ordinary and partial differential equations. The topic of integral equations is one of the most useful mathematical tools in both pure and practical mathematics and it has vast applications in a variety of scientific situations.

Two-dimensional integral equations provide an important tool for modeling several problems in engineering and research [5, 8]. Many processes in physics and engineering domains give rise to two-dimensional integral equations and are frequently difficult to solve analytically. In many circumstances, it is needed to find the approximate solutions. As we know, substantial effort has been done on creating and studying numerical methods for solving one-dimensional integral equations of the second sort, but in two-dimensional cases, a very little amount of work has been done [19].

An equation is considered integral if the unknown function appears inside the integral sign. The various forms of integral equations primarily depend on the equation's kernel and the integration's limits. According to [19], an integral equation is referred to as a Volterra integral equation if at least one of the limits is variable and a Fredholm integral equation if the limits of integration are fixed. The Fredholm integral equation is characterized by fixed integration limits, whereas the Volterra integral equation exhibits at least one variable integration limit.

An essential tool for modeling a wide range of phenomena and resolving various boundary value issues involving ordinary and partial differential equations is the integral equation. One of the most helpful mathematical fields in both pure and applied mathematics is integral equations, which has numerous applications in science, engineering, and so on [11]. An equation that combines the Fredholm integral and the Volterra integral in one equation is known as the Volterra–Fredholm integral equation.

Numerous methods have been developed for solving one-dimensional integral equations and two-dimensional mixed Volterra–Fredholm integral equations (2D MVFIEs). These methods include perturbed collocation method [18], collocation method [2] and [3], boukakar collocation method [1] and [1], multiquadric radial basis functions [4], Two-dimensional Legendre wavelets method [6], applications of two-dimensional triangular functions [12], series solution methods [15], successive approximation method and method of successive substitutions [16], and Adomian decomposition method [17]. In this study, we develop the polynomial collocation method to solve 2D MVFIE of the form:

$$m(x,t) = h(x,t) + \rho \int_0^t \int_a^b k(x,t,y,z) m(y,z) dydz \tag{1}$$

and

$$m^n(x,t) = h(x,t) + \rho \int_0^t \int_a^b k(x,t,y,z) m(y,z) dydz, \tag{2}$$

where $m(x,t)$ is considered an unknown function to be determined, the functions $h(x,t)$ is analytic on $C\left([0,1]^2, \mathbb{R}\right)$, $k(x,t,y,z)$ is analytic on $C\left([0,1]^4, \mathbb{R}\right)$, $m(y,z)$ is a continuous function with respect to $m(y,z)$, and ρ is a constant coefficient.

Definition 1. In order to apply the Bernstein polynomials in the interval $[0,1]$, $B_{i,n}(x)$ is defined as [10]

$$B_{i,n}(x) = \binom{n}{i} x^i (1-x)^{n-i}, \qquad i = 0,1,2,\ldots,n. \tag{3}$$

Definition 2. Bernstein polynomials of degree n in the interval $[0,1]$ can also be written in the following equivalent form:

$$B_{i,n}(x) = \sum_{p=0}^{n-i} \binom{n}{i} \binom{n-i}{p} (-1)^p x^{i+p}. \tag{4}$$

Definition 3. Bernstein polynomials of degree n can be defined recursively by blending together two Bernstein polynomials of degree $n-1$. That is, the kth nth-degree Bernstein polynomial can be written as

$$B_{k,n}(x) = (1-x)B_{k,n-1}(x) + xB_{k-1,n-1}(x), \quad k = 0(1)n, \quad n \geq 1. \tag{5}$$

Definition 4 (Standard Collocation Method (SCM))**.** This method is used to determine the desired collocation points within an interval, that is, $[a,b]$ and is given by

$$\begin{aligned} x_i &= a + \frac{(b-a)i}{N}, \quad i = 0(1)N, \\ t_j &= a + \frac{(b-a)j}{N}, \quad j = 0(1)N. \end{aligned} \tag{6}$$

Definition 5.

(i) ***Lipschitzian*** [7]

Let $(X, \|.\|)$ be a norm space. Mapping $T : X \longrightarrow X$ is L-Lipschitz if there exists $L > 0$ such that $\|Tx - Ty\|_\infty \leq L\|x - y\|_\infty$,$q \in [0,1]$ for all $x, y \in X$.

(ii) ***Lipschitz continuity*** [14]

A function f is Lipschitz continuous if there exists $K < \infty$ such that $\|f(y) - f(x)\| \leq K\|y - x\|$.

Definition 6 (Infinity norm $\|v\|_\infty$)**.** [14]

The infinity norm (also known as the L_∞-norm, l_∞, max norm, or uniform norm) of a vector v is denoted by $\|v\|_\infty$ and is defined as the maximum of the absolute values of its components, that is,

$$\|v\|_\infty = \max\{|v_i| : i = 1, 2, \ldots, n\}$$

2 Uniqueness, convergence, error analysis and method of solution

2D MVFIEs can be solved numerically using the polynomial collocation method, which is based on the collocation approach and takes into account the linear combination of the Bernstein polynomial as our approximated solution. In this section, we will develop a method by using standard collocation points to reduce the 2D MVFIE to a system of algebraic equations.

2.1 Integral form

Let $M_N(x,t)$ be the approximate solution of

$$m^n(x,t) = h(x,t) + \rho \int_0^t \int_a^b k(x,t,y,z) m(y,z) dydz, \tag{7}$$

with initial condition given as $m^{n-1}(x_0,t) = m_{n-1}$, where $m^n(x,t) = \frac{d^n}{dx^n} m(x,t)$ is the nth order derivative of $m(x,t)$, $m(x,t)$ is an unknown function to be determined, $h(x,t)$ and $k(x,t,y,z)$ are analytic function on $[a,b]$.

