Iranian Journal of Numerical Analysis and Optimization
Vol. ??, No. ??, ??, pp ??
https://doi.org/??
https://ijnao.um.ac.ir/

Research Article

Approximate symmetries of the perturbed KdV-KS equation

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Abstract

The analysis of approximate symmetries in perturbed nonlinear partial differential equations (PDEs) stands as a cornerstone for unraveling complex physical behaviors and solution patterns. This paper delves into the investigation of approximate symmetries inherent in the perturbed

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Received ??? ; revised ??? ; accepted ???

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How to cite this article
Mohammadpouri, A., Hashemi, M.S., Abbasi, R. and Abbasi, R., Approximate symmetries of the perturbed KdV-KS equation. *Iran. J. Numer. Anal. Optim.*, ??; ??(??): ??-??. ??

Korteweg-de Vries and Kuramoto-Sivashinsky (KdV-KS) equation, fundamental models in the realm of fluid dynamics and wave phenomena. Our study commences by detailing the method to derive approximate vector Lie symmetry generators that underpin the approximate symmetries of the perturbed KdV-KS equation. These generators, while not exact, provide invaluable insights into the equation's dynamics and solution characteristics under perturbations. A comprehensive approximate commutator table is subsequently constructed, elucidating the relationships and interplay between these approximate symmetries and shedding light on their algebraic structure. Leveraging the power of the adjoint representation, we examine the stability of these approximate symmetries when subjected to perturbations. This analysis enables us to discern the most resilient symmetries, instrumental in identifying intrinsic features that persist even in the face of disturbances. Furthermore, we harness the concept of approximate symmetry reductions, a pioneering technique that allows us to distill crucial dynamics from the complexity of the perturbed equation. Through this methodology, we uncover invariant solutions and reduced equations that serve as effective surrogates for the original system, capturing its essential behavior and facilitating analytical and numerical investigations. In summary, our exploration into the approximate symmetries of the perturbed KdV-KS equation not only advances our comprehension of the equation's intricate dynamics but also offers a comprehensive framework for studying the impact of perturbations on approximate symmetries, all while opening new avenues for tackling nonlinear PDEs in diverse scientific disciplines.

AMS subject classifications (2020): Primary 53A10; Secondary 22E70, 26M60.

Keywords: Approximate Lie symmetries; Commutator table; Adjoint representation; Reductions.

1 Introduction

The Korteweg-de Vries and Kuramoto-Sivashinsky (KdV-KS) equation is a notable partial differential equation (PDE) that amalgamates the Korteweg-de Vries equation, renowned for describing long, weakly nonlinear waves, with the Kuramoto-Sivashinsky equation, which captures spatiotemporal chaos in pattern-forming systems. This fusion yields a versatile equation capable of modeling a diverse array of physical phenomena. The KdV-KS equation finds application in various fields, including fluid dynamics, combustion, and nonlinear optics. In fluid dynamics, it can depict the evolution of complex wave patterns on fluid interfaces, while in combustion processes, it may illuminate the behavior of flame fronts and combustion instabilities. Additionally, the equation's presence in the realm of nonlinear optics can aid in understanding pulse propagation in optical fibers. Its broad applicability underscores the KdV-KS equation's significance as a tool for investigating intricate dynamics in real-world systems and its role in advancing our comprehension of nonlinear phenomena across multiple scientific disciplines [10, 5, 12].

Analytical methods for solving PDEs constitute a vital framework in understanding the behavior of various physical, mathematical, and engineering systems. These methods encompass a range of techniques, such as Lie symmetry method [14, 2, 18, 19, 17, 16, 22], Kudryashov's method [16, 20, 9], Nucci's reduction method [23, 24], invariant subspace method [8, 21, 4], and Tanh method [7, 1, 6].

