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Research Article

Two-step inertial Tseng's extragradient methods for a class of bilevel split variational inequalities

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Abstract

This work presents a two-step inertial Tseng's extragradient method with a self-adaptive step size for solving a bilevel split variational inequality problem (BSVIP) in Hilbert spaces. This algorithm only requires two projections per iteration, enhancing its practicality. We establish a strong convergence theorem for the method, showing that it effectively tackles the BSVIP without necessitating prior knowledge of the Lipschitz or strongly monotone constants associated with the mappings. Additionally, the implementation of this method removes the need to compute or estimate the

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norm of the given operator, a task that can often be challenging in practical situations. We also explore specific cases to demonstrate the versatility of the method. Finally, we present an application of the split minimum norm problem in production and consumption systems and provide several numerical experiments to validate the practical implementability of the proposed algorithms.

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1 Introduction

Let C and Q be nonempty closed convex subsets of the real Hilbert spaces $\mathcal{H}_1$ and $\mathcal{H}_2$, respectively. Let $A : \mathcal{H}_1 \longrightarrow \mathcal{H}_2$ be a bounded linear operator. Define the mappings $F_1 : \mathcal{H}_1 \longrightarrow \mathcal{H}_1$ and $F_2 : \mathcal{H}_2 \longrightarrow \mathcal{H}_2$ on $\mathcal{H}_1$ and $\mathcal{H}_2$, respectively. The split variational inequality problem (SVIP), initially proposed by Censor, Gibali, and Reich [15], can be expressed as follows:

$$\text{Find } x^* \in C : \langle F_1(x^*), x - x^* \rangle \geq 0, \quad \text{for all } x \in C \tag{1}$$

such that

$$y^* = Ax^* \in Q : \langle F_2(y^*), y - y^* \rangle \geq 0, \quad \text{for all } y \in Q. \tag{2}$$

When $F_1 = 0$ and $F_2 = 0$, the SVIP reduces to a special case known as the split feasibility problem (SFP), which is formulated as

$$\text{Find } x^* \in C \text{ such that } Ax^* \in Q. \tag{3}$$

This problem was first introduced by Censor and Elfving [13] as a model for inverse problems in finite-dimensional Hilbert spaces. Recently, its applicability has been extended to fields such as intensity-modulated radiation therapy [12, 16, 14] and other practical scenarios. For additional details on the SFP, refer to [1, 3, 5, 14, 7, 8, 11, 10, 9, 23, 24, 34, 36, 43] and the sources cited within those references.

In this paper, our primary objective is to solve a variational inequality problem (VIP) defined over the solution set of the SVIP. Specifically, we aim to address the following problem:

$$\text{Find } x^* \in \Omega_{\text{SVIP}} \text{ such that } \langle F(x^*), x - x^* \rangle \geq 0, \quad \text{for all } x \in \Omega_{\text{SVIP}}, \tag{4}$$

where $F : \mathcal{H}_1 \longrightarrow \mathcal{H}_1$ is η-strongly monotone and L-Lipschitz continuous on $\mathcal{H}_1$, and Ω_{SVIP} represents the solution set of the SVIP defined by equations (1) and (2). Problem (4) is referred to as the bilevel split variational inequality problem (BSVIP) in [1]. Suppose that $\mathcal{H}_1 = \mathcal{H}$, $F : \mathcal{H} \longrightarrow \mathcal{H}$ is strongly monotone and Lipschitz continuous on $\mathcal{H}$, that $F_1 = G : \mathcal{H} \longrightarrow \mathcal{H}$ is a mapping on $\mathcal{H}$, that $F_2 = 0$, and that $Q = \mathcal{H}_2$. Then the BSVIP (4) simplifies to the following bilevel VIP:

$$\text{Find } x^* \in \text{Sol}(C, G) \text{ such that } \langle F(x^*), y - x^* \rangle \geq 0, \quad \text{for all } y \in \text{Sol}(C, G), \tag{5}$$

where $\text{Sol}(C, G)$ represents the set of all solutions to the VIP given by

$$\text{Find } y^* \in C \text{ such that } \langle G(y^*), z - y^* \rangle \geq 0, \quad \text{for all } z \in C. \tag{6}$$

Bilevel VIPs (5)–(6) encompass various types of bilevel optimization problems [20, 37, 6], minimum norm problems related to the solution set of variational inequalities [42, 44], and other variational inequalities [28, 29, 21, 19, 22]. In recent years, numerous approaches have been developed to solve the BVIP (5)-(6) in both finite and infinite-dimensional spaces. For a comprehensive overview, see [2, 4, 38] and the references therein.

One of the most famous methods for solving VIPs is the extragradient method, first proposed by Korpelevich [30] for saddle problems. However, the extragradient method may be costly, since it requires two projections at each step. To improve this, Tseng [39] introduced an alternative extragradient method that reduces the number of projections required. Instead of performing two projections, Tseng's method requires only one projection onto C per iteration. Tseng's extragradient method is described as follows:

$$\begin{cases} x^0 \in \mathcal{H}, \\ y^n = P_C(x^n - \lambda F_1(x^n)), \\ x^{n+1} = y^n - \lambda(F_1(y^n) - F_1(x^n)), \end{cases} \tag{7}$$

where F_1 is L_1-Lipschitz continuous and $\lambda \in \left(0, \dfrac{1}{L_1}\right)$.

