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Research Article

Convergence analysis of triangular and symmetric splitting method for fuzzy stochastic linear systems

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Abstract

In this article, the triangular and symmetric splitting iterative method is suggested for solving linear homogeneous systems of equations $\pi Q = 0$, where Q is the stochastic rate matrix and π is the steady state vector. The

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homogeneous system is converted to the nonhomogeneous regularized fuzzy linear system $Ax = b$ with the small perturbation parameter $0 < r \leq 1$. The regularized fuzzy linear system is converted into an embedded linear system. The iterative scheme is established; convergence criteria and its sensitivity analysis are analyzed using the numerical examples and convergence theorems. From the numerical results, it is evident to conclude that the proposed method is effective and efficient compared to the theoretical results.

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1 Introduction

Fuzzy linear systems (FLS) and Fully Fuzzy linear systems (FFLS) have great applications in various areas of engineering, science, and social sciences, such as physics, statistics, operational research, control problems, neural networks, communication systems, sensors, and economics. The mathematical modeling of a physical problem is formulated into the system of fuzzy linear homogeneous or nonhomogeneous equations. There are many methods in the literature to solve the nonhomogeneous FLSs. The homogeneous FLS has either a trivial solution or an infinite number of solutions. For a unique nontrivial solution, the homogeneous system $\pi Q = 0$, where π is the steady state vector and Q is the stochastic rate matrix, is converted into the regularized nonhomogeneous fuzzy linear system $Ax = b$ with the small perturbation $0 < r \leq 1$. The regularized FLS is converted into an embedded linear system $\Theta X = \Upsilon$, where Θ and Υ are fuzzy matrices and X is an unknown fuzzy vector. Many straight forward methods are existing in the literature to find the unique non-zero solution of pertinent to linear systems when the coefficient matrix is a crisp matrix. However, in actual cases, the parameters may be uncertain or vague. So, to overcome the uncertainty and vagueness, the coefficient matrix of the system $Ax = b$ is assumed as a fuzzy stochastic matrix instead of the crisp matrix.

Many researchers proposed direct and iterative methods to solve FLSs. The first iterative model with an embedding technique for computing a class of $n \times n$ FLS was triggered by Friedman, Ming, and Kandel [11]. For solving a system of fuzzy linear equations, a few numerical methods were developed and discussed for the existence of solution, provided that the diagonal elements are positive and satisfy the diagonal dominance property by Dehghan, Hashemi and Ezzati [6, 9]. The steepest descent method and LU decomposition method were developed in [1, 2]. Allahviranloo [3, 4] used the Jacobi, Gauss–Seidel, SOR, iterative methods for finding the approximate solution of the FLS. A fuzzy system of linear equations with crisp coefficients was proposed by Chakraverty and Behera [5]. The inherited LU factorization method was proposed by Fariborzi Araghi and Fallahzadeh [10] for solving a fuzzy systems of linear equations. Koam et al. [13] used the LU decomposition scheme for solving m-polar fuzzy system of linear equations. Block SOR method for FLSs was proposed by Miao, Zheng, and Wang [14], and the QR-decomposition method was developed by S.H. Nasseri, Matinfar, and Sohrabi [15]; Wang and Wu [17] introduced the Uzawa-SOR method. Symmetric successive over relaxation method, block iterative method, and splitting iterative methods were established by Wang, Zheng and Yin [18, 19, 21]. Wang and Chen [20] suggested a modified Jacobi iterative method for large-size linear systems, and a new method based on Jacobi iteration was proposed to solve the FLSs by Zhen et al. [12]. If the coefficient matrix is crisp, then it will restrict the modeling of the real-time problems. In the system of linear equations, both the coefficient matrix and right-hand side matrices are fuzzy matrices. then it is defined as an FFLS. FFLS gives wide scope in real-time applications by removing the crispness in the left-hand side coefficient matrix. The iterative solution of general FFLS is proposed in [7]. Edalatpanah [8] proposed a modified iterative method for finding the solution of FFLS. Classical triangular and symmetric (TS) splitting methods are simple to implement and suitable to find the steady state probability vector and performance measures in many real time systems [16]. In this paper, a new improved method based on TS iteration is provided for solving FFLSs. The rest of the paper is organized as follows: Section 2 gives some basic definitions and results of FLS. In section 3, the new method is established with convergence theorems. Perturbation analysis is discussed in section 4. Numerical examples are presented in section 5, and the conclusions are in section 6.

2 Basic definitions of FLSs and convergence analysis of TS method

In this section, we have defined the FLS and some basic definitions such as fuzzy numbers, fuzzy solutions, arithmetic operations on fuzzy numbers, and embedded model of FFLS, which are useful in the numerical solution of FLS.

Fuzzy number: An arbitrary form of fuzzy number is an ordered pair of functions $(\underline{v}(r), \overline{v}(r))$, $0 < r \leq 1$, satisfying

- $\underline{v}(r)$ is a bounded monotonic increasing left continuous function over $[0, 1]$,
- $\overline{v}(r)$ is a bounded monotonic decreasing left continuous function over $[0, 1]$,
- $\underline{v}(r) \leq \overline{v}(r)$, $0 < r \leq 1$.