In cases where PDEs possess specific geometrical or algebraic properties, separation of variables can yield exact solutions by decomposing the equation into simpler ordinary differential equations. Similarity transformations assist in reducing complex PDEs to canonical forms that admit analytical solutions. Integral transforms, like the Fourier and Laplace transforms, provide a powerful means to convert differential equations into algebraic equations that can be more easily solved. Perturbation methods, including the method of matched asymptotic expansions and multiple scales analysis, are particularly useful when dealing with systems that exhibit small parameter deviations from simpler cases, allowing the derivation of approximate solutions.

These analytical techniques not only offer insights into the underlying dynamics of diverse systems but also serve as benchmarks for numerical methods. However, their applicability is often constrained by the complexity of the equations and the presence of nonlinear terms. In such cases, a combination of these methods, along with innovations in mathematical analysis, plays a crucial role in uncovering solutions that enrich our understanding of the intricate interplay between mathematics and the physical world.

We shall research the perturbed KdV-KS equations's vector fields, approximate symmetry, and symmetry reductions. A perturbed form of the KdV-KS equations are

$$u_t + uu_x + u_{xxx} + \epsilon(u_{xx} + u_{xxxx}) = 0, \tag{1}$$

where $0 < \epsilon \ll 1$ is a small parameter, $x \in \mathbb{R}$, and $t \geq 0$.

The structure of this work is as follows. We find the approximate symmetry and optimal system of the perturbed KdV-KS equation in section 2. Ordinary differential equation symmetry reductions are covered in section 3. Finally, section 4 will provide the conclusions.

2 Analysis of the approximate Lie symmetries

Approximate Lie symmetries play a crucial role in various scientific and mathematical contexts, particularly in the study of dynamical systems and differential equations. Unlike exact symmetries, which lead to conserved quantities and well-defined transformations, approximate Lie symmetries emerge in situations where the underlying system's behavior is influenced by small perturbations or deviations from ideal conditions. These symmetries provide insights into the system's response to fluctuations and disturbances, contributing to our understanding of stability, chaos, and the emergence of complex patterns. By analyzing the behavior of systems under approximate Lie symmetries, researchers gain valuable insights into the underlying dynamics and are better equipped to model real-world phenomena with a more comprehensive perspective.

Let $\triangle(t, x, u, \epsilon) = \triangle_0(t, x, u) + \epsilon\triangle_1(t, x, u) = u_t + uu_x + u_{xxx} + \epsilon(u_{xx} + u_{xxxx})$. If an operator $Y = Y_0 + \epsilon Y_1$ satisfies

$$\left[Y^{(4)}\triangle(t, x, u, \epsilon)\right]_{\triangle(t,x,u,\epsilon)=0} = 0, \tag{2}$$

then it is referred to as an approximation Lie symmetry generator. Here, $Y^{(4)}$ is the forth-order prolongation of the forth-order approximate Lie symmetry Y, and

$$Y_0 = \xi_0^1(t,x,u)\frac{\partial}{\partial t} + \xi_0^2(t,x,u)\frac{\partial}{\partial x} + \eta_0(t,x,u)\frac{\partial}{\partial u},$$
$$Y_1 = \xi_1^1(t,x,u)\frac{\partial}{\partial t} + \xi_1^2(t,x,u)\frac{\partial}{\partial x} + \eta_1(t,x,u)\frac{\partial}{\partial u}.$$

The prolongation formula is a fundamental tool within the realm of Lie symmetry methods, a powerful mathematical approach used to analyze and solve differential equations. In this context, the prolongation formula extends the Lie derivative to higher-order derivatives and introduces a systematic way of calculating symmetries of a given differential equation. By iteratively applying the prolongation formula, one can uncover hidden symmetries that may not be immediately apparent. This process allows researchers to determine transformations that leave the equation invariant and identify conserved quantities or transformations that simplify its solutions.