Inspired the Tseng's extragradient method for solving VIPs, Huy et al. [25] introduced the modified Tseng's extragradient method for solving the BSVIP (4), where $F : \mathcal{H}_1 \longrightarrow \mathcal{H}_1$ is η-strongly monotone and L-Lipschitz continuous on $\mathcal{H}_1$, $F_1 : \mathcal{H}_1 \longrightarrow \mathcal{H}_1$ and $F_2 : \mathcal{H}_2 \longrightarrow \mathcal{H}_2$ are pseudomonotone and Lipschitz continuous mappings. Specifically, they proposed the following algorithm

$$\begin{cases} x^0 \in \mathcal{H}_1, \\ u^n = A(x^n), \\ v^n = P_Q(u^n - \mu_n F_2(u^n)), \\ w^n = v^n - \mu_n (F_2(v^n) - F_2(u^n)), \\ \mu_{n+1} = \begin{cases} \min\left\{ \dfrac{\mu \|u^n - v^n\|}{\|F_2(u^n) - F_2(v^n)\|}, \mu_n \right\} & \text{if } F_2(u^n) \neq F_2(v^n), \\ \mu_n & \text{if } F_2(u^n) = F_2(v^n), \end{cases} \\ y^n = x^n + \delta_n A^*(w^n - u^n), \\ \delta_n = \begin{cases} \dfrac{\|w^n - u^n\|^2}{2\|A^*(w^n - u^n)\|^2} & \text{if } A^*(w^n - u^n) \neq 0, \\ 0 & \text{if } A^*(w^n - u^n) = 0. \end{cases} \\ z^n = P_C(y^n - \lambda_n F_1(y^n)), \\ t^n = z^n - \lambda_n (F_1(z^n) - F_1(y^n)), \\ \lambda_{n+1} = \begin{cases} \min\left\{ \dfrac{\lambda \|y^n - z^n\|}{\|F_1(y^n) - F_1(z^n)\|}, \lambda_n \right\} & \text{if } F_1(y^n) \neq F_1(z^n), \\ \lambda_n & \text{if } F_1(y^n) = F_1(z^n), \end{cases} \\ x^{n+1} = t^n - \varepsilon_n F(t^n), \end{cases} \tag{8}$$

where $\mu_0 > 0$, $\lambda_0 > 0$, $\mu \in (0,1)$, $\lambda \in (0,1)$, $\{\varepsilon_n\} \subset (0,1)$, $\lim\limits_{n\to\infty} \varepsilon_n = 0$, and $\sum\limits_{n=0}^{\infty} \varepsilon_n = \infty$. The author demonstrated that the sequence $\{x^n\}$ produced by the algorithm (8) converges strongly to the unique solution of the BSVIP (4), provided that the solution set of the SVIP (1)–(2) is nonempty.

To enhance the convergence rate of algorithms, inertial acceleration is commonly utilized. Originally introduced by Polyak [33] in 1964 for solving smooth convex minimization problems, the inertial algorithm distinguishes itself by leveraging the previous two iterates to generate the next one. Numerous researchers have explored and implemented the inertial scheme to accelerate algorithmic convergence (see [32, 40] and references therein). These studies primarily employ a single inertial parameter to achieve acceleration. However, recent works by some authors [26, 27] have investigated multi-step inertial algorithms, demonstrating that incorporating multi-step inertial terms, such as the two-step inertial term, further enhances algorithmic speed.

In this paper, drawing inspiration from the aforementioned studies, we introduce a novel iterative scheme that combines the two-step inertial technique with a modified Tseng's extragradient method, as employed by Huy et al. [25], to solve the BSVIP in (4). We demonstrate that the sequence produced by our method converges strongly to the unique solution of (4), with the stepsize determined at each iteration. Consequently, our approach does not necessitate prior knowledge of the Lipschitz or strong monotonicity constants for the mappings involved. Additionally, the implementation of this method eliminates the need to compute or estimate the norm of the bounded linear operator.

The structure of the paper is organized as follows. Section 2 presents essential definitions and lemmas that will be utilized in section 3, where we outline the algorithm and demonstrate its strong convergence. We conclude this section by exploring various applications of our results to the bilevel VIPs, the simple bilevel optimization problem and VIPs with the SF constraints. Lastly, we apply the split minimum norm problem (SMNP) to production and consumption systems and conduct numerical experiments to evaluate the effectiveness of the proposed algorithms.

2 Preliminaries

In the following discussion, we denote the strong convergence of a sequence $\{x^n\}$ to x in a real Hilbert space $\mathcal{H}$ as $x^n \to x$ and the weak convergence as $x^n \rightharpoonup x$. Recall that for a nonempty closed convex subset C of $\mathcal{H}$, the metric projection P_C is a mapping from $\mathcal{H}$ to C. For each $x \in \mathcal{H}$, $P_C(x)$ is defined as the unique point in C that minimizes the distance to x, satisfying the condition:

$$\|x - P_C(x)\| = \min\{\|x - y\| : y \in C\}.$$

Let us also recall some well-known definitions which will be used in this paper.

Definition 1. ([35]) Consider two Hilbert spaces, denoted as $\mathcal{H}_1$ and $\mathcal{H}_2$. Let $A : \mathcal{H}_1 \longrightarrow \mathcal{H}_2$ be a bounded linear operator. The adjoint of this operator, represented as $A^* : \mathcal{H}_2 \longrightarrow \mathcal{H}_1$, is characterized by the following relationship:

$$\langle A(x), y\rangle = \langle x, A^*(y)\rangle, \text{ for all } x \in \mathcal{H}_1, \text{ for all } y \in \mathcal{H}_2.$$

The adjoint operator of a bounded linear operator A between Hilbert spaces $\mathcal{H}_1$ and $\mathcal{H}_2$ is both well-defined and unique. Moreover, the adjoint operator A^* is also a bounded linear operator, satisfying the property that $\|A^*\| = \|A\|$.

Definition 2. ([17, 29])
A mapping $F : \mathcal{H} \longrightarrow \mathcal{H}$ is said to be
(i) η-strongly monotone on $\mathcal{H}$ if there exists $\eta > 0$ such that

$$\langle F(x) - F(y), x - y\rangle \geq \eta\|x - y\|^2, \text{ for all } x, y \in \mathcal{H};$$

(ii) L-Lipschitz continuous on $\mathcal{H}$ if there exists $L > 0$ such that

$$\|F(x) - F(y)\| \leq L\|x - y\|, \text{ for all } x, y \in \mathcal{H};$$

(iii) monotone on $\mathcal{H}$ if

$$\langle F(x) - F(y), x - y\rangle \geq 0, \text{ for all } x, y \in \mathcal{H};$$

(iv) pseudomonotone on C if

$$\langle F(y), x - y\rangle \geq 0 \Rightarrow \langle F(x), x - y\rangle \geq 0, \text{ for all } x, y \in C.$$

To demonstrate the convergence of the proposed algorithm, we will require the following lemmas.