Arithmetic operations on fuzzy numbers: If $u = (\underline{u}(r), \overline{u}(r))$ and $v = (\underline{v}(r), \overline{v}(r))$ are arbitrary fuzzy numbers, then the arithmetic operations of arbitrary fuzzy numbers for $0 < r \leq 1$ and real number k, are defined as follows:

- $u = v$ if and only if $\underline{u}(r) = \underline{v}(r)$ and $\overline{u}(r) = \overline{v}(r)$,
- $u + v = (\underline{u}(r) + \underline{v}(r), \overline{u}(r) + \overline{v}(r))$, and
- $ku = \begin{cases} (k\underline{u}(r), k\overline{u}(r)), & k > 0, \\ (k\overline{u}(r), k\underline{u}(r)), & k < 0. \end{cases}$

Fuzzy linear system: The $n \times n$ FLS is defined as

$$
\begin{aligned}
a_{11}x_1 + a_{12}x_2 + \cdots + a_{1n}x_n &= b_1, \\
a_{21}x_1 + a_{22}x_2 + \cdots + a_{2n}x_n &= b_2, \\
&\vdots
\end{aligned}
$$

$$a_{n1}x_1 + a_{n2}x_2 + \cdot + a_{nn}x_n = b_n.$$

The matrix form of the above linear system is

$$Ax = b, \tag{1}$$

where

$$A = \begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{n1} & a_{n2} & \cdots & a_{nn} \end{bmatrix}.$$

is a crisp matrix, $b = [b_1, b_2, \ldots, b_n]^T$ is a fuzzy vector, and $x = [x_1, x_2, \ldots, x_n]^T$ is unknown.

Solution of an FLS: A fuzzy vector $x = (x_1, x_2, \ldots, x_n)^T$ given by $x_i = (\underline{x}_i(r), \overline{x}_i(r))$, $1 \le i \le n$, $0 < r \le 1$, is called a solution of the FLS (1) if

$$\begin{cases} \underline{\sum_{j=1}^{n} a_{ij}x_j} = \sum_{j=1}^{n} \underline{a_{ij}x_j} = \underline{b}_i, \\ \\ \overline{\sum_{j=1}^{n} a_{ij}x_j} = \sum_{j=1}^{n} \overline{a_{ij}x_j} = \overline{b}_i. \end{cases} \tag{2}$$

Embedded Model of FLS: The embedding model of extended FLS (1) into the $2n \times 2n$ crisp linear system is defined as

$$\begin{aligned} \theta_{1,1}\underline{x}_1 + \cdots + \theta_{1,n}\underline{x}_n + \theta_{1,n+1}(-\overline{x}_1) + \cdots + \theta_{1,2n}(-\overline{x}_n) &= \underline{b}_1, \\ \theta_{2,1}\underline{x}_1 + \cdots + \theta_{2,n}\underline{x}_n + \theta_{2,n+1}(-\overline{x}_1) + \cdots + \theta_{2,2n}(-\overline{x}_n) &= \underline{b}_2, \\ &\vdots \\ \theta_{n,1}\underline{x}_1 + \cdots + \theta_{n,n}\underline{x}_n + \theta_{n,n+1}(-\overline{x}_1) + \cdots + \theta_{n,2n}(-\overline{x}_n) &= \underline{b}_n, \\ \theta_{n+1,1}\underline{x}_1 + \cdots + \theta_{n+1,n}\underline{x}_n + \theta_{n+1,n+1}(-\overline{x}_1) + \cdots + \theta_{n+1,2n}(-\overline{x}_n) &= \overline{-b}_1, \\ &\vdots \\ \theta_{2n,1}\underline{x}_1 + \cdots + \theta_{2n,n}\underline{x}_n + \theta_{2n,n+1}(-\overline{x}_1) + \cdots + \theta_{2n,2n}(-\overline{x}_n) &= \overline{-b}_n. \end{aligned}$$

The matrix form of above $2n \times 2n$ linear system is

$$\Theta X = \Upsilon, \tag{3}$$

where $\Theta = (\theta_{kl})$, θ_{kl} are determined as follows:

1. For $a_{ij} > 0, \theta_{ij} = a_{ij}, \theta_{n+i,n+j} = a_{ij}$.
2. For $a_{ij} < 0, \theta_{i,n+j} = a_{ij}, \theta_{n+i,j} = a_{ij}, 1 \leq i, j \leq 2n$.
3. $\theta_{kl} = 0$ if it is not presented in above system, and

$$X = \begin{bmatrix} \underline{x}_1 \\ \vdots \\ \underline{x}_n \\ \overline{x}_1 \\ \vdots \\ \overline{x}_n \end{bmatrix} \text{ and } \Upsilon = \begin{bmatrix} \underline{b}_1 \\ \vdots \\ \underline{b}_n \\ \bar{b}_1 \\ \vdots \\ \bar{b}_n \end{bmatrix}.$$

Furthermore, the matrix Θ has the structure $\begin{bmatrix} \Theta_1 & \Theta_2 \\ \Theta_2 & \Theta_1 \end{bmatrix}$, $\Theta = \Theta_1 + \Theta_2$, and (2) can be written as

$$\begin{cases} \Theta_1 \underline{X} + \Theta_2 \overline{X} = \underline{\Upsilon}, \\ \Theta_2 \underline{X} + \Theta_1 \overline{X} = \overline{\Upsilon}, \end{cases} \tag{4}$$

where

$$\underline{X} = \begin{bmatrix} \underline{x}_1 \\ \underline{x}_2 \\ \vdots \\ \underline{x}_n \end{bmatrix}, \overline{X} = \begin{bmatrix} \overline{x}_1 \\ \overline{x}_2 \\ \vdots \\ \overline{x}_n \end{bmatrix},$$

$$\underline{\Upsilon} = \begin{bmatrix} \underline{b}_1 \\ \underline{b}_2 \\ \vdots \\ \underline{b}_n \end{bmatrix}, \text{ and } \overline{\Upsilon} = \begin{bmatrix} \bar{b}_1 \\ \bar{b}_2 \\ \vdots \\ \bar{b}_n \end{bmatrix}.$$

In the next section, a new iterative scheme based on TS iteration is presented for regularized linear system with nonsingular coefficient matrix [16].