Equation (2) divides two parts into

$$\left[Y_0^{(4)}\triangle_0(t,x,u,\epsilon)\right]_{\triangle_0(t,x,u,\epsilon)=0} = 0, \tag{3}$$
$$\left[Y_1^{(4)}\triangle_0(t,x,u,\epsilon) + Y_0^{(4)}\triangle_1(t,x,u,\epsilon)\right]_{\triangle(t,x,u,\epsilon)=0} = 0. \tag{4}$$

By conditions (3) and (4), we arrive at the set of determining equations below:

$$\xi^1_{0,t} = \xi^1_{0,x} = \xi^1_{0,u} = \xi^2_{0,x} = \xi^2_{0,u} = \eta_{0,u} = 0, \quad \eta_{0,xxx} + \eta_{0,t} + u\eta_{0,x} = 0,$$
$$\eta_0 - \xi^2_{0,t} = 0, \quad \xi^1_{1,x} = \xi^1_{1,u} = \xi^2_{1,u} = 0, \quad \eta_{1,uu} = 0, \quad \eta_{1,xxx} + \eta_{1,t} + u\eta_{1,x} = 0,$$
$$2u\xi^2_{1,x} - \xi^2_{1,xxx} - \xi^2_{1,t} + 3\eta_{1,xxu} + \eta_1 = 0, \quad \xi^1_{1,t} - 3\xi^2_{1,x} = 0, \quad \eta_{1,xu} - \xi^2_{1,xx} = 0.$$

Solving this PDE system gives us

$$\xi_0^1 = a_0, \quad \xi_0^2 = b_0 t + c_0, \quad \eta_0 = b_0,$$
$$\xi_1^1 = -\frac{3}{2}a_1 t + b_1, \quad \xi_1^2 = -\frac{1}{2}a_1 x + c_1 t + d_1, \quad \eta_1 = a_1 u + c_1.$$

Therefore

$$X = (a_0 + \epsilon(-\frac{3}{2}a_1 t + b_1))\partial_t + (b_0 t + c_0 + \epsilon(-\frac{1}{2}a_1 x + c_1 t + d_1))\partial_x + (b_0 + \epsilon(a_1 u + c_1))\partial_u$$

where $a_0, b_0, c_0, a_1, b_1, c_1$, and d_1 are constants. Consequently, the following seven independent approximate operators span infinitesimal symmetries of

Table 1: Approximate commutator table for symmetries in (1)

$[y_i, y_j]$	y_1	y_2	y_3	y_4	y_5	y_6	y_7
y_1	0	0	y_2	0	$-\frac{3}{2}y_4$	0	y_6
y_2	0	0	0	0	$-\frac{1}{2}y_6$	0	0
y_3	$-y_2$	0	0	$-y_6$	y_7	0	0
y_4	0	0	y_6	0	0	0	0
y_5	$\frac{3}{2}y_4$	$\frac{1}{2}y_6$	$-y_7$	0	0	0	0
y_6	0	0	0	0	0	0	0
y_7	$-y_6$	0	0	0	0	0	0

equation (1)

$$y_1 = \partial_t, \quad y_2 = \partial_x, \quad y_3 = t\partial_x + \partial_u, \quad y_4 = \epsilon\partial_t, \quad y_5 = \epsilon(-\frac{3}{2}t\partial_t - \frac{1}{2}x\partial_x + u\partial_u),$$
$$y_6 = \epsilon\partial_x, \quad y_7 = \epsilon(t\partial_x + \partial_u).$$

In the field of Lie symmetry methods, a commutator table serves as a fundamental tool for analyzing the algebraic structure of Lie symmetries associated with a system of differential equations. The commutator of two vector fields, representing different symmetry transformations, is calculated and organized in a table format. This table provides valuable information about the Lie algebra generated by these vector fields, revealing how they interact and combine. By determining the commutators, researchers can discern the algebraic relationships between symmetries, uncover hidden patterns, and ultimately construct a Lie algebra that captures the system's inherent symmetries. The commutator table thus acts as a guiding compass in the exploration of differential equations, aiding in the classification, solution, and deeper comprehension of complex dynamical systems. Table 1 contains the approximate commutator table for symmetries in equation (1). The adjoint representation involves mapping each element of a Lie group to an associated automorphism of its corresponding Lie algebra. This representation provides insights into how transformations in the group relate to transformations in the algebra, offering a way to study the Lie group through its associated