Lemma 1. ([25]) Let C be a nonempty closed convex subset of a real Hilbert space $\mathcal{H}$. Let $F : \mathcal{H} \longrightarrow \mathcal{H}$ be pseudomonotone on C and L-Lipschitz continuous on $\mathcal{H}$ such that the solution set $\text{Sol}(C, F)$ of the $VIP(C, F)$ is nonempty. Let let $x \in \mathcal{H}$, and let $\mu \in (0,1)$, $\lambda > 0$, and define

$$\begin{aligned} &y = P_C(x - \lambda F(x)), \\ &z = y - \lambda(F(y) - F(x)), \\ &\gamma = \begin{cases} \min\left\{\dfrac{\mu\|x-y\|}{\|F(x)-F(y)\|}, \lambda\right\} & \text{if } F(x) \neq F(y), \\ \lambda & \text{if } F(x) = F(y). \end{cases} \end{aligned}$$

Then for all $x^* \in \mathrm{Sol}(C,F)$

$$\|z - x^*\|^2 \leq \|x - x^*\|^2 - \Big(1 - \mu^2\frac{\lambda^2}{\gamma^2}\Big)\|x-y\|^2.$$

Lemma 2. ([25]) Let C be a nonempty closed convex subset of a real Hilbert space $\mathcal{H}$. Let $F : \mathcal{H} \longrightarrow \mathcal{H}$ be a mapping such that $\limsup\limits_{n\to\infty}\langle F(x^n), z - y^n\rangle \leq \langle F(\overline{x}), z - \overline{y}\rangle$ for every sequences $\{x^n\}$, $\{y^n\}$ in $\mathcal{H}$ converging weakly to $\overline{x}$ and $\overline{y}$, respectively. Assume that $\lambda_n \geq a > 0$ for all n, $\{x^n\}$ is a sequence in $\mathcal{H}$ satisfying $x^n \rightharpoonup \overline{x}$ and $\lim\limits_{n\to\infty}\|x^n - y^n\| = 0$, where $y^n = P_C(x^n - \lambda_n F(x^n))$ for all n. Then $\overline{x} \in \mathrm{Sol}(C,F)$.

Lemma 3. ([31, Remark 4.4]) Let $\{a_n\}$ be a sequence of nonnegative real numbers. Suppose that for any integer m, there exists an integer p such that $p \geq m$ and $a_p \leq a_{p+1}$. Let n_0 be an integer such that $a_{n_0} \leq a_{n_0+1}$ and define, for all integer $n \geq n_0$, by

$$\tau(n) = \max\{k \in \mathbb{N} : n_0 \leq k \leq n, a_k \leq a_{k+1}\}.$$

Then $\{\tau(n)\}_{n \geq n_0}$ is a nondecreasing sequence satisfying $\lim\limits_{n\to\infty}\tau(n) = \infty$ and the following inequalities hold true:

$$a_{\tau(n)} \leq a_{\tau(n)+1}, a_n \leq a_{\tau(n)+1}, \quad \text{for all } n \geq n_0.$$

Lemma 4. ([41]) Let $\{a_n\}$ be a sequence of nonnegative real numbers, let $\{\varepsilon_n\}$ be a sequence in $(0,1)$ such that $\sum\limits_{n=0}^{\infty}\varepsilon_n = \infty$, and let $\{b_n\}$ be a sequence of real numbers with $\limsup\limits_{n\to\infty} b_n \leq 0$. Suppose that

$$a_{n+1} \leq (1-\varepsilon_n)a_n + \varepsilon_n b_n, \quad \text{for all } n \geq 0.$$

Then $\lim_{n\to\infty} a_n = 0$.

3 The algorithm and convergence analysis

In this section, we propose a strong convergence algorithm for solving BSVIP by using two-step inertial Tseng's extragradient methods with self-adaptive step size. We impose the following assumptions concerning the mappings F, F_1, and F_2 related to the BSVIP.

Assumption 1. ([1, 25]) Let the following hold:

A_1) $F : \mathcal{H}_1 \longrightarrow \mathcal{H}_1$ is η-strongly monotone and L-Lipschitz continuous on $\mathcal{H}_1$.

A_2) $F_1 : \mathcal{H}_1 \longrightarrow \mathcal{H}_1$ is pseudomonotone on C and L_1-Lipschitz continuous on $\mathcal{H}_1$.

A_3) $\limsup_{n\to\infty}\langle F_1(x^n), y - y^n\rangle \leq \langle F_1(\overline{x}), y - \overline{y}\rangle$ holds for any sequences $\{x^n\}$ and $\{y^n\}$ in $\mathcal{H}_1$ that converge weakly to $\overline{x}$ and $\overline{y}$, respectively.

A_4) $F_2 : \mathcal{H}_2 \longrightarrow \mathcal{H}_2$ is pseudomonotone on Q and L_2-Lipschitz continuous on $\mathcal{H}_2$.

A_5) $\limsup_{n\to\infty}\langle F_2(u^n), v - v^n\rangle \leq \langle F_2(\overline{u}), v - \overline{v}\rangle$ holds for any sequences $\{u^n\}$ and $\{v^n\}$ in $\mathcal{H}_2$ that converge weakly to $\overline{u}$ and $\overline{u}$, respectively.

One can see that in finite-dimensional spaces, the conditions A_3 and A_5 automatically result from the Lipschitz continuity of F_1 and F_2.

Remark 1. In Algorithm 1, we introduce a two-step inertial version of Tseng's extragradient method. The inertial update is applied in Step 2, where we replace x^n with $y^n = x^n + \alpha_n(x^n - x^{n-1}) + \beta_n(x^{n-2} - x^{n-1})$ for the next step. Starting from Step 3, our algorithm closely follows [25, Algorithm 3.1], as described in (8). The key differences between our approach and [25, Algorithm 3.1] lie in the order of applying the modified Tseng's extragradient method in the two spaces, as well as the inclusion of the two-step inertial update. In [25, Algorithm 3.1], the authors first transform to space $\mathcal{H}_2$, apply the modified Tseng's extragradient method to the mapping F_2,

Algorithm 1

Step 0. Choose $\mu_0 > 0$, $\lambda_0 > 0$, $\mu \in (0,1)$, $\lambda \in (0,1)$, $\{\rho_n\} \subset [a,b] \subset (0,1)$, $\{\gamma_n\} \subset [0,\infty)$, $\{\xi_n\} \subset [0,\infty)$, $\{\eta_n\} \subset (0,\infty)$, $\{\varepsilon_n\} \subset (0,1)$ such that $\lim_{n\to\infty} \frac{\eta_n}{\varepsilon_n} = 0$, $\lim_{n\to\infty} \varepsilon_n = 0$, $\sum_{n=0}^{\infty} \varepsilon_n = \infty$.