Here, L is an operator defined as $L = \frac{d^n}{dx^n}$ and $L^{-1} = \int_0^x \int_0^x \cdots \int_0^x dxdx\ldots dx$ operating L^{-1} on both sides of (7) is given by

$$L^{-1}\left(m^n(x,t)\right) = L^{-1}\left(h(x,t)\right) + L^{-1}\left(\rho \int_0^t \int_a^b k(x,t,y,z) m(y,z) dydz\right). \tag{8}$$

Integrating (7) n times from 0 to x gives

$$\int_0^x \int_0^x \cdots \int_0^x m^n(x,t) dxdx\ldots dx$$
$$= \int_0^x \int_0^x \cdots \int_0^x h(x,t) dxdx\ldots dx$$

$$+\int_0^x\int_0^x\cdots\int_0^x\left(\rho\int_0^t\int_a^b k(x,t,y,z)m(y,z)dydz\right)dxdx\ldots dx \tag{9}$$

$$\begin{aligned}\int_0^x\int_0^x\cdots\int_0^x m^n(x,t)dxdx\ldots dx\\ =\int_0^x\int_0^x\cdots\int_0^x h(x,t)dxdx\ldots dx \\ +\rho\int_0^x\int_0^x\cdots\int_0^x\left(\int_0^t\int_a^b k(x,t,y,z)m(y,z)dydz\right)dxdx\ldots dx.\end{aligned} \tag{10}$$

Converting multiple integrals to single integral from (10) gives

$$\begin{aligned}m(x,t)=&\frac{x^{n-1}}{(n-1)!}u_0+\frac{x^{n-2}}{(n-2)!}u_1+\frac{x^{n-3}}{(n-3)!}u_2\\ &+\cdots+u_{n-1}+\frac{1}{(n-1)!}\int_0^x(x-t)^{n-1}h(x,t)dt\\ &+\rho\frac{1}{(n-1)!}\int_0^x(x-t)^{n-1}\left(\int_0^t\int_a^b k(x,t,y,z)m(y,z)dydz\right)dt\end{aligned} \tag{11}$$

Simplifying (11) gives

$$\begin{aligned}m(x,t)=&\sum_{i=1}^{n-1}\frac{1}{i!}u_ix^i+\frac{1}{(n-1)!}\int_0^x(x-t)^{n-1}h(x,t)dt\\ &+\rho\frac{1}{(n-1)!}\int_0^x(x-t)^{n-1}\left(\int_0^t\int_a^b k(x,t,y,z)m(y,z)dydz\right)dt,\end{aligned} \tag{12}$$

where

$$\begin{aligned}H(x,t)=&\frac{x^{n-1}}{(n-1)!}u_0+\frac{x^{n-2}}{(n-2)!}u_1+\frac{x^{n-3}}{(n-3)!}u_2\\ &+\cdots+u_{n-1}+\frac{1}{(n-1)!}\int_0^x(x-t)^{n-1}h(x,t)dt\end{aligned} \tag{13}$$

or

$$H(x,t)=\sum_{i=1}^{n-1}\frac{1}{i!}u_ix^i+\frac{1}{(n-1)!}\int_0^x(x-t)^{n-1}h(x,t)dt$$

and

$$\rho(x,t) = \frac{\rho}{(n-1)!}\int_0^x (x-t)^{n-1}dt, \tag{14}$$

$$m(x,t) = H(x,t) + \rho\int_0^t\int_a^b K(x,t,y,z)m(y,z)dydz. \tag{15}$$

Therefore, (15) is a 2D MVFIE of the second kind, which is the integral form of (2).

2.2 Method of solution to 2D MVFIE

We recall that (1) and (2) can be written as

$$m(x,t) = h(x,t) + \lambda\int_0^t\int_a^b k(x,t,y,z)m(y,z)dydz. \tag{16}$$

Let $M_N(x,t)$ be the approximate solution to (15), where

$$m_N(x,t) = \sum_{i=0}^{N}\sum_{j=0}^{N} c_{i,j}B_{i,N}(x)B_{j,N}(t) = \phi(x,t)C. \tag{17}$$

Substituting (17) into (16) gives

$$\phi(x,t)C = h(x,t) + \rho\int_0^t\int_a^b k(x,t,y,z)\left(\phi(y,z)C\right)dydz, \tag{18}$$

$$\phi(x,t)C - \rho\int_0^t\int_a^b k(x,t,y,z)\left(\phi(y,z)C\right)dydz = h(x,t), \tag{19}$$

$$\left\{\phi(x,t) - \rho\int_0^t\int_a^b k(x,t,y,z)\phi(y,z)dydz\right\}C = h(x,t). \tag{20}$$

Collocating (20) and using standard collocation points at $x = x_i$ and $t = t_j$ with

$$\begin{aligned} x_i &= a + \frac{(b-a)i}{N}, \quad i = 0(1)N,\\ t_j &= a + \frac{(b-a)j}{N}, \quad j = 0(1)N, \end{aligned} \tag{21}$$

we have

$$\left\{\phi(x_i,t_j)-\rho\int_0^t\int_a^b k(x_i,t_j,y,z)\phi(y,z)dydz\right\}C=h(x_i,t_j), \tag{22}$$

where $\gamma(x_i,t_j)=\left\{\phi(x_i,t_j)-\rho\int_0^t\int_a^b k(x_i,t_j,y,z)\phi(y,z)dydz\right\}$ and $C=[c_{0,0},c_{0,1},c_{0,2},\ldots,c_{0,N},\ldots,c_{N,0},c_{N,1},c_{N,2},\ldots,c_{N,N}]$,

$$\gamma(x_i,t_j)C=h(x_i,t_j). \tag{23}$$

Multiplying both sides of (23) by $\gamma(x_i,t_j)^{-1}$ gives

$$C=\gamma(x_i,t_j)^{-1}h(x_i,t_j). \tag{24}$$

Substituting C into the approximate solution to (17) gives

$$M_N(x,t)=\phi(x,t)\gamma(x_i,t_j)^{-1}h(x_i,t_j), \quad i,j=0(1)N. \tag{25}$$

The system of equations is then solved using Maple 18 software and the unknown constants obtained are then substituted back into the approximate solution to get the required solution.