3 Fuzzy TS splitting iterative method for regularized linear system

In this section, we find the steady state probability vector π of a homogeneous equation $\pi Q = 0$. The solution of the homogeneous system is either a trivial solution or an infinite number of solutions. For a unique, nonzero solution, the above homogeneous system is converted to the regularized FLS and is equivalent embedded crisp system $\Theta x = b$, using the small perturbation $0 < r \leq 1$. Now, the TS splitting method for the transition matrix is adopted in a fuzzy environment. Let the coefficient matrix Θ of the embedded regularized linear system can be split in the form:
$\Theta = (L + D - U^T) + (U + U^T) = T + S$,
where $T = L + D - U^T$ and $S = (U + U^T)$ are TS matrices.

The regularized system, (3) can be expressed as $(T + S)X = b$. Consider $D = \begin{bmatrix} D_1 & 0 \\ 0 & D_1 \end{bmatrix}$, $L = \begin{bmatrix} L_1 & 0 \\ -S_2 & L_1 \end{bmatrix}$, $U = \begin{bmatrix} U_1 & -S_2 \\ 0 & U_1 \end{bmatrix}$,

where $D_1 = diag(s_{ii})$, L_1, and U_1 are diagonal, lower, and upper triangular matrices, respectively. Now,

$$T = L + D - U^T = \begin{bmatrix} D_1 & 0 \\ 0 & D_1 \end{bmatrix} + \begin{bmatrix} L_1 & 0 \\ -S_2 & L_1 \end{bmatrix} - \begin{bmatrix} U_1 & 0 \\ -S_2 & U_1 \end{bmatrix}$$
$$= \begin{bmatrix} L_1 + D_1 - U_1 & 0 \\ 0 & L_1 + D_1 - U_1 \end{bmatrix},$$

and

$$S = U + U^T = \begin{bmatrix} 2U_1 & -S_2 \\ -S_2 & 2U_1 \end{bmatrix}.$$

The TS splitting iterative scheme is as follows [16]:

$$(\alpha I + T)X^{(k+1/2)} = (\alpha I - S)X^{(k)} + b,$$
$$(\alpha I + S)X^{(k+1)} = (\alpha I - T)X^{(k+1/2)} + b.$$

The above iterative scheme could be written as

$$X^{(k+1)} = M(\alpha)X^{(k)} + N(\alpha)b, \qquad \text{for} \quad k = 0, 1, 2, \ldots,$$

where

$$X^{(k+1)} = \begin{bmatrix} \underline{X}^{(k+1)} \\ \overline{X^{(k+1)}} \end{bmatrix}$$

$$M(\alpha) = (\alpha I_n + S)^{-1}(\alpha I_n - T)(\alpha I_n + T)^{-1}(\alpha I_n - S)$$

and

$$N(\alpha) = 2\alpha(\alpha I_n + S)^{-1}(\alpha I_n + T)^{-1}.$$

We have

$$\alpha I_n + S = \begin{bmatrix} \alpha I_n + 2U_1 & -S_2 \\ -S_2 & \alpha I_n + 2U_1 \end{bmatrix},$$

$$\alpha I_n - S = \begin{bmatrix} \alpha I_n - 2U_1 & S_2 \\ S_2 & \alpha I_n - 2U_1 \end{bmatrix},$$

$$\alpha I_n + T = \begin{bmatrix} \alpha I_n + T_1 & 0 \\ 0 & \alpha I_n + T_1 \end{bmatrix},$$

$$\alpha I_n - T = \begin{bmatrix} \alpha I_n - T_1 & 0 \\ 0 & \alpha I_n - T_1 \end{bmatrix}.$$

Thus,

$$M(\alpha) = \frac{\alpha I_n - T_1}{(\alpha I_n + T_1)[(\alpha I_n + 2U_1)^2 - S_2^2]} \begin{bmatrix} (\alpha I_n)^2 - 4U_1^2 + S_2^2 & 2\alpha I_n S_2 \\ 2\alpha I_n S_2 & (\alpha I_n)^2 - 4U_1^2 + S_2^2 \end{bmatrix}$$

$$\text{and } N(\alpha) = \frac{2\alpha}{(\alpha I_n + 2U_1)^2 - S_2^2} \begin{bmatrix} \frac{\alpha I_n + 2U_1}{\alpha I_n + T_1} & \frac{S_2}{\alpha I_n + T_1} \\ \frac{S_2}{\alpha I_n + T_1} & \frac{\alpha I_n + 2U_1}{\alpha I_n + T_1} \end{bmatrix}.$$

Theorem 1. If $\Theta X = b$ is the regularized FLS, then the convex solution $X = \{r\underline{X_j} + (1-r)\overline{X_j} / 0 < r \leq 1\}$ is the solution.