Step 1. Let $x^{-2}, x^{-1}, x^0 \in \mathcal{H}_1$. Set $n := 0$.

Step 2. Compute $y^n = x^n + \alpha_n(x^n - x^{n-1}) + \beta_n(x^{n-2} - x^{n-1})$, where

$$\alpha_n = \begin{cases} \min\left\{\dfrac{\eta_n}{\|x^n - x^{n-1}\|}, \gamma_n\right\} & \text{if } x^n \neq x^{n-1}, \\ \gamma_n & \text{if } x^n = x^{n-1}, \end{cases}$$

and

$$\beta_n = \begin{cases} \min\left\{\dfrac{\eta_n}{\|x^{n-2} - x^{n-1}\|}, \xi_n\right\} & \text{if } x^{n-2} \neq x^{n-1}, \\ \xi_n & \text{if } x^{n-2} = x^{n-1}. \end{cases}$$

Step 3. Compute

$$z^n = P_C(y^n - \lambda_n F_1(y^n)),$$
$$t^n = z^n - \lambda_n(F_1(z^n) - F_1(y^n)),$$

where

$$\lambda_{n+1} = \begin{cases} \min\left\{\dfrac{\lambda\|y^n - z^n\|}{\|F_1(y^n) - F_1(z^n)\|}, \lambda_n\right\} & \text{if } F_1(y^n) \neq F_1(z^n), \\ \lambda_n & \text{if } F_1(y^n) = F_1(z^n). \end{cases}$$

Step 4. Compute $u^n = A(t^n)$ and

$$v^n = P_Q(u^n - \mu_n F_2(u^n)),$$
$$w^n = v^n - \mu_n(F_2(v^n) - F_2(u^n)),$$

where

$$\mu_{n+1} = \begin{cases} \min\left\{\dfrac{\mu\|u^n - v^n\|}{\|F_2(u^n) - F_2(v^n)\|}, \mu_n\right\} & \text{if } F_2(u^n) \neq F_2(v^n), \\ \mu_n & \text{if } F_2(u^n) = F_2(v^n). \end{cases}$$

Step 5. Compute

$$s^n = t^n + \delta_n A^*(w^n - u^n),$$

where the stepsize δ_n is chosen in such a way that

$$\delta_n = \begin{cases} \dfrac{\rho_n\|w^n - u^n\|^2}{\|A^*(w^n - u^n)\|^2} & \text{if } A^*(w^n - u^n) \neq 0, \\ 0 & \text{if } A^*(w^n - u^n) = 0. \end{cases}$$

Step 6. Compute

$$x^{n+1} = s^n - \varepsilon_n F(s^n).$$

Step 7. Set $n := n+1$, and go to **Step 2**.

return to space $\mathcal{H}_1$, and then apply the method again to the mapping F_1. In contrast, our algorithm first applies this modified extragradient method to the mapping F_1 in space $\mathcal{H}_1$ (Step 3), then transforms to space $\mathcal{H}_2$ and applies it to F_2 (Step 4), before returning to space $\mathcal{H}_1$ in Step 5. Notably, before applying the method to F_1 in space $\mathcal{H}_1$, we use the two-step inertial update $y^n = x^n + \alpha_n(x^n - x^{n-1}) + \beta_n(x^{n-2} - x^{n-1})$ instead of x^n.

The following lemma is part of the proof of [25, Algorithm 3.1], but we have made a slight modification to better suit the new proof.

Lemma 5. Assume that the conditions $(A_1) - (A_5)$ are satisfied and that $\Omega_{\text{SVIP}} \neq \emptyset$. Let μ, λ as in Algorithm 1, let $\varepsilon \in \left(0, \frac{2\eta}{L^2}\right)$, and let the sequences $\{\mu^n\}$ and $\{\lambda^n\}$be generated by Algorithm 1. We show that there exists $n_0 \in \mathbb{N}$ such that

$$1-\mu^2\frac{\mu_n^2}{\mu_{n+1}^2} > \frac{1-\mu^2}{2} > 0, \;\; 1-\lambda^2\frac{\lambda_n^2}{\lambda_{n+1}^2} > \frac{1-\lambda^2}{2} > 0, \;\; \varepsilon_n < \varepsilon, \;\; \text{for all } n \geq n_0.$$

Proof. With F_2 being L_2-Lipschitz continuous on $\mathcal{H}_2$, it follows that $\|F_2(u^n) - F_2(v^n)\| \leq L_2\|u^n - v^n\|$. Consequently, employing induction, we have $\mu_n \geq \min\left(\frac{\mu}{L_2}, \mu_0\right) > 0$ for all $n \geq 0$. The definition of μ_{n+1} implies $\mu_{n+1} \leq \mu_n$ for all $n \geq 0$. Combining this with $\mu_n \geq \min\left(\frac{\mu}{L_2}, \mu_0\right) > 0$ for all $n \geq 0$, we infer the existence of the limit of the sequence $\{\mu_n\}$. Let us denote $\lim_{n\to\infty} \mu_n = \mu^*$. It is evident that $\mu^* \geq \min\left(\frac{\mu}{L_2}, \mu_0\right) > 0$.