2.3 Uniqueness, convergence and error analysis

Hypothesis

The following assumptions were made:

Z_1:Let $(C([0,1]\times[0,1]),\|.\|)$ be the space of all continuous functions on the interval $[0,1]\times[0,1]$ with the norm $\|M\|_\infty=\underbrace{\underbrace{Max}_{x\in[0,1]}}_{t\in[0,1]}|M(x,t)|$.

$Z_2: M(x,t)\neq 0$.

$Z_3: |K(x,t,y,z)|\leq L$ (L is a positive real number) for all $(x,t)\in[0,1]\times[0,1]$, and

$Z_4:$ for all $(x,t)\in[0,1]\times[0,1]$ and $\beta=\{(x,t,y,z): 0\leq z\leq t\leq 1; 0\leq y\leq x\leq 1\}$.

With this conditions, we present the uniqueness and convergence of the solution.

Theorem 1 (Uniqueness of solution for 2D MVFIE)**.** Let $M(x,t)$ be an exact solution to (1), and let $M_{N,N}(x,t)$ be the approximate solution to (1), where

$$M_{N,N}(x,t) = \sum_{i=0}^{N}\sum_{j=0}^{N} c_{i,j} B_{i,N}(x) B_{j,N}(t).$$

Then (1) has a unique solution whenever $0 \leq \alpha \leq 1$ and $\alpha = 1 - L_1 \lambda (b-a) t$.

Proof. Equation (1) can be written in the form

$$M(x,t) = h(x,t) + \rho \int_0^t \int_a^b F\left(x,t,y,z,M(y,z)\right) dydz$$

such that the linear term $F(M)$ is Lipschitz continuous with $|F(M) - F(V)| \leq L_1 |M - V|$.

Let $M_{N,N}$ and $M'_{N,N}$ be any two different approximate solutions to (1). Then

$$\begin{aligned} M_{N,N}(x,t) - M'_{N,N}(x,t) =& h(x,t) + \rho \int_0^t \int_a^b F\left(x,t,y,z,m_{N,N}(y,z)\right) dydz - h(x,t) \\ & - \rho \int_0^t \int_a^b F\left(x,t,y,z,M'_{N,N}(y,z)\right) dydz \end{aligned}$$

$$\begin{aligned} & \left| M_{N,N}(x,t) - M'_{N,N}(x,t) \right| \\ & = \left| \rho \int_0^t \int_a^b F\left(x,t,y,z,M_{N,N}(y,z)\right) dydz - \rho \int_0^t \int_a^b F\left(x,t,y,z,M'_{N,N}(y,z)\right) dydz \right|, \end{aligned}$$

$$\begin{gathered} \left| M_{N,N}(x,t) - M'_{N,N}(x,t) \right| \\ \leq |\rho| \int_0^t \int_a^b \left| F\left(x,t,y,z,M_{N,N}(y,z)\right) - F\left(x,t,y,z,M'_{N,N}(y,z)\right) \right| dydz, \\ \left| M_{N,N}(x,t) - M'_{N,N}(x,t) \right| \leq |\rho| \int_0^t \int_a^b \left| F\left(M_{N,N}\right) - F\left(M'_{N,N}\right) \right| dydz, \\ \left| m_{N,N} - M'_{N,N} \right| \leq |\rho| L_1 \int_0^t \int_a^b \left| M_{N,N} - M'_{N,N} \right| dydz, \\ \left| M_{N,N} - M'_{N,N} \right| - |\rho| L_1 (b-a) t \left| M_{N,N} - M'_{N,N} \right| \leq 0, \\ \left\{ 1 - |\rho| L_1 (b-a) t \right\} \left| M_{N,N} - M'_{N,N} \right| \leq 0. \end{gathered}$$

If $\alpha = \{1 - |\rho| L_1 (b-a) t\}$, then

$$\alpha \left| M_{N,N} - M'_{N,N} \right| \leq 0.$$

As $0 \leq \alpha \leq 1$, $\left|M_{N,N} - M'_{N,N}\right| = 0$, which implies $M_{N,N} = M'_{N,N}$. Hence, the uniqueness proof is complete.

□

Theorem 2 (Convergence of the method for 2D MVFIE)**.** Let $U(x,t)$ be an exact solution to (1), and let $M_{N,N}(x,t)$ be the approximate solution to (1), where

$$M_{N,N}(x,t) = \sum_{i=0}^{N}\sum_{j=0}^{N} c_{i,j} B_{i,N}(x) B_{j,N}(t).$$

Then, the solution of L2D-LMVFIE by using Bernstein polynomial as a basis function is unique and convergent if $0 \leq \eta_1 \leq 1$.