Table 2: Adjoint representation spanned by the the basis approximate symmetries of the KdV-KS equation

$Ad(\exp(ay_i))y_j$	y_1	y_2	y_3	y_4	y_5	y_6	y_7
y_1	y_1	y_2	$y_3 - ay_2$	y_4	$y_5 + \frac{3}{2}ay_4$	y_6	$y_7 - ay_6$
y_2	y_1	y_2	y_3	y_4	$y_5 + \frac{1}{2}ay_6$	y_6	y_7
y_3	$y_1 + ay_2$	y_2	y_3	$y_4 + ay_6$	$y_5 - ay_7$	y_6	y_7
y_4	y_1	y_2	$y_3 - ay_6$	y_4	y_5	y_6	y_7
y_5	$y_1 - \frac{3}{2}ay_4$	$y_2 - \frac{1}{2}ay_6$	$y_3 - \frac{1}{2}ay_7$	y_4	y_5	y_6	y_7
y_6	y_1	y_2	y_3	y_4	y_5	y_6	y_7
y_7	$y_1 + ay_6$	y_2	y_3	y_4	y_5	y_6	y_7

Lie algebra. By analyzing the adjoint representation, researchers can explore the relationships between different Lie group elements and understand the symmetries and transformations that underlie a given system of differential equations. This representation is a key tool for investigating the symmetries, conservation laws, and invariants inherent in complex dynamical systems, ultimately facilitating the application of Lie symmetry methods to a wide range of scientific and mathematical problems. Each y_i, $i = 1, \ldots, 7$ of the basis approximate infinitesimal symmetries spans an adjoint representation $Ad(\exp(ay_i))y_j$, a is a parameter, given by

$$Ad(\exp(ay_i))y_j = y_j - a\,[y_i, y_j] + \frac{a^2}{2}\,[y_i, [y_i, y_j]] - \cdots .$$

Table 2 lists each adjoint representation of the Lie approximate symmetry of the KdV-KS equation. Here, in the following, we find that the group approximate transformation h_i, which is generated by the y_i for $i = 1, 2, \ldots, 7$ for the KdV-KS equation (1)

$$\begin{cases} h_1.(t,x,u)\mapsto(t+a,x,u),\\ h_2.(t,x,u)\mapsto(t,x+a,u),\\ h_3.(t,x,u)\mapsto(t,x+ta,u+a),\\ h_4.(t,x,u)\mapsto(t+a\epsilon,x,u),\\ h_5.(t,x,u)\mapsto((1-\frac{3}{2}a\epsilon)t,((1-\frac{1}{2}a\epsilon)x,((1+a\epsilon)u),\\ h_6.(t,x,u)\mapsto(t,x+a\epsilon,u),\\ h_7.(t,x,u)\mapsto(t,x+a\epsilon t,u+a\epsilon). \end{cases}$$

Consequently, the invariant solutions of a solution $u=g(t,x)$ for the KdV-KS equation is given by

$$\begin{cases} h_1.g(t,x)=g(t-a,x),\\ h_2.g(t,x)=g(t,x-a),\\ h_3.g(t,x)=g(t,x-ta)+a,\\ h_4.g(t,x)=g(t-a\epsilon,x),\\ h_5.g(t,x)=(1+a\epsilon)g((1+\frac{3}{2}a\epsilon)t,(1+\frac{1}{2}a\epsilon)x),\\ h_6.g(t,x)=g(t,x-\epsilon a),\\ h_7.g(t,x)=\epsilon g(t,x-a\epsilon t)+a. \end{cases}$$