Using the same reasoning as before, we find that

$$\lambda_0 \geq \lambda_n \geq \min\left(\frac{\lambda}{L_1}, \lambda_0\right) > 0, \;\; \text{for all } n \geq 0$$

and

$$\lim_{n\to\infty} \lambda_n = \lambda^* \geq \min\left(\frac{\lambda}{L_1}, \lambda_0\right) > 0.$$

From $\lim_{n\to\infty} \mu_n = \mu^* > 0$ and $\lim_{n\to\infty} \lambda_n = \lambda^* > 0$, we get $\lim_{n\to\infty}\left(1 - \mu^2\frac{\mu_n^2}{\mu_{n+1}^2}\right) = 1 - \mu^2 > 0$, $\lim_{n\to\infty}\left(1 - \lambda^2\frac{\lambda_n^2}{\lambda_{n+1}^2}\right) = 1 - \lambda^2 > 0$. Since $\lim_{n\to\infty} \varepsilon_n = 0$, there exists $n_0 \in \mathbb{N}$ such that

$$1-\mu^2\frac{\mu_n^2}{\mu_{n+1}^2} > \frac{1-\mu^2}{2} > 0, \;\; 1-\lambda^2\frac{\lambda_n^2}{\lambda_{n+1}^2} > \frac{1-\lambda^2}{2} > 0, \;\; \varepsilon_n < \varepsilon, \;\; \text{for all } n \geq n_0.$$

□

Lemma 6. Let $\{t^n\}$, $\{u^n\}$, $\{w^n\}$ and $\{s^n\}$ be the sequences generated by Algorithm 1. Then, for all $n \geq n_0$, where n_0 is given in Lemma 5, the following inequalities hold:

$$0 \leq \frac{a^2}{(\|A\|+1)^2}\|w^n - u^n\|^2 \leq \|s^n - t^n\|^2 \leq \frac{b}{1-b}\big(\|t^n - x^*\|^2 - \|s^n - x^*\|^2\big),$$

where x^* is the unique solution to the problem (4).

Proof. As Ω_{SVIP} is nonempty, problem (4) has a unique solution denoted by x^*. Specifically, $x^* \in \Omega_{\text{SVIP}}$, implying that it satisfies $x^* \in \text{Sol}(C, F_1)$ and $Ax^* \in \text{Sol}(Q, F_2)$. According to Lemma 1, for all $n \geq 0$, we have

$$\|w^n - Ax^*\|^2 \leq \|u^n - Ax^*\|^2 - \Big(1 - \mu^2\frac{\mu_n^2}{\mu_{n+1}^2}\Big)\|u^n - v^n\|^2, \tag{9}$$

$$\|t^n - x^*\|^2 \leq \|y^n - x^*\|^2 - \Big(1 - \lambda^2\frac{\lambda_n^2}{\lambda_{n+1}^2}\Big)\|y^n - z^n\|^2. \tag{10}$$

From Lemma 5, (9) and (10), we get

$$\|w^n - Ax^*\| \leq \|u^n - Ax^*\|, \text{ for all } n \geq n_0, \tag{11}$$

$$\|t^n - x^*\| \leq \|y^n - x^*\|, \text{ for all } n \geq n_0. \tag{12}$$

From (11), since $u^n = A(t^n)$, we obtain, for all $n \geq n_0$

$$\begin{aligned} 2\langle t^n - x^*, A^*(w^n - u^n)\rangle &= 2\langle A(t^n - x^*), w^n - u^n\rangle \\ &= 2\langle u^n - Ax^*, w^n - u^n\rangle \\ &= 2\big[\langle w^n - Ax^*, w^n - u^n\rangle - \|w^n - u^n\|^2\big] \\ &= \big(\|w^n - Ax^*\|^2 - \|u^n - Ax^*\|^2\big) - \|w^n - u^n\|^2 \\ &\leq -\|w^n - u^n\|^2. \end{aligned} \tag{13}$$

Case 1. $A^*(w^n - u^n) = 0$.
In this case, $s^n = t^n$. Also, it follows from (13) that $\|w^n - u^n\| = 0$. Thus, the inequalities in Lemma 6 hold.
Case 2. $A^*(w^n - u^n) \neq 0$.
From (13) and $\{\rho_n\} \subset [a, b] \subset (0, 1)$, we get for all $n \geq n_0$ that

$$\begin{aligned}\|s^n - x^*\|^2 &= \|(t^n - x^*) + \delta_n A^*(w^n - u^n)\|^2 \\ &= \|t^n - x^*\|^2 + \delta_n^2 \|A^*(w^n - u^n)\|^2 + 2\delta_n \langle t^n - x^*, A^*(w^n - u^n\rangle \\ &\leq \|t^n - x^*\|^2 + \delta_n^2 \|A^*(w^n - u^n)\|^2 - \delta_n \|w^n - u^n\|^2 \\ &= \|t^n - x^*\|^2 + \frac{\rho_n^2 \|w^n - u^n\|^4}{\|A^*(w^n - u^n)\|^2} - \frac{\rho_n \|w^n - u^n\|^4}{\|A^*(w^n - u^n)\|^2} \\ &= \|t^n - x^*\|^2 - \frac{\rho_n^2 \|w^n - u^n\|^4}{\|A^*(w^n - u^n)\|^2} \cdot \frac{1-\rho_n}{\rho_n} \\ &\leq \|t^n - x^*\|^2 - \frac{\rho_n^2 \|w^n - u^n\|^4}{\|A^*(w^n - u^n)\|^2} \cdot \frac{1-b}{b}. \end{aligned} \tag{14}$$

Then, from (14), we have

$$\begin{aligned}\|s^n - t^n\|^2 = \delta_n^2 \|A^*(w^n - u^n)\|^2 &= \frac{\rho_n^2 \|w^n - u^n\|^4}{\|A^*(w^n - u^n)\|^2} \\ &\leq \frac{b}{1-b}\big(\|t^n - x^*\|^2 - \|s^n - x^*\|^2\big). \end{aligned} \tag{15}$$

On the other hand,

$$0 < \|A^*(w^n - u^n)\| \leq \|A^*\| \|w^n - u^n\| = \|A\| \|w^n - u^n\| \leq (\|A\| + 1) \|w^n - u^n\|.$$

Taking into account the last inequality together with (15), we find

$$\begin{aligned}\|s^n - t^n\|^2 \geq \frac{\rho_n^2 \|w^n - u^n\|^4}{(\|A\|+1)^2 \|w^n - u^n\|^2} &= \frac{\rho_n^2}{(\|A\|+1)^2} \|w^n - u^n\|^2 \\ &\geq \frac{a^2}{(\|A\|+1)^2} \|w^n - u^n\|^2. \end{aligned}$$

□

Lemma 7. Let $\{x^n\}$, $\{y^n\}$, $\{t^n\}$ and $\{s^n\}$ be the sequences generated by Algorithm 1. Then the sequences $\{x^n\}$, $\{y^n\}$, $\{t^n\}$, $\{s^n\}$, and $\{F(s^n)\}$ are bounded.