Proof. Since we have already proved for the uniqueness, we now prove the convergence using the definition of norms and our assumptions $Z_1 - Z_4$. We have

$$\|M(x,t) - M_{N,N}(x,t)\|_\infty = \underbrace{\underbrace{Max}_{x\in[0,1]}}_{t\in[0,1]} |M(x,t) - M_{N,N}(x,t)|$$

$$\|M(x,t) - M_{N,N}(x,t)\|_\infty$$
$$\underbrace{\underbrace{Max}_{x\in[0,1]}}_{t\in[0,1]}$$
$$= \left| h(x,t) + \rho \int_0^t \int_a^b k(x,t,y,z) M(y,z) dydz - h(x,t) \right.$$
$$\left. -\rho \int_0^t \int_a^b k(x,t,y,z) M_{N,N}(y,z) dydz \right|,$$

$$\|M(x,t) - M_{N,N}(x,t)\|_\infty$$
$$\leq |\rho| \underbrace{\underbrace{Max}_{x\in[0,1]}}_{t\in[0,1]} \int_0^t \int_a^b |k(x,t,y,z)| \, |M(y,z) - M_{N,N}(y,z) dydz| \,,$$

$$\|M(x,t) - M_{N,N}(x,t)\|_\infty \leq |\rho| \, L\beta \, \|M(y,z) - m_{N,N}(y,z)\|_\infty \,,$$

$$\|M(x,t)-M_{N,N}(x,t)\|_{\infty}(1-|\rho|L\beta)\leq 0.$$

If $\eta_1=(|\rho|L\beta)$, then

$$(1-\eta_1)\|M(x,t)-M_{N,N}(x,t)\|_{\infty}\leq 0.$$

Then, if $0\leq\eta_1\leq 1$ and $N\to\infty$, then $\underbrace{\lim}_{N\to\infty}\|M(x,t)-M_{N,N}(x,t)\|_{\infty}=0$. □

Theorem 3 (Error bound of 2D MVFIE)**.** Let $U(x,t)$ be an exact solution to (1), and let $M_{N,N}(x,t)$ be the approximate solution to (1), where

$$M_{N,N}(x,t)=\sum_{i=0}^{N}\sum_{j=0}^{N}c_{i,j}B_{i,N}(x)B_{j,N}(t).$$

Then, the error of L2D-LMVFIE by using Bernstein polynomial as a basis function is

$$\frac{\|e_{N,N}(x,t)\|_{\infty}}{\|e_{N,N}(y,z)\|_{\infty}}\leq|\rho|M_{\alpha}\beta_{\alpha}.$$

Proof. In establishing the error bound of this method, we substitute the approximate solution into (1), which gives

$$,M_{N,N}(x,t)=h(x,t)+\rho\int_0^t\int_a^b k(x,t,y,z)M_{N,N}(y,z)dydz,$$

and the exact solution is given by

$$M(x,t)=h(x,t)+\rho\int_0^t\int_a^b k(x,t,y,z)M(y,z)dydz,$$

$$M_{N,N}(x,t)-M(x,t)=e_N(x,t),$$

$$\begin{aligned}M_{N,N}(x,t)-M(x,t)=&h(x,t)+\rho\int_0^t\int_a^b k(x,t,y,z)M_{N,N}(y,z)dydz\\&-h(x,t)-\rho\int_0^t\int_a^b k(x,t,y,z)M(y,z)dydz,\end{aligned}$$

$$|M_{N,N}(x,t)-M(x,t)|=\left|\rho\int_0^t\int_a^b k(x,t,y,z)M_{N,N}(y,z)dydz\right.$$

$$-\rho\int_0^t\int_a^b k(x,t,y,z)M(y,z)dydz\Bigg|,$$

$$|M_{N,N}(x,t)-M(x,t)|\leq|\rho|\int_0^t\int_a^b|k(x,t,y,z)|\,|M_{N,N}(y,z)-M(y,z)|\,dydz,$$

$$\frac{|M_{N,N}(x,t)-M(x,t)|}{|M_{N,N}(y,z)-M(y,z)|}\leq\frac{|\rho|\int_0^t\int_a^b|k(x,t,y,z)|\,|M_{N,N}(y,z)-M(y,z)|\,dydz}{|M_{N,N}(y,z)-M(y,z)|},$$

$$\frac{|e_{N,N}(x,t)|}{|e_{N,N}(y,z)|}\leq|\rho|\int_0^t\int_a^b|k(x,t,y,z)|\,dydz,$$

$$\frac{\|e_{N,N}(x,t)\|_\infty}{\|e_{N,N}(y,z)\|_\infty}\leq|\rho|\int_0^t\int_a^b|k(x,t,y,z)|\,dydz,$$

$$\frac{\|e_{N,N}(x,t)\|_\infty}{\|e_{N,N}(y,z)\|_\infty}\leq|\rho|\,M_\alpha\beta_\alpha.$$

Therefore the error is bounded and hence the solution of the method is convergent. □

Theorem 4. Let $M(x,t)$ be the solution to (1). Then the solution is

$$M_N(x,t)=\phi(x,t)\gamma(x_i,t_j)^{-1}h(x_i,t_j);\quad i,j=0(1)N,$$

where

$$\gamma(x_i,t_j)=\left\{\phi(x_i,t_j)-\rho\int_0^t\int_a^b k(x_i,t_j,y,z)\phi(y,z)dydz\right\}.$$

Proof. The approximate solution to (1) is

$$m_N(x,t)=\sum_{i=0}^{N}\sum_{j=0}^{N}c_{i,j}B_{i,N}(x)B_{j,N}(t)=\phi(x,t)C.$$

From (23)

$$C=\gamma(x_i,t_j)^{-1}h(x_i,t_j),$$

where

$$\gamma(x_i,t_j)=\left\{\phi(x_i,t_j)-\rho\int_0^t\int_a^b k(x_i,t_j,y,z)\phi(y,z)dydz\right\}.$$

Substituting for C in the approximate solution gives

$$M_N(x,t) = \phi(x,t)\gamma(x_i,t_j)^{-1}h(x_i,t_j), \quad i,j = 0(1)N.$$

□

3 Numerical examples

In this research, numerical examples are utilized to assess the simplicity and efficiency of the method and are presented in tables except where it delivers the exact solution. All computations are done with the help of the MAPLE 18 program. Let $M_N(x,t)$ and $M(x,t)$ be the approximate and exact solution, respectively. Then $Error_N = |M_N(x,t) - M(x,t)|$. Table 1 gives a brief description of some abbreviations made.