Proof. Let $\underline{X_j}$ and $\overline{X_j}$ be the solutions corresponding to right-hand side vector $\underline{b_j}$ and $\overline{b_j}$ of

$$\sum_{j=1}^{n} a_{ij} X_j = b_j,$$

which implies,

$$\sum_{j=1}^{n} a_{ij}\underline{X_j} = \underline{b_j},$$
$$\sum_{j=1}^{n} a_{ij}\overline{X_j} = \overline{b_j}.$$

We prove that $X_j = r\underline{X_j} + (1-r)\overline{X_j}$ is the solution.

Now,

$$\begin{aligned}\sum_{j=1}^{n} a_{ij}X_j &= \sum_{j=1}^{n} a_{ij}[r\underline{X_j} + (1-r)\overline{X_j}]\\ &= r\sum_{j=1}^{n} a_{ij}\underline{X}_j + (1-r)\sum_{j=1}^{n} a_{ij}\overline{X_j}\\ &= r\underline{b}_j + (1-r)\overline{b}_j\\ &= b_j.\end{aligned}$$

Therefore, $X_j = r\underline{X_j} + (1-r)\overline{X_j}$ is the solution of the given system. □

Theorem 2. If $\Theta X = b$, then the convex solution $X = \{r\underline{X} + (1-r)\overline{X}/0 < r \leq 1\}$ is the solution vector of the system $\Theta\{\underline{X} + \overline{X}\} = \{\underline{b} + \overline{b}\}$.

Proof. The regularized linear system $\Theta X = b$ can be written as

$$\sum_{j=1}^{n} a_{ij}X_j = b_i \qquad \text{for} \quad i = 1, 2, \ldots n.$$

Let

$$X_j = r\underline{X_j} + (1-r)\overline{X_j}$$

and

$$b_j = [\underline{b_i}, \overline{b_i}] \qquad \text{for} \quad i = 1, 2, \ldots, n.$$

Now

$$\sum_{j=1}^{n} a_{ij}X_j = \sum_{j=1}^{n} a_{ij}\left[r\underline{X_j} + (1-r)\overline{X_j}\right] = \left[\underline{b_i}, \overline{b_i}\right],$$

$$\sum_{a_{ij}>0} ra_{ij}\underline{X_j} + \sum_{a_{ij}<0} (1-r)a_{ij}\overline{X_j} = \underline{b_i},$$

and

$$\sum_{a_{ij}>0} (1-r)a_{ij}\underline{X_j} + \sum_{a_{ij}<0} ra_{ij}\overline{X_j} = \overline{b_i}.$$

Consider

$$\begin{aligned}
\Theta\left[\underline{X}+\overline{X}\right] &= \sum_{a_{ij}>0} a_{ij}\left[r\underline{X_j}+(1-r)\overline{X_j}\right] + \sum_{a_{ij}<0} a_{ij}\left[(1-r)\overline{X_j}+r\underline{X_j}\right] \\
&= \left[\sum_{a_{ij}>0} ra_{ij}\underline{X_j} + \sum_{a_{ij}<0} (1-r)a_{ij}\overline{X_j}\right] \\
&\quad + \left[\sum_{a_{ij}>0} (1-r)a_{ij}\overline{X_j} + \sum_{a_{ij}<0} ra_{ij}\underline{X_j}\right] \\
&= \left[\underline{b_i}+\overline{b_i}\right], \\
\Theta\left[\underline{X}+\overline{X}\right] &= \left[\underline{b}+\overline{b}\right].
\end{aligned}$$

This proves that X is the solution. □

Theorem 3. If $\Theta X = b$, then the convex solution $X = \{r\underline{X} - (1-r)\overline{X}\}$, $0 < r \leq 1$ is the solution vector of the system $\Theta\{\underline{X} - \overline{X}\} = \{\underline{b} - \overline{b}\}$

Proof. The system $\Theta X = b$ can be written as

$$\sum_{j=1}^{n} a_{ij}X_j = b_i \qquad \text{for} \quad i = 1, 2, \ldots, n.$$

Now, we may write the real fuzzy unknown and the right-hand real fuzzy number vectors as

$$X_j = r\underline{X_j} - (1-r)\overline{X_j},$$

and

$$b_j = \left[\underline{b_i}, \overline{b_i}\right] \qquad \text{for} \quad i = 1, 2, \ldots, n,$$

which implies

$$\sum_{j=1}^{n} [a_{ij}r\underline{X_j} + (1-r)\overline{X_j}] = \left[\underline{b_i}, \overline{b_i}\right],$$

$$\Rightarrow \sum_{a_{ij}>0} r a_{ij}\underline{X_j} + \sum_{a_{ij}<0} (1-r) a_{ij}\overline{X_j} = \underline{b_i}$$

and

$$\sum_{a_{ij}>0} (1-r) a_{ij}\underline{X_j} + \sum_{a_{ij}<0} r a_{ij}\overline{X_j} = \overline{b_i}.$$

Consider

$$\begin{aligned}
\Theta\left[\underline{X}-\overline{X}\right] &= \sum_{a_{ij}>0} a_{ij}\left[r\underline{X_j} + (1-r)\overline{X_j}\right] - \sum_{a_{ij}<0} a_{ij}\left[(1-r)\overline{X_j} + r\underline{X_j}\right] \\
&= \left[\sum_{a_{ij}>0} r a_{ij}\underline{X_j} + \sum_{a_{ij}<0} (1-r) a_{ij}\overline{X_j}\right] \\
&\quad - \left[\sum_{a_{ij}>0} (1-r) a_{ij}\overline{X_j} + \sum_{a_{ij}<0} r a_{ij}\underline{X_j}\right] \\
&= \left[\underline{b_i} - \overline{b_i}\right], \\
\Theta\left[\underline{X}-\overline{X}\right] &= \left[\underline{b}-\overline{b}\right].
\end{aligned}$$

□

Theorem 4. If $\Theta X = b$, then the mid point solution $X = \dfrac{\underline{\underline{X_j}} + \overline{X_j}}{2}$, is the solution vector of the system $\Theta X = b$.