Proof. By the η-strong monotonicity and the L-Lipschitz continuity of F on $\mathcal{H}_1$, we have

$$\begin{aligned}&\|s^n - x^* - \varepsilon(F(s^n) - F(x^*))\|^2 \\ &= \|s^n - x^*\|^2 - 2\varepsilon \langle s^n - x^*, F(s^n) - F(x^*)\rangle + \varepsilon^2 \|F(s^n) - F(x^*)\|^2 \\ &\leq \|s^n - x^*\|^2 - 2\varepsilon\eta \|s^n - x^*\|^2 + \varepsilon^2 L^2 \|s^n - x^*\|^2 \end{aligned}$$

$$= \left[1 - \varepsilon(2\eta - \varepsilon L^2)\right] \|s^n - x^*\|^2. \tag{16}$$

From (16), we obtain, for all $n \geq n_0$,

$$\begin{aligned}
&\|s^n - \varepsilon_n F(s^n) - (x^* - \varepsilon_n F(x^*))\| \\
&= \|(s^n - x^*) - \varepsilon_n (F(s^n) - F(x^*))\| \\
&= \left\| \left(1 - \frac{\varepsilon_n}{\varepsilon}\right)(s^n - x^*) + \frac{\varepsilon_n}{\varepsilon}\left[s^n - x^* - \varepsilon(F(s^n) - F(x^*))\right] \right\| \\
&\leq \left(1 - \frac{\varepsilon_n}{\varepsilon}\right)\|s^n - x^*\| + \frac{\varepsilon_n}{\varepsilon}\|s^n - x^* - \varepsilon(F(s^n) - F(x^*))\| \\
&\leq \left(1 - \frac{\varepsilon_n}{\varepsilon}\right)\|s^n - x^*\| + \frac{\varepsilon_n}{\varepsilon}\sqrt{1 - \varepsilon(2\eta - \varepsilon L^2)}\|s^n - x^*\| \\
&= \left[1 - \frac{\varepsilon_n}{\varepsilon}\left(1 - \sqrt{1 - \varepsilon(2\eta - \varepsilon L^2)}\right)\right]\|s^n - x^*\| \\
&= \left(1 - \frac{\varepsilon_n \tau}{\varepsilon}\right)\|s^n - x^*\|,
\end{aligned} \tag{17}$$

where

$$\tau = 1 - \sqrt{1 - \varepsilon(2\eta - \varepsilon L^2)} \in (0, 1].$$

Alternatively, we have

$$0 \leq \alpha_n \|x^n - x^{n-1}\| \leq \eta_n, \quad \text{for all } n \geq 0 \tag{18}$$

and

$$\lim_{n\to\infty} \frac{\alpha_n}{\varepsilon_n}\|x^n - x^{n-1}\| = 0. \tag{19}$$

Indeed, if $x^n = x^{n-1}$, then inequality (18) holds. Otherwise, we get

$$\begin{aligned}
0 \leq \alpha_n = \min\left\{\frac{\eta_n}{\|x^n - x^{n-1}\|}, \gamma_n\right\} \leq \frac{\eta_n}{\|x^n - x^{n-1}\|} \\
\Rightarrow 0 \leq \alpha_n \|x^n - x^{n-1}\| \leq \eta_n.
\end{aligned}$$

From (18), we have

$$0 \leq \frac{\alpha_n}{\varepsilon_n}\|x^n - x^{n-1}\| \leq \frac{\eta_n}{\varepsilon_n}, \quad \text{for all } n \geq 0.$$

Since $\lim_{n\to\infty} \frac{\eta_n}{\varepsilon_n} = 0$, it can be inferred from the above inequality that $\lim_{n\to\infty} \frac{\alpha_n}{\varepsilon_n}\|x^n - x^{n-1}\| = 0$.

Using a similar argument, we arrive at

$$0 \le \beta_n \|x^{n-2} - x^{n-1}\| \le \eta_n, \quad \text{for all } n \ge 0 \tag{20}$$

and

$$\lim_{n\to\infty} \frac{\beta_n}{\varepsilon_n} \|x^{n-2} - x^{n-1}\| = 0. \tag{21}$$

From $\lim_{n\to\infty} \frac{\alpha_n}{\varepsilon_n} \|x^n - x^{n-1}\| = 0$ and $\lim_{n\to\infty} \frac{\beta_n}{\varepsilon_n} \|x^{n-2} - x^{n-1}\| = 0$, we can infer that there exist positive constants K_1 and K_2 such that $\frac{\alpha_n}{\varepsilon_n} \|x^n - x^{n-1}\| \le K_1$ and $\frac{\beta_n}{\varepsilon_n} \|x^{n-2} - x^{n-1}\| \le K_2$ for all $n \ge 0$. So, we have

$$\begin{aligned}
\|y^n - x^*\| &= \|(x^n - x^*) + \alpha_n(x^n - x^{n-1}) + \beta_n(x^{n-2} - x^{n-1})\| \\
&\le \|x^n - x^*\| + \varepsilon_n \cdot \frac{\alpha_n}{\varepsilon_n} \|x^n - x^{n-1}\| + \varepsilon_n \cdot \frac{\beta_n}{\varepsilon_n} \|x^{n-2} - x^{n-1}\| \\
&\le \|x^n - x^*\| + \varepsilon_n K_1 + \varepsilon_n K_2 \\
&= \|x^n - x^*\| + \varepsilon_n K_3, \quad \text{for all } n \ge 0,
\end{aligned} \tag{22}$$

where $K_3 = K_1 + K_2$.