Table 1: Notations

Tag	$Description$
$Error_{OurMethod}$	$AbsoluteErrorofOurMethod$
$Error_{NKH}$	$AbsoluteErrorFrom$[9]
$Error_{AM}$	$AbsoluteErrorFrom$[4]

Example 1. Consider a linear 2D MVFIE of the second kind [9]

$$m(x,t) = x^2 + e^t + \frac{2}{3}x^3t^2 - \int_0^t \int_0^1 t^2e^{-z}m(y,z)dydz, \tag{26}$$

which has an exact solution given as $m(x,t) = x^2 + e^t$ in the interval $x,t = [0,1]$.

Let the approximate solution to (26) for $N = 5$ be

$$m_N(x,t) = \sum_{i=0}^{5}\sum_{j=0}^{5} c_{i,j}B_{i,5}(x)B_{j,5}(t). \tag{27}$$

Substituting (27) in (26) gives

$$\sum_{i=0}^{5}\sum_{j=0}^{5} c_{i,j}B_{i,5}(x)B_{j,5}(t) = x^2 + e^t + \frac{2}{3}x^3t^2 \tag{28}$$

$$- \int_0^t \int_0^1 t^2 e^{-z} \left(\sum_{i=0}^{5} \sum_{j=0}^{5} c_{i,j} B_{i,5}(y) B_{j,5}(z) \right) dydz,$$

$$\sum_{i=0}^{5} \sum_{j=0}^{5} c_{i,j} B_{i,5}(x) B_{j,5}(t) + \int_0^t \int_0^1 t^2 e^{-z} \left(\sum_{i=0}^{5} \sum_{j=0}^{5} c_{i,j} B_{i,5}(y) B_{j,5}(z) \right) dydz$$
$$= x^2 + e^t + \frac{2}{3} x^3 t^2. \tag{29}$$

Collocating (29) and using standard collocation points at $x = x_i$ and $t = t_j$ with

$$x_i = \frac{i}{5}; \quad i = 0(1)5,$$
$$t_j = \frac{j}{5}; \quad j = 0(1)5,$$

we have

$$\sum_{i=0}^{5} \sum_{j=0}^{5} c_{i,j} B_{i,5}(x_i) B_{j,5}(t_j) + \int_0^t \int_0^1 t_j^2 e^{-z} \left(\sum_{i=0}^{5} \sum_{j=0}^{5} c_{i,j} B_{i,5}(y) B_{j,5}(z) \right) dydz$$
$$= x_i^2 + e^{t_j} + \frac{2}{3} x_i^3 t_j^2. \tag{30}$$

The method was implemented using MAPLE 18 software, and $M_5(x,t)$ was obtained as

$$\begin{aligned} M_5(x,t) = &- 2.912943530 \times 10^{-8} t^3 x + 2.240094596 \times 10^{-7} t^4 x \\ &- 1.948800244 \times 10^{-7} t^5 x + 4.595065836 \times 10^{-8} t^2 x \\ &+ 2.000000000 \times 10^{-8} t x^5 + 1.000000000 \times 10^{-8} t x^3 + \\ &- 3.000000000 \times 10^{-8} t x^4 + 1.000082530 t x^2 + .49906830 t^2 x^2 \\ &+ 0.4866 \times 10^{-4} t^2 x^4 - 0.4651 \times 10^{-4} t^2 x^3 + x^2 \\ &+ 0.1385710011 \times 10^{-1} t^5 x^2 - 0.1187446705 \times 10^{-3} t^4 x^3 \\ &- 0.2236348902 \times 10^{-3} t^3 x^4 + 0.230 \times 10^{-5} t^2 x^5 \\ &+ 0.1086523352 \times 10^{-3} t^3 x^3 - 0.1622 \times 10^{-4} t^5 x^5 \end{aligned}$$

$$
\begin{aligned}
&- 0.8256489016 \times 10^{-4} t^5 x^4 - 0.11620 \times 10^{-3} t^4 x^5 \\
&+ 0.5474233524 \times 10^{-4} t^5 x^3 + 0.3259097803 \times 10^{-3} t^4 x^4 \\
&+ 0.5115 \times 10^{-4} t^3 x^5 \\
&+ 0.1704088951 t^3 x^2 + 0.3486396978 \times 10^{-1} t^4 x^2. \qquad (31)
\end{aligned}
$$

Table 2: Results using Bernstein polynomial for Example 1

(x,t)	$Exact$	$OurMethod_{N=5}$	$Error_{OurMethod}$
$(0,0)$	0.0000000000	0.0000000000	0.0000000000
$(0.1,0.1)$	0.01105170918	0.01105172941	2.023×10^{-8}
$(0.2,0.2)$	0.04885611032	0.04885610125	9.07×10^{-9}
$(0.3,0.3)$	0.1214872927	0.1214871746	1.181×10^{-7}
$(0.4,0.4)$	0.2386919517	0.2386917676	1.841×10^{-7}
$(0.5,0.5)$	0.4121803178	0.4121800093	3.085×10^{-7}
$(0.6,0.6)$	0.6559627680	0.6559619662	8.018×10^{-7}
$(0.7,0.7)$	0.9867388264	0.9867372530	1.5734×10^{-6}
$(0.8,0.8)$	1.424346194	1.424344379	1.815×10^{-6}
$(0.9,0.9)$	1.992278520	1.992276313	2.207×10^{-6}
$(1.0,1.0)$	2.718281828	2.718268380	1.3448×10^{-5}