It would be useful to determine the minimal collection of subgroups that will produce all potential group invariant solutions since a solution can be utilized to construct additional solutions using various groups. An optimal system, which is a collection of such solutions, is created by analyzing the manner in which group invariant solutions change one another via the adjoint operation. Thus we will construct a one-dimensional optimal system of approximate Lie subalgebra of perturbed KdV-KS equation by considering an arbitrary element $y=\sum_{i=1}^{7}s_iy_i$ of KdV-KS equation lie algebra $\mathfrak{g}$. The map $G_i^{a_i}:\mathfrak{g}\to\mathfrak{g}$ given by $y\to Ad(\exp(a_iy_i))y$ is a linear, $i=1,\dots,7$. By using Table 2 the matrix $M_i^{a_i}$ of $G_i^{a_i}$ with respect to the approximate basis is given by

$$M_1^{a_1}=\begin{bmatrix}1&0&0&0&0&0&0\\0&1&0&0&0&0&0\\0&-a_1&1&0&0&0&0\\0&0&0&1&0&0&0\\0&0&0&\frac{3}{2}a_1&1&0&0\\0&0&0&0&0&1&0\\0&0&0&0&0&-a_1&1\end{bmatrix},\quad M_2^{a_2}=\begin{bmatrix}1&0&0&0&0&0&0\\0&1&0&0&0&0&0\\0&0&1&0&0&0&0\\0&0&0&1&0&0&0\\0&0&0&0&1&\frac{1}{2}a_2&0\\0&0&0&0&0&1&0\\0&0&0&0&0&0&1\end{bmatrix},$$

$$M_3^{a_3}=\begin{bmatrix}1&a_3&0&0&0&0&0\\0&1&0&0&0&0&0\\0&0&1&0&0&0&0\\0&0&0&1&0&a_3&0\\0&0&0&0&1&0&-a_3\\0&0&0&0&0&1&0\\0&0&0&0&0&0&1\end{bmatrix},\quad M_4^{a_4}=\begin{bmatrix}1&0&0&0&0&0&0\\0&1&0&0&0&0&0\\0&0&1&0&0&-a_4&0\\0&0&0&1&0&0&0\\0&0&0&0&1&0&0\\0&0&0&0&0&1&0\\0&0&0&0&0&0&1\end{bmatrix},$$

$$M_5^{a_5}=\begin{bmatrix}1&0&0&-\frac{3}{2}a_5&0&0&0\\0&1&0&0&0&-\frac{1}{2}a_5&0\\0&0&1&0&0&0&-\frac{1}{2}a_5\\0&0&0&a_5&0&0&0\\0&0&0&0&1&0&0\\0&0&0&0&0&1&0\\0&0&0&0&0&0&1\end{bmatrix},\quad M_6^{a_6}=I_7,\quad M_7^{a_7}=\begin{bmatrix}1&0&0&0&0&a_7&0\\0&1&0&0&0&0&0\\0&0&1&0&0&0&0\\0&0&0&1&0&0&0\\0&0&0&0&1&0&0\\0&0&0&0&0&1&0\\0&0&0&0&0&0&1\end{bmatrix}.$$

Then it is seen that

$$\begin{aligned}G_7^{a_7}\circ G_6^{a_6}\circ\cdots\circ G_1^{a_1}: y \mapsto & s_1y_1+(a_3s_1+s_2-a_1s_3)y_2+s_3y_3\\ &+(\frac{3}{2}(a_1s_5+a_5s_1)+s_4)y_4+s_5y_5+[a_3(s_4+\frac{3}{2}a_1s_5)\\ &-\frac{1}{2}a_5(a_3s_1+s_2-a_1s_3)-a_4s_3\\ &+\frac{1}{2}a_2s_5+s_6-a_1s_7+a_7s_1]y_6\\ &+(s_7-a_3s_5-\frac{1}{2}a_5s_3)y_7.\end{aligned}$$