From Lemma 6, (12) and (22), we get

$$\|s^n - x^*\| \le \|t^n - x^*\| \le \|y^n - x^*\| \le \|x^n - x^*\| + \varepsilon_n K_3, \quad \text{for all } n \ge n_0. \tag{23}$$

Employing (17) and (23), we derive, for all $n \ge n_0$,

$$\begin{aligned}
\|x^{n+1} - x^*\| &= \|s^n - \varepsilon_n F(s^n) - (x^* - \varepsilon_n F(x^*)) - \varepsilon_n F(x^*)\| \\
&\le \|s^n - \varepsilon_n F(s^n) - (x^* - \varepsilon_n F(x^*))\| + \varepsilon_n \|F(x^*)\| \\
&\le \Big(1 - \frac{\varepsilon_n \tau}{\varepsilon}\Big) \|s^n - x^*\| + \varepsilon_n \|F(x^*)\| \qquad (24) \\
&\le \Big(1 - \frac{\varepsilon_n \tau}{\varepsilon}\Big) \big(\|x^n - x^*\| + \varepsilon_n K_3\big) + \varepsilon_n \|F(x^*)\| \\
&\le \Big(1 - \frac{\varepsilon_n \tau}{\varepsilon}\Big) \|x^n - x^*\| + \varepsilon_n K_3 + \varepsilon_n \|F(x^*)\| \\
&= \Big(1 - \frac{\varepsilon_n \tau}{\varepsilon}\Big) \|x^n - x^*\| + \frac{\varepsilon_n \tau}{\varepsilon} \cdot \frac{\varepsilon\big(K_3 + \|F(x^*)\|\big)}{\tau}. \qquad (25)
\end{aligned}$$

From (25), we have, for every $n \ge n_0$,

$$\|x^{n+1} - x^*\| \le \max\left\{\|x^n - x^*\|, \frac{\varepsilon\big(K_3 + \|F(x^*)\|\big)}{\tau}\right\}.$$

So, by induction, we obtain

$$\|x^n - x^*\| \le \max\left\{\|x^{n_0} - x^*\|, \frac{\varepsilon\big(K_3 + \|F(x^*)\|\big)}{\tau}\right\}, \quad \text{for all } n \ge n_0.$$

Therefore, the sequence $\{x^n\}$ is bounded, and so are the sequences $\{y^n\}$, $\{t^n\}$, $\{s^n\}$, and $\{F(s^n)\}$ due to $\{\varepsilon_n\} \subset (0,1)$, (23), and the Lipschitz continuity of F. □

Lemma 8. Let $\{x^n\}$ be the sequence generated by Algorithm 1. Then, there exists a constant $K > 0$ such that for all $n \ge n_0$, we have

$$\begin{aligned}\|x^{n+1} - x^*\|^2 \le& \Big(1 - \frac{\varepsilon_n \tau}{\varepsilon}\Big)\|x^n - x^*\|^2 + 2\varepsilon_n\langle F(x^*), x^* - x^{n+1}\rangle \\ &+ K\alpha_n\|x^n - x^{n-1}\| + K\beta_n\|x^{n-2} - x^{n-1}\|.\end{aligned}$$

Proof. From (23), we have

$$\begin{aligned}\|y^n - x^*\|^2 &\le \big(\|x^n - x^*\| + \varepsilon_n K_3\big)^2 \\ &= \|x^n - x^*\|^2 + \varepsilon_n\big(2K_3\|x^n - x^*\| + \varepsilon_n K_3^2\big) \\ &\le \|x^n - x^*\|^2 + \varepsilon_n K_4, \end{aligned} \tag{26}$$

where $K_4 = \sup_{n\ge 0}\big\{2K_3\|x^n - x^*\| + \varepsilon_n K_3^2\big\}$.

From $\dfrac{\alpha_n}{\varepsilon_n}\|x^n - x^{n-1}\| \le K_1$, $\dfrac{\beta_n}{\varepsilon_n}\|x^{n-2} - x^{n-1}\| \le K_2$ for all $n \ge 0$ and $\{\varepsilon_n\} \subset (0,1)$, we deduce that $\alpha_n\|x^n - x^{n-1}\| \le K_1$, $\beta_n\|x^{n-2} - x^{n-1}\| \le K_2$ for all $n \ge 0$. This, in conjunction with the boundedness of the sequence $\{x^n\}$, implies the existence of a constant $K > 0$ such that

$$2\|x^n - x^*\| + \alpha_n\|x^n - x^{n-1}\| + \beta_n\|x^{n-2} - x^{n-1}\| \le K, \quad \text{for all } n \ge 0. \tag{27}$$

From (27), we get

$$\begin{aligned}\|y^n - x^*\|^2 =& \|(x^n - x^*) + \alpha_n(x^n - x^{n-1}) + \beta_n(x^{n-2} - x^{n-1})\|^2 \\ =& \|x^n - x^*\|^2 + 2\alpha_n\langle x^n - x^*, x^n - x^{n-1}\rangle \\ &+ 2\beta_n\langle x^n - x^*, x^{n-2} - x^{n-1}\rangle \\ &+ \|\alpha_n(x^n - x^{n-1}) + \beta_n(x^{n-2} - x^{n-1})\|^2 \\ \le& \|x^n - x^*\|^2 + 2\alpha_n\|x^n - x^*\|.\|x^n - x^{n-1}\| \\ &+ 2\beta_n\|x^n - x^*\|.\|x^{n-2} - x^{n-1}\| \\ &+ \alpha_n^2\|x^n - x^{n-1}\|^2 + 2\alpha_n\beta_n\|x^n - x^{n-1}\|.\|x^{n-2} - x^{n-1}\|\end{aligned}$$

$$
\begin{aligned}
&+\beta_n^2\|x^{n-2}-x^{n-1}\|^2\\
=&\|x^n-x^*\|^2+\alpha_n\|x^n-x^{n-1}\|\big(2\|x^n-x^*\|\\
&+\alpha_n\|x^n-x^{n-1}\|+\beta_n\|x^{n-2}-x^{n-1}\|\big)\\
&+\beta_n\|x^{n-2}-x^{n-1}\|\big(2\|x^n-x^*\|+\alpha_n\|x^n-x^{n-1}\|\\
&+\beta_n\|x^{n-2}-x^{n-1}\|\big)\\
\le&\|x^n-x^*\|^2+K\alpha_n\|x^n-x^{n-1}\|+K\beta_n\|x^{n-2}-x^{n-1}\|. \qquad (28)
\end{aligned}
$$