Table 3: Comparison Absolute Error for Example 1

(x,t)	$Exact$	$Error_{OurMethod}$	$Error_{NKH}$
$(0.1,0)$	0.01	0.0000000000	0.0000000000
$(0.1,0.1)$	0.01105170918	2.023×10^{-8}	3.34691×10^{-6}
$(0.1,0.3)$	0.01349858808	9.43×10^{-9}	3.03472×10^{-5}
$(0.1,0.5)$	0.01648721271	1.5×10^{-10}	8.22639×10^{-5}
$(0.1,0.7)$	0.02013752707	1.460×10^{-8}	1.48971×10^{-4}
$(0.1,0.9)$	0.02459603111	1.745×10^{-8}	2.05545×10^{-4}

Example 2. Consider a linear 2D MVFIE of the second kind [4]

$$m(x,t) = t^2 e^x + \frac{1}{3}t^3 x^2 + \int_0^t \int_0^1 x^2 e^{-y} m(y,z) dy dz, \tag{32}$$

which has an exact solution given as $m(x,t) = t^2 e^x$ in the interval $(x,t) = [0,1]$.

Let the approximate solution to (32) for $N = 5$ be

$$m_N(x,t) = \sum_{i=0}^{5}\sum_{j=0}^{5} c_{i,j} B_{i,5}(x) B_{j,5}(t). \tag{33}$$

Substituting (33) into (32) gives

$$\begin{aligned} &\sum_{i=0}^{5}\sum_{j=0}^{5} c_{i,j} B_{i,5}(x) B_{j,5}(t) \\ &= t^2 e^x + \frac{1}{3}t^3 x^2 + \int_0^t \int_0^1 x^2 e^{-y} \left(\sum_{i=0}^{5}\sum_{j=0}^{5} c_{i,j} B_{i,5}(y) B_{j,5}(z) \right) dy dz, \end{aligned} \tag{34}$$

$$\begin{aligned} &\sum_{i=0}^{5}\sum_{j=0}^{5} c_{i,j} B_{i,5}(x) B_{j,5}(t) - \int_0^t \int_0^1 x^2 e^{-y} \left(\sum_{i=0}^{5}\sum_{j=0}^{5} c_{i,j} B_{i,5}(y) B_{j,5}(z) \right) dy dz \\ &= t^2 e^x + \frac{1}{3}t^3 x^2. \end{aligned} \tag{35}$$

Collocating (35) and using standard collocation points at $x = x_i$ and $t = t_j$ with

$$\begin{aligned} x_i &= \frac{i}{5}; \quad i = 0(1)5, \\ t_j &= \frac{j}{5}; \quad j = 0(1)5, \end{aligned}$$

we have

$$\begin{aligned} &\sum_{i=0}^{5}\sum_{j=0}^{5} c_{i,j} B_{i,5}(x_i) B_{j,5}(t_j) - \int_0^t \int_0^1 x_i^2 e^{-y} \left(\sum_{i=0}^{5}\sum_{j=0}^{5} c_{i,j} B_{i,5}(y) B_{j,5}(z) \right) dy dz \\ &= t_j^2 e^{x_i} + \frac{1}{3}t_j^3 x_i^2. \end{aligned} \tag{36}$$

The method was implemented using MAPLE 18 software, and $M_5(x,t)$ was obtained as

$$\begin{aligned}M_5(x,t) =& 1. \times 10^{-8}t^3x + 1.000082530t^2x + 0.171638e^{-3}t^5x^5\\ &- 0.11186e^{-3}t^4x^5 - 0.2793619605e^{-4}t^3x^5 + 0.1385620746e^{-1}t^2x^5\\ &- 0.32928e^{-3}t^5x^4 + 0.26711e^{-3}t^5x^3 + 0.24133e^{-3}t^4x^4\\ &- 0.18882e^{-3}t^4x^3 + 0.1951239210e^{-4}t^3x^4 + 0.3486292507e^{-1}t^2x^4\\ &- 0.5431e^{-4}t^5x^2 - 0.1628619605e^{-4}t^3x^3 + 0.1704115775t^2x^3\\ &+ 0.2330e^{-4}t^4x^2 + 0.117e^{-5}t^3x^2 + .499067752t^2x^2\\ &+ 1.468779972 \times 10^{-7}tx^2 + 2.0 \times 10^{-8}t^5x - 3.0 \times 10^{-8}t^4x + 1.0t^2.\end{aligned} \tag{37}$$

Table 4: Result of Absolute Error for Example 2

(x,t)	$Exact$	$Error_{OurMethod}$	$Error_{AM}$
$(0,0)$	0.0000000000	0.0000000000	2.46×10^{-5}
$(0.1,0.1)$	0.01105170918	2.064×10^{-8}	1.46×10^{-5}
$(0.2,0.2)$	0.04885611032	1.22×10^{-9}	3.37×10^{-4}
$(0.3,0.3)$	0.1214872927	8.31×10^{-8}	2.45×10^{-3}
$(0.4,0.4)$	0.2386919517	1.192×10^{-7}	1.00×10^{-2}
$(0.5,0.5)$	0.4121803178	3.200×10^{-7}	3.05×10^{-2}
$(0.6,0.6)$	0.6559627680	1.2541×10^{-6}	7.58×10^{-2}
$(0.7,0.7)$	0.9867388264	3.1681×10^{-6}	1.63×10^{-1}
$(0.8,0.8)$	1.424346194	5.121×10^{-6}	3.17×10^{-1}
$(0.9,0.9)$	1.992278520	5.352×10^{-6}	5.69×10^{-1}
$(1.0,1.0)$	2.718281828	5.121×10^{-6}	5.70×10^{-1}

Example 3. Consider a linear 2D MVFIE of the second kind [19]

$$m'(x,t) = 2x - \frac{1}{4}t^2 + \frac{1}{6}t^4 + \int_0^t \int_0^1 rtm(r,s)drds, \tag{38}$$

with initial condition $m(0,t) = -t^2$ which has an exact solution given as $m(x,t) = x^2 - t^2$ in the interval $(x,t) = [0,1]$.