Proof. The system $\Theta X = b$ can be written as

$$\sum_{j=1}^{n} a_{ij}X_j = b_k \qquad \text{for} \quad k = 1,2,\ldots,n.$$

The real fuzzy unknown and the right-hand real fuzzy number vectors can be written as

$$X_j = \frac{\underline{\underline{X_j}} + \overline{X_j}}{2} \quad \text{and} \quad b_j = \left[\underline{b_j}, \overline{b_j}\right] \qquad \text{for} \quad j = 1,2,\ldots,n,$$

$$\sum_{j=1}^{n} a_{ij}\frac{\underline{\underline{X_j}} + \overline{X_j}}{2} = \left[\underline{b_j}, \overline{b_j}\right],$$

which implies $\dfrac{1}{2}\left[\sum_{a_{ij}>0} a_{ij}\underline{X_j} + \sum_{a_{ij}<0} a_{ij}\overline{X_j}\right] = \underline{b_j}$,

and

$$\frac{1}{2}\left[\sum_{a_{ij}>0} a_{ij}\overline{X_j} + \sum_{a_{ij}<0} a_{ij}\underline{X_j}\right] = \overline{b_j}.$$

Consider

$$\Theta\frac{\left[\underline{X}+\overline{X}\right]}{2} = \sum_{j=1}^{n} a_{ij}\frac{\left[\underline{x_j}+\overline{x_j}\right]}{2}$$

$$\sum_{j=1}^{n} a_{ij}\frac{\left[\underline{X_j}+\overline{X_j}\right]}{2} = \sum_{a_{ij}>0} a_{ij}\frac{\left[\underline{x_j}+\overline{x_j}\right]}{2} + \sum_{a_{ij}<0} a_{ij}\frac{\left[\overline{x_j}+\underline{x_j}\right]}{2}$$

$$= \frac{\left[\sum_{a_{ij}>0} a_{ij}\underline{x_j} + \sum_{a_{ij}<0} a_{ij}\overline{x_j}\right]}{2} + \frac{\left[\sum_{a_{ij}>0} a_{ij}\overline{x_j} + \sum_{a_{ij}<0} a_{ij}\underline{x_j}\right]}{2}$$

$$= \frac{\left[\underline{b_j}+\overline{b_j}\right]}{2},$$

$$\Theta\frac{\left[\underline{X}+\overline{X}\right]}{2} = \frac{\left[\underline{b}+\overline{b}\right]}{2}.$$

Hence, $X = \dfrac{\underline{x_j}+\overline{x_j}}{2}$ is solution of the regularized linear system. □

4 Perturbation analysis of FLS

As discussed in the previous section, the homogeneous system $\pi Q = 0$, where the coefficient matrix Q is circulant stochastic rate matrix, is converted into the regularized FLS $Ax = b$ using small perturbation $0 < r \leq 1$. In this section, perturbation to the FLS is added, in which both the coefficient matrix and right-hand side matrices are perturbed and the sensitivity analysis of $Ax = b$ is discussed by using the FFTS method. The well-posed and ill-posed solution of the system $\Theta X = \Upsilon$ depends on the membership value $0 < r \leq 1$. If the coefficient matrix Θ or the right-hand side fuzzy vector Υ or both are slightly disturbed with the membership value r, then the solution will be changed as well. The relative error and absolute by the FFTS method are evaluated between the exact solution and numerical solution with the perturbed FLS. The following theorems are proved in preparation for investigating the sensitivity analysis of the regularized fuzzy system.

Theorem 5. If $A \in R^{n\times n}$ is a circulant stochastic matrix of the regularized linear system $Ax = b$ and Θ is the embedded matrix of the embedded system $\Theta X = \Upsilon$, then Θ is positive definite.

Proof. For proving the matrix Θ is positive definite, it is sufficient to prove that $\frac{\Theta+\Theta^T}{2}$ is positive definite.

We have

$$\Theta = \begin{pmatrix} c_1+r & 0 & \dots & 0 & 0 & c_2 & \dots & c_n \\ 0 & c_1+r & \dots & 0 & c_n & 0 & \dots & c_{n-1} \\ \vdots & \vdots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & \dots & c_1+r & c_2 & c_3 & 0\dots & 0 \\ 0 & c_2 & 0\dots & c_n & c_1+r & 0 & \dots & 0 \\ c_n & 0 & 0\dots & c_{n-1} & 0 & c_1+r & \dots & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ c_2 & c_3 & \dots & 0 & 0 & 0 & \dots & c_1+r \end{pmatrix},$$