Now, by setting suitable a_i, we can easily omit the coefficient of y_j in several cases, so y can be reduced and one-dimensional optimal system is provided by

$y_1+\delta y_3+\beta y_4,\quad y_1+\alpha y_3+\gamma y_5,\quad y_1+\alpha y_3+\lambda y_7,\quad y_2+\upsilon y_4+\lambda y_7,\quad y_2+\gamma y_5,$
$y_3+\beta y_4+\gamma y_5,\quad y_4+\lambda y_7,\quad y_5,\quad y_6,\quad y_7,$

where α, β, γ, λ, and $\delta, \upsilon \neq 0$ are real numbers.

3 Approximate symmetry reductions

In this section, we consider some reductions of equation (1) corresponding to some approximate Lie symmetries [13, 11, 15, 25]. The reduction of PDEs by approximate symmetries is a valuable technique used to simplify the complexity of solving these equations while retaining essential features of their behavior.

Reduction 3.1. Similarity variable respect to the symmetry y_1, is $u(t,x)=s_0(x)+\epsilon s_1(x)+O(\epsilon^2)$, substituting into equation (1). Comparing of the constant and ϵ coefficients, we get the following ordinary differential equation (ODE) system:

$$\begin{cases} (\frac{{s_0}^2}{2})'+s_0^{(3)}=0,\\ (s_0s_1)'+{s_1}^{(3)}+s_0''-(\frac{{s_0}^2}{2})''=0, \end{cases}$$

where $'$ shows the derivative respect to x.

Note that the symmetry y_2 produces the trivial solutions.

Reduction 3.2. Similarity variable of y_3 is $u(t,x)=\frac{x}{t}+s_0(t)+\epsilon s_1(t)+O(\epsilon^2)$, where $s_0(t)$ and $s_1(t)$ admit the following first order ODE system:

$$\begin{cases} ts_0'+s_0=0,\\ ts_1'+s_1=0, \end{cases}$$

where $'$ shows the derivative respect to t. Therefore, we find that $s_0(t)=\frac{c_0}{t}$ and $s_1(t)=\frac{c_1}{t}$. Thus we have

$$u(t,x)=\frac{x}{t}+\frac{c_0}{t}+\epsilon\frac{c_1}{t}+O(\epsilon^2).$$

Reduction 3.3. Similarity variable respect to the symmetry y_4, is $u(t,x)=s_0(x)+\epsilon s_1(t,x)+O(\epsilon^2)$. Substituting into (1), it satisfies the following PDE:

$$\begin{cases} (\frac{{s_0}^2}{2})' + s_0^{(3)} = 0, \\ s_{1,t} + (s_0 s_1)' + {s_1}^{(3)} + s_0'' - (\frac{{s_0}^2}{2})'' = 0, \end{cases}$$

where $'$ shows the derivative respect to x.

Reduction 3.4. For the approximate operator y_5 the similarity variables are $\eta = \dfrac{t}{x^3}$, $u(t,x) = \frac{s_0(\eta)}{x^2} + \epsilon \frac{s_1(\eta)}{x^\alpha} + O(\epsilon^2)$. Therefore, s_0 satisfies the following reduced equation:

$$27\eta^3 s_0^{(3)} + 90\eta^2 s_0'' + 186\eta s_0' + 3\eta s_0' s_0 - s_0' + 2s_0^2 + 24 s_0 = 0,$$

where $s_0' = \dfrac{ds_0}{d\eta}$. Also, s_1 and α may be determined by following equation:

$$(\frac{s_1}{x^\alpha})_{,t} + (\frac{s_0 s_1}{x^{\alpha+2}})_{,x} + (\frac{s_1}{x^\alpha})_{,xxx} + (\frac{s_0}{x^2})_{,xx} + (\frac{s_0}{x^2})_{,xxxx} = 0.$$