Let the approximate solution to (38) for $N = 5$ be

$$m_N(x,t) = \sum_{i=0}^{5}\sum_{j=0}^{5} c_{i,j} B_{i,5}(x) B_{j,5}(t). \tag{39}$$

Integrating both sides of (38) from 0 from x

$$\int_0^x (m'(x,t))\,dx = \int_0^x \left(2x - \frac{1}{4}t^2 + \frac{1}{6}t^4\right) + \int_0^x \left(\int_0^t \int_0^1 rtm(r,s)drds\right) dx, \tag{40}$$

$$m(x,t) - m(0,t) = x^2 - \frac{1}{4}xt^2 + \frac{1}{6}xt^4 + \int_0^x \left(\int_0^t \int_0^1 rtm(r,s)drds\right) dx, \tag{41}$$

$$m(x,t) = x^2 - t^2 - \frac{1}{4}xt^2 + \frac{1}{6}xt^4 + \int_0^x \left(\int_0^t \int_0^1 rtm(r,s)drds\right) dx. \tag{42}$$

substituting (39) into (42) gives

$$\begin{aligned}\sum_{i=0}^{5}\sum_{j=0}^{5} c_{i,j} B_{i,5}(x) B_{j,5}(t) =& x^2 - t^2 - \frac{1}{4}xt^2 + \frac{1}{6}xt^4 \\ &+ \int_0^x \left(\int_0^t \int_0^1 rt \left(\sum_{i=0}^{5}\sum_{j=0}^{5} c_{i,j} B_{i,5}(r) B_{j,5}(s)\right) drds\right) dx,\end{aligned} \tag{43}$$

$$\begin{aligned}&\sum_{i=0}^{5}\sum_{j=0}^{5} c_{i,j} B_{i,5}(x) B_{j,5}(t) - \int_0^x \left(\int_0^t \int_0^1 rt \left(\sum_{i=0}^{5}\sum_{j=0}^{5} c_{i,j} B_{i,5}(r) B_{j,5}(s)\right) drds\right) dx \\ &= x^2 - t^2 - \frac{1}{4}xt^2 + \frac{1}{6}xt^4.\end{aligned} \tag{44}$$

Collocating (44) and using standard collocation points at $x = x_i$ and $t = t_j$ with

$$\begin{aligned} x_i &= \frac{i}{5}; \quad i = 0(1)5, \\ t_j &= \frac{j}{5}; \quad j = 0(1)5, \end{aligned}$$

we have

$$\sum_{i=0}^{5}\sum_{j=0}^{5} c_{i,j} B_{i,5}(x_i) B_{j,5}(t_j)$$

$$- \int_0^x \left(\int_0^t \int_0^1 rt_j \left(\sum_{i=0}^{5} \sum_{j=0}^{5} c_{i,j} B_{i,5}(r) B_{j,5}(s) \right) drds \right) dx$$
$$= x_i^2 - t_j^2 - \frac{1}{4} x_i t_j^2 + \frac{1}{6} x_i t_j^4. \quad (45)$$

The method was implemented using MAPLE 18 software and $M_5(x,t)$ was obtained as

$$M_5(x,t) = x^2 - t^2, \quad (46)$$

is the exact solution.

Example 4. Consider a linear 2D MVFIE of the second kind [19]

$$m'(x,t) = 1 - \frac{1}{6}t^2 - \frac{1}{6}t^3 + \int_0^t \int_0^1 rsm(r,s)drds \quad (47)$$

with initial condition $m(0,t) = t$ that has an exact solution given as $m(x,t) = x + t$ in the interval $(x,t) = [0,1]$.
Let the approximate solution to (47) for $N = 5$ be

$$m_N(x,t) = \sum_{i=0}^{5} \sum_{j=0}^{5} c_{i,j} B_{i,5}(x) B_{j,5}(t). \quad (48)$$

Integrating both sides of (47) from 0 to x, we have

$$\int_0^x (m'(x,t))\, dx = \int_0^x \left(1 - \frac{1}{6}t^2 - \frac{1}{6}t^3 \right) + \int_0^x \left(\int_0^t \int_0^1 rsm(r,s)drds \right) dx, \quad (49)$$

$$m(x,t) = x + t - \frac{1}{6}xt^2 - \frac{1}{6}xt^3 + \int_0^x \left(\int_0^t \int_0^1 rsm(r,s)drds \right) dx. \quad (50)$$

Substituting (48) into (50) gives

$$\sum_{i=0}^{5} \sum_{j=0}^{5} c_{i,j} B_{i,5}(x) B_{j,5}(t)$$
$$- \int_0^x \left(\int_0^t \int_0^1 rs \left(\sum_{i=0}^{5} \sum_{j=0}^{5} c_{i,j} B_{i,5}(r) B_{j,5}(s) \right) drds \right) dx$$
$$= x + t - \frac{1}{6}xt^2 - \frac{1}{6}xt^3. \quad (51)$$

Collocating (51) and using standard collocation points at $x = x_i$ and $t = t_j$ with

$$x_i = \frac{i}{5}; \quad i = 0(1)5,$$
$$t_j = \frac{j}{5}; \quad j = 0(1)5,$$

we have

$$\begin{aligned}
&\sum_{i=0}^{5}\sum_{j=0}^{5} c_{i,j} B_{i,5}(x_i) B_{j,5}(t_j) \\
&- \int_0^x \left(\int_0^t \int_0^1 rs \left(\sum_{i=0}^{5}\sum_{j=0}^{5} c_{i,j} B_{i,5}(r) B_{j,5}(s) \right) drds \right) dx \\
&\quad = x_i + t_j - \frac{1}{6} x_i t_j^2 - \frac{1}{6} x_i t_j^3 \qquad (52)
\end{aligned}$$