From (17), (23), and (28), we obtain, for all $n \ge n_0$,

$$
\begin{aligned}
\|x^{n+1}-x^*\|^2 &\le \|x^{n+1}-x^*\|^2+\varepsilon_n^2\|F(x^*)\|^2\\
&=\|x^{n+1}-x^*+\varepsilon_n F(x^*)\|^2-2\langle\varepsilon_n F(x^*),x^{n+1}-x^*\rangle\\
&=\|s^n-\varepsilon_n F(s^n)-(x^*-\varepsilon_n F(x^*))\|^2-2\varepsilon_n\langle F(x^*),x^{n+1}-x^*\rangle\\
&\le\left[\left(1-\frac{\varepsilon_n\tau}{\varepsilon}\right)\|s^n-x^*\|\right]^2-2\varepsilon_n\langle F(x^*),x^{n+1}-x^*\rangle\\
&\le\left(1-\frac{\varepsilon_n\tau}{\varepsilon}\right)\|s^n-x^*\|^2-2\varepsilon_n\langle F(x^*),x^{n+1}-x^*\rangle \qquad (29)\\
&\le\left(1-\frac{\varepsilon_n\tau}{\varepsilon}\right)\|y^n-x^*\|^2+2\varepsilon_n\langle F(x^*),x^*-x^{n+1}\rangle\\
&\le\left(1-\frac{\varepsilon_n\tau}{\varepsilon}\right)\big(\|x^n-x^*\|^2+K\alpha_n\|x^n-x^{n-1}\|\\
&\quad+K\beta_n\|x^{n-2}-x^{n-1}\|\big)+2\varepsilon_n\langle F(x^*),x^*-x^{n+1}\rangle\\
&\le\left(1-\frac{\varepsilon_n\tau}{\varepsilon}\right)\|x^n-x^*\|^2+2\varepsilon_n\langle F(x^*),x^*-x^{n+1}\rangle\\
&\quad+K\alpha_n\|x^n-x^{n-1}\|+K\beta_n\|x^{n-2}-x^{n-1}\|.
\end{aligned}
$$

□

The theorem presented here establishes the validity and convergence of Algorithm 1.

Theorem 1. Assume that Assumption 1 is satisfied. Then the sequence $\{x^n\}$ generated by Algorithm 1 converges strongly to the unique solution of the BSVIP (4), provided the solution set $\Omega_{\text{SVIP}} = \{x^* \in \text{Sol}(C, F_1) : Ax^* \in \text{Sol}(Q, F_2)\}$ of the SVIP (1)–(2) is nonempty.

Proof. We prove that the sequence $\{x^n\}$ converges strongly to the unique solution x^* of the problem (4). Let us consider two cases.

Case 1. There exists $n_1 \in \mathbb{N}$ such that $\{\|x^n - x^*\|\}$ is decreasing for all $n \geq n_1$. Consequently, the limit of $\|x^n - x^*\|$ exists. Therefore, it follows from (23), (26), and (29), for all $n \geq n_0$, that

$$\begin{aligned}-\varepsilon_n K_4 &\leq \|y^n - x^*\|^2 - \|s^n - x^*\|^2 - \varepsilon_n K_4\\ &\leq \|x^n - x^*\|^2 - \|s^n - x^*\|^2\\ &\leq \left(\|x^n - x^*\|^2 - \|x^{n+1} - x^*\|^2\right) - \frac{\varepsilon_n \tau}{\varepsilon}\|s^n - x^*\|^2\\ &\quad - 2\varepsilon_n \langle F(x^*), x^{n+1} - x^*\rangle.\end{aligned}$$

Given the limit of $\|x^n - x^*\|$ exists, along with $\lim_{n\to\infty} \varepsilon_n = 0$, and both $\{x^n\}$ and $\{s^n\}$ being bounded sequences, the above inequalities imply that

$$\begin{aligned}&\lim_{n\to\infty}\left(\|y^n - x^*\|^2 - \|s^n - x^*\|^2 - \varepsilon_n K_4\right) = 0\\ &\qquad\Rightarrow \lim_{n\to\infty}\left(\|y^n - x^*\|^2 - \|s^n - x^*\|^2\right) = 0, \qquad (30)\end{aligned}$$

$$\lim_{n\to\infty}\left(\|x^n - x^*\|^2 - \|s^n - x^*\|^2\right) = 0. \qquad (31)$$

From (23), we get

$$0 \leq \|y^n - x^*\|^2 - \|t^n - x^*\|^2 \leq \|y^n - x^*\|^2 - \|s^n - x^*\|^2, \quad \text{for all } n \geq 0,$$

from which, by (30), it follows that

$$\lim_{n\to\infty}\left(\|y^n - x^*\|^2 - \|t^n - x^*\|^2\right) = 0. \qquad (32)$$

From Lemma 5 and (10), we have

$$\frac{1-\lambda^2}{2}\|y^n - z^n\|^2 \leq \|y^n - x^*\|^2 - \|t^n - x^*\|^2, \quad \text{for all } n \geq n_0,$$

which together with (32) implies

$$\lim_{n\to\infty}\|y^n - z^n\| = 0. \qquad (33)$$

From (30) and (32), it follows that

$$\lim_{n\to\infty}\left(\|t^n - x^*\|^2 - \|s^n - x^*\|^2\right) = 0.$$

Hence, by combining Lemma 6, we obtain

$$\lim_{n\to\infty} \|w^n - u^n\| = 0, \tag{34}$$

$$\lim_{n\to\infty} \|s^n - t^n\| = 0. \tag{35}$$

Using the triangle inequality and the L_1-Lipschitz continuity of F_1 on $\mathcal{H}_1$, we get

$$\begin{aligned}\|y^n - s^n\| &\le \|y^n - z^n\| + \|z^n - t^n\| + \|t^n - s^n\| \\ &= \|y^n - z^n\| + \|\lambda_n(F_1(z^n) - F_1(y^n))\| + \|t^n - s^n\| \\ &\le \|y^n - z^n\| + \lambda_n L_1 \|z^n - y^n\| + \|t^n - s^n\| \\ &\le (1 + \lambda_0 L_1)\|y^n - z^n\| + \|t^n - s^n\|,\end{aligned}$$

which together with (33), (35) implies

$$\lim_{n\to\infty} \|y^n - s^n\| = 0. \tag{36}$$

Now, observe that

$$\begin{aligned}\|w^n - Ax^*\|^2 &= \|u^n - Ax^* + (w^n - u^n