$$\frac{\Theta+\Theta^T}{2} = \begin{pmatrix} c_1+r & 0 & \dots & 0 & 0 & \frac{c_2+c_n}{2} & \dots & \frac{c_2+c_n}{2} \\ 0 & c_1+r & \dots & 0 & \frac{c_2+c_n}{2} & 0 & \dots & \frac{c_3+c_{n-1}}{2} \\ \vdots & \vdots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & \dots & c_1+r & \frac{c_2+c_n}{2} & \frac{c_3+c_{n-1}}{2} & 0\dots & 0 \\ 0 & \frac{c_2+c_n}{2} & 0\dots & \frac{c_2+c_n}{2} & c_1+r & 0 & \dots & 0 \\ \frac{c_2+c_n}{2} & 0 & 0\dots & \frac{c_3+c_{n-1}}{2} & 0 & c_1+r & \dots & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ \frac{c_2+c_n}{2} & \frac{c_3+c_{n-1}}{2} & \dots & 0 & 0 & 0 & \dots & c_1+r \end{pmatrix}$$

$$= (c_1+r)I_{n^2} - R,$$

where

$$R = \begin{pmatrix} 0 & 0 & \dots & 0 & 0 & \frac{c_2+c_n}{2} & \dots & \frac{c_2+c_n}{2} \\ 0 & 0 & \dots & 0 & \frac{c_2+c_n}{2} & 0 & \dots & \frac{c_3+c_{n-1}}{2} \\ \vdots & \vdots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & \dots & 0 & \frac{c_2+c_n}{2} & \frac{c_3+c_{n-1}}{2} & 0\dots & 0 \\ 0 & \frac{c_2+c_n}{2} & 0\dots & \frac{c_2+c_n}{2} & 0 & 0 & \dots & 0 \\ \frac{c_2+c_n}{2} & 0 & 0\dots & \frac{c_3+c_{n-1}}{2} & 0 & 0 & \dots & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ \frac{c_2+c_n}{2} & \frac{c_3+c_{n-1}}{2} & \dots & 0 & 0 & 0 & \dots & 0 \end{pmatrix} \geq 0.$$

From the theorem in [16], we have

$$\begin{aligned} &\Rightarrow \rho(R) = c_2 + c_3 + \cdots + c_n = c_1, \\ &\Rightarrow c_1 + \epsilon > \rho(R). \end{aligned}$$

Therefore, $\frac{\Theta+\Theta^T}{2}$ is positive definite.

□

Theorem 6. [16] For any nonsymmetric stochastic circulant rate matrix $Q \in R^{n\times n}$, there exists a constant $\epsilon > 0$ such that $A = Q^T + \epsilon I_n$ is positive definite if and only if all its eigenvalues are nonnegative real numbers.

Theorem 7. [16] Let $A \in R^{n\times n}$ be the regularized matrix, and splitting into TS matrices. Then the spectral radius of the iterative matrix $M(\alpha)$ is less than one.

Theorem 8. Let $\Theta \in R^{2n\times 2n}$ be a fuzzy matrix of regularized linear system and let $M(\alpha)$ be the iteration matrix of the FFTS iteration method. Then the spectral radius of $M(\alpha)$ is less than 1.

Proof. The proof of the theorem is on the similar lines of Theorems 6 and 7. □

5 Numerical results

In this section, we examine the effectiveness of the FFTS iteration method with the numerical solution of stochastic matrices under a fuzzy environment and compare the error analysis of fuzzy iterative solution with the TS and TSS iteration methods. For the numerical illustration, we consider the homogeneous system $\pi Q = 0$, where Q is the 3×3 doubly stochastic rate matrix given below:

$$Q = \begin{bmatrix} 0.6 & -0.35 & -0.25 \\ -0.25 & 0.6 & -0.35 \\ -0.35 & -0.25 & 0.6 \end{bmatrix}.$$

We convert the above homogeneous system $\pi Q = 0$, into a regularized linear system $Ax = b$ with the small perturbation parameter $0 < r \leq 1$. The regularized linear system is converted to a 6×6 embedded linear system $\Theta X = \Upsilon$, where

$$\Theta = \begin{bmatrix} 0.6+r & 0 & 0 & 0 & -0.35 & -0.25 \\ 0 & 0.6+r & 0 & -0.25 & 0 & -0.35 \\ 0 & 0 & 0.6+r & -0.35 & -0.25 & 0 \\ 0 & -0.35 & -0.25 & 0.6+r & 0 & 0 \\ -0.25 & 0 & -0.35 & 0 & 0.6+r & 0 \\ -0.35 & -0.25 & 0 & 0 & 0 & 0.6+r \end{bmatrix}.$$

Let the initial distribution vector bw $x^{(0)} = [0\ \ 0\ \ 0\ \ 0\ \ 0\ \ 1]^T$ and let right-hand side vector be $\Upsilon = [0\ \ 0\ \ 1-r\ \ 0\ \ 0\ \ r-1]$, where $0 < r \leq 1$ is the membership function. Only one case $\Theta = (L + D - U^T) + (U + U^T) = T_1 + S_1$ of FFTS splitting method is considered, and other methods would follow the same. A fuzzy iterative solution to the system (1) is computed, and a classical solution of TS and TSS iterative methods is evaluated. It is illustrated the result for the case of contraction factor $\alpha = 0.6 + r$, which is numerically equivalent to the diagonal elements of the matrix Θ, for different values of r. The steady state distribution vector x of the preconditioned linear system is obtained, and the results are presented in Figures 1–7. The numerical solutions of the FFTS method are presented in Figure 1. From this figure, it is concluded that the numerical solutions FFTS method coincides with the theoretical results. The average and linear convex solutions of lower bound and upper bound solutions are depicted in Figures 2–4. From these two figures, it is concluded that the center and convex solution curves lie within the monotonically increasing and monotonically decreasing curve. The error analysis of the FFTS method and classical iterative solution is presented in Figure 5. From this figure, it concluded that the FFTS iterative solution converges rapidly when compared with classical TS, TSS methods. The convergence and sensitivity analysis of the FFTS method are presented in Figures 6–7. From Figure 6, it is evidently concluded that the condition number of the FFTS method is very low compared with classical TS and TSS methods. From this figure, one can conclude that the iterative solution obtained using the FFTS method is well conditioned and the regularized matrix is nonsingular for larger membership values. It is depicted in Figure 7 that the spectral radius of the FFTS method and spectral radius are clearly less than one. From this figure, it is concluded that the FFTS method converges to a unique nonzero solution, and it is evident by the theoretical solution.