The method was implemented using MAPLE 18 software and $M_5(x,t)$ was obtained as

$$\begin{aligned}
M_5(x,t) =& 0.166e^{-5}t^4x - 0.141e^{-5}t^3x - 0.61e^{-5}t^2x^2 - 0.240e^{-5}tx^3 \\
&+ 0.170e^{-4}t^3x^2 + 0.211e^{-4}t^2x^3 + 0.318e^{-5}tx^4 - 0.595e^{-4}t^3x^3 \\
&- 0.285e^{-4}t^2x^4 - 0.1435e^{-5}tx^5 - 0.3660e^{-4}t^3x^5 + 0.681e^{-4}t^4x^3 \\
&+ 0.801e^{-4}t^3x^4 + 0.1294e^{-4}t^2x^5 - 0.196e^{-4}t^4x^2 + 0.826e^{-5}t^5x^2 \\
&- 0.16695e^{-4}t^5x^5 + 0.3678e^{-4}t^5x^4 - 0.2778e^{-4}t^5x^3 \\
&+ 0.41300e^{-4}t^4x^5 - 0.914e^{-4}t^4x^4 + 1.000000000t + 1.000000000x \\
&- 7.1e^{-7}t^5x + 7.0e^{-7}tx^2 + 5.0e^{-7}t^2x - 6.0e^{-8}tx \qquad (53)
\end{aligned}$$

4 Conclusion

In this section, a new numerical method was developed for solving 2D MV-FIEs of the second kind utilizing polynomial collocation. The findings obtained from each case were compared with the exact solution and some existing studies in the literature, the new approach established is simple, reliable, and efficient to compute. Maple 18 software is utilized for all computations

Table 5: Results using Bernstein polynomial for Example 4

(x,t)	$Exact$	$OurMethodN=5$	$Error_{OurMethod}$
$(0,0)$	0.0000000000	0.0000000000	0.0000000000
$(0.1,0.1)$	0.0200000000	0.0200000000	0.0000000000
$(0.2,0.2)$	0.0400000000	0.3999999998	$2.0e{-10}$
$(0.3,0.3)$	0.6000000000	0.5999999993	$7.0e{-10}$
$(0.4,0.4)$	0.8000000000	0.7999999978	$2.2e{-9}$
$(0.5,0.5)$	0.1000000000	0.9999999930	$7.0e{-9}$
$(0.6,0.6)$	1.2000000000	1.199999981	$1.9e{-8}$
$(0.7,0.7)$	1.4000000000	1.3999999564	$4.4e{-8}$
$(0.8,0.8)$	1.6000000000	1.599999896	$1.04e{-7}$
$(0.9,0.9)$	1.8000000000	1.799999750	$2.50e{-7}$
$(1.0,1.0)$	2.0000000000	1.999999430	$5.70e{-7}$

in this work. The accuracy of the method is proved by considering various examples, which shows that the method is efficient and appropriate for this type of situations. We compare our absolute errors of Example 1 with [9] as shown in Table 2 and also absolute errors of Example 2 with [4] as shown in Table 4. We can therefore conclude that our method is superior and more preferable than the existing methods.

The results obtained from problem 1 at $N=5$ and at different values of (x,t) shows clearly that the developed method is better than the method presented by [9]. From Table 3 for $(x,t)=(0.1,0.1)$ and $N=5$, for instance the absolute errors are $Error_B = 2.023\times 10^{-8}$ and $Error_{NKH} = 3.3469\times 10^{-6}$. Again from Table 3 for $(x,t)=(0.1,0.3)$ and $N=5$, the absolute errors are $Error_B = 9.43\times 10^{-9}$ and $Error_{NKH} = 3.03472\times 10^{-5}$ which shows clearly that the developed method is consistent, reliable, and performs favorably.

The results obtained from problem 2 at $N=5$ and at different values of (x,t) shows clearly that the developed method is better than the method presented by [4]. From Table 4, for instance the absolute errors for $(x,t) = (0.0,0.0)$ and at $N=5$ gives $Error_B = 0.0000000$ and $Error_{AM} = 2.46\times 10^{-5}$, for $(x,t)=(0.1,0.1)$ and at $N=5$, gives $Error_B = 2.064\times 10^{-8}$ and

$Error_{AM} = 1.46 \times 10^{-5}$. Again From Table 3 for $(x, t) = (1.0, 1.0)$ and at $N = 5$, the absolute errors are $Error_B = 5.121 \times 10^{-6}$ and $Error_{AM} = 5.70 \times 10^{-1}$ shows clearly that the developed method is consistent, efficient and converges faster than the method presented by [4].

It was observed that the results obtained for Example 3 at $N = 5$ give the exact solution, hence the reason it is not in tabular form. This clearly indicates that the method is efficient and convergent.

The solution obtained from Example 4 at $N = 5$ and at various values of (x, t) indicates the method is stable and converges to the exact solution. From Table 5 for instance, the result obtained at $N = 5$ and $(x, t) = (0, 0)$, $(x, t) = (0.1, 0.1)$, $(x, t) = (0.2, 0.2)$ and $(x, t) = (0.3, 0.3)$ gives 0.0000000, 0.0000000, 2.0×10^{-10}, 7.0×10^{-10} respectively.

It has been observed and examined that when the values of N increase, the error decreases and the approximate solution converges rapidly to the exact solution, the value of $N = 5$ was chosen arbitrarily and for simplicity.

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