Reduction 3.5. Similarity variable of y_6 is $u(t,x) = c_0 + \epsilon s_1(t,x) + O(\epsilon^2)$, where $s_1(t,x)$ admits the following PDE equation:

$$s_{1,t} + c_0 s_{1,x} + s_{1,xxx} = 0.$$

Reduction 3.6. Similarity variable of y_7 is $u(t,x) = \frac{x}{t} + \frac{c_0}{t} + \epsilon s_1(t,x) + O(\epsilon^2)$, where $s_1(t,x)$ admits the following PDE equation:

$$s_1 + t s_{1,t} + (c_0 + x) s_{1,x} + t s_{1,xxx} = 0.$$

Reduction 3.7. Similarity variables respect to the symmetry $y_1 + y_6$ are $u(t,x) = s(\eta)$, $\eta = x - \epsilon t$. We see that the parameter ϵ does not appear directly, but it is instead implicitly contained within the relevant variables. Substituting it into equation (1), we obtain the following reduced approximate ODE:

$$ss' + s^{(3)} + \epsilon(-s' + s'' + s^{(4)}) = 0,$$

where $s' = \dfrac{ds}{d\eta}$. Integrating this ODE under the conditions $s(\mp\infty) = 0$, $s'(\mp\infty) = 0$, $s''(\mp\infty) = 0$, and $s^{(3)}(\mp\infty) = 0$, and setting the integral constant to zero result in

$$\frac{s^2}{2} + s'' + \epsilon(-s + s' + s^{(3)}) = 0.$$

This equation can be given as

$$\begin{cases} \frac{ds}{d\eta} = v, \\ \frac{dv}{d\eta} = w, \\ \epsilon \frac{dw}{d\eta} = -\frac{s^2}{2} - w - \epsilon v + \epsilon s. \end{cases} \tag{5}$$

By putting $\epsilon = 0$, in the above slow system, the critical manifold M_0 is any compact subset contained in the set of critical points $\{(s, v, w) \mid w = -\frac{s^2}{2}\}$ (see a complete information about critical manifold and Fenichel's theorems in [3]). Therefore, the slow flow on M_0 is given by the following system:

$$\begin{cases} \frac{ds}{d\eta} = v, \\ \frac{dv}{d\eta} = -\frac{s^2}{2}. \end{cases} \tag{6}$$

Figure 1 shows the orbit of (6) to the critical point $(0, 0)$.

By Fenichel's invariant manifold theorem, for sufficiently small ϵ, the slow manifold M_ϵ located within $O(\epsilon)$ of M_o, that is,

$$w = -\frac{s^2}{2} + \epsilon g(s, v) + O(\epsilon^2).$$

Substituting this relation into the last equation of slow system (5) gives $g(s, v) = (s - 1)v + s$.

4 Conclusion

In this research, the perturbed KdV-KS equation was studied using Lie approximation symmetry analysis. We were able to reduce this problem using similarity Lie approximation algebra. Based on the optimal system approach, all of the group-invariant solutions to equation (1) are taken into consideration. Wide classes of nonlinear differential equations can be effectively solved using the fundamental concept provided in this study.

Conflict of interest

The authors have no conflict of interest to declare that are relevant to this article.

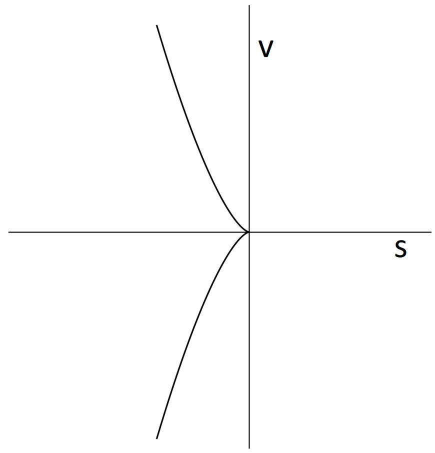

Figure 1: The orbit of (6) to the critical point $(0, 0)$

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