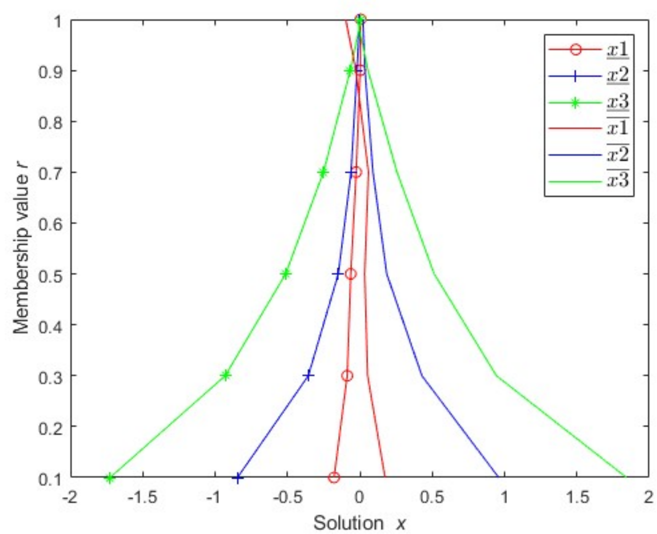

Figure 1: Solution x for the contraction factor $\alpha = 0.6$ over the membership value r.

6 Conclusions

In this paper, a new iterative method was suggested based on the TS iteration to the solution of a class of fuzzy linear systems of equations with a coeffi-

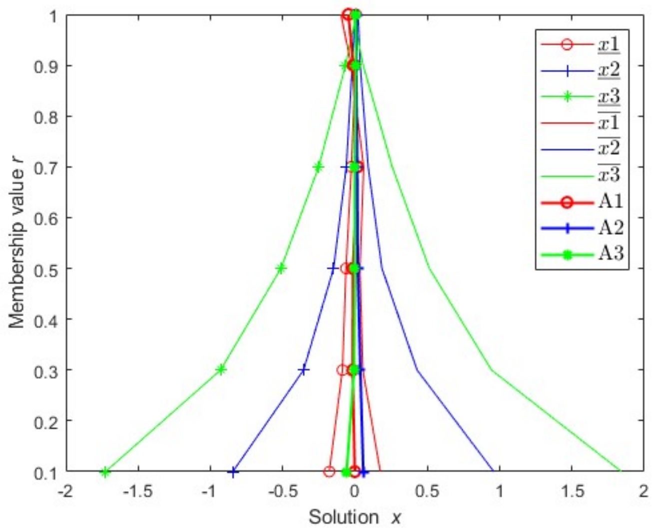

Figure 2: Solution x and average solution for the contraction factor $\alpha = 0.6$ over the membership value r.

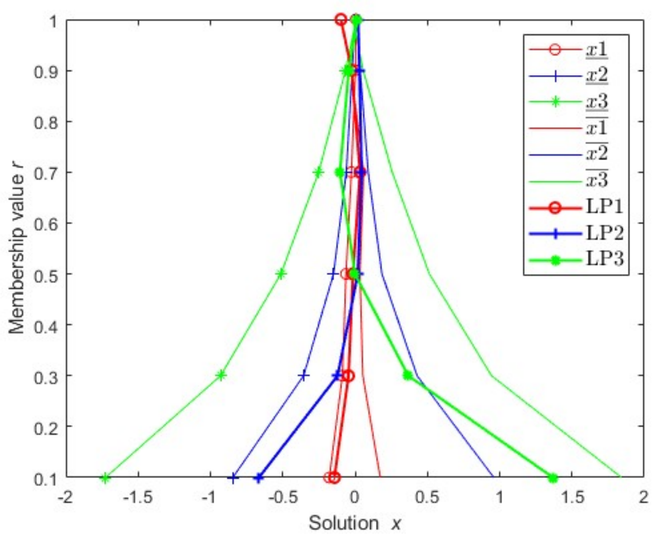

Figure 3: Solution x and LP solution for the contraction factor $\alpha = 0.6$ over the membership value r.

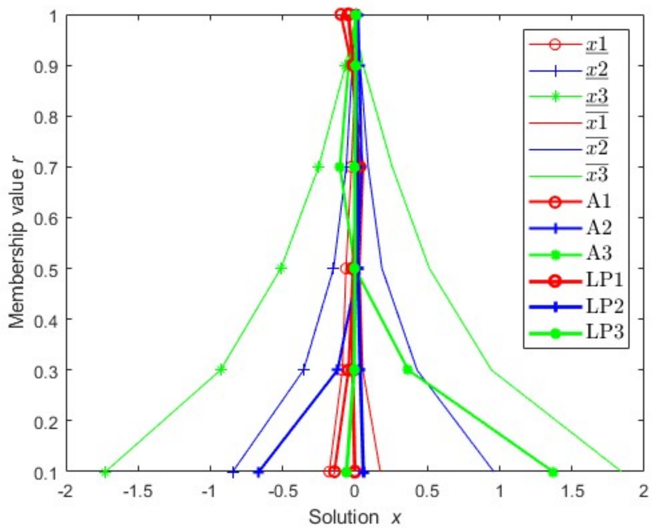

Figure 4: Solution x, average solution, LP solution for the contraction factor $\alpha = 0.6$ over the membership value r.

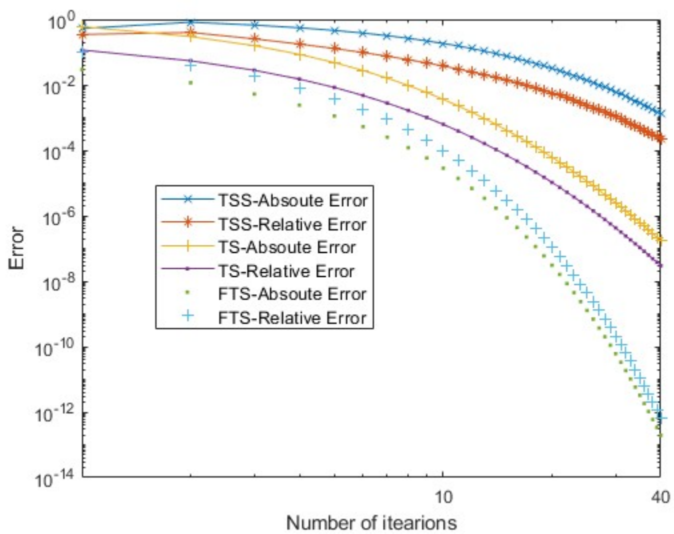

Figure 5: Absolute error and Relative error of the TSS, TS, and FTS iterative solution for the contraction factor $\alpha = 0.6$

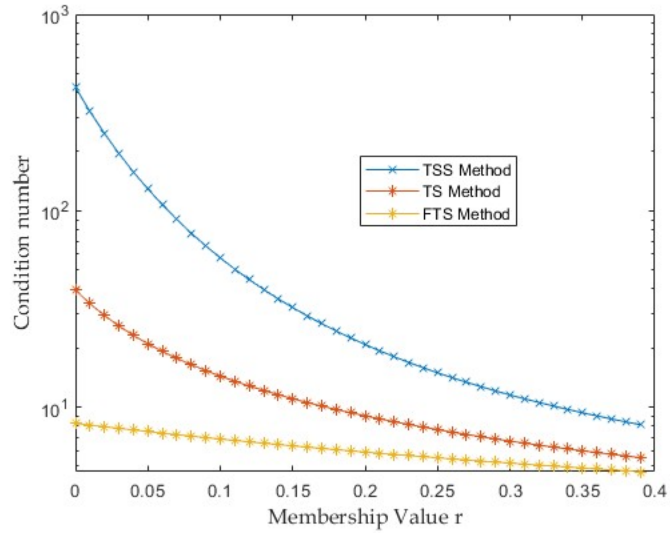

Figure 6: Condition number of TSS, TS, and FTS methods for the contraction factor $\alpha = 0.6$ over the membership values r.

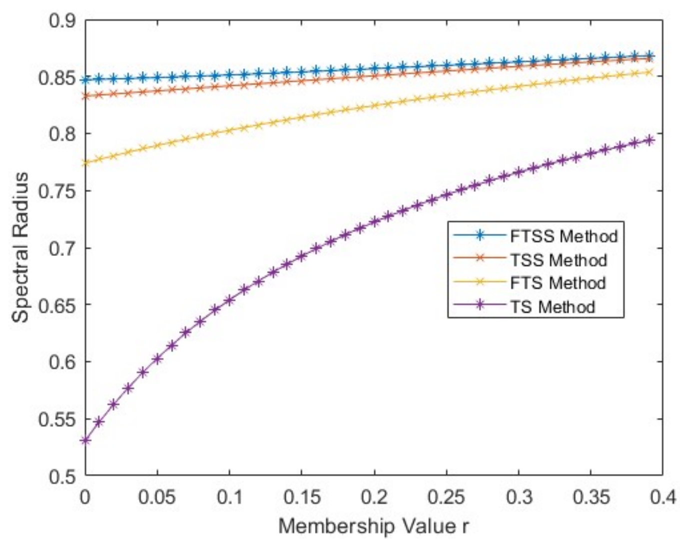

Figure 7: Spectral radius of TSS, FTSS, TS, FTS methods for the contraction factor $\alpha = 0.6$ over the membership values r.

cient matrix as a fuzzy stochastic matrix and fuzzy right-hand side matrix. The iterative scheme was established, and the convergence theorems were presented. FTS iterative solutions with classical TS and TSS methods were compared. Numerical examples showed that the method is effective and efficient when compared with the classical iterative methods. The convergence and sensitivity analysis were discussed. The numerical value of spectral radius concluded that the solution of FTS method converges to unique nonzero solution. The numerical value of condition number gave the sensitivity analysis of regularized linear system and concluded that the iterative solution of regularized FLS is well conditioned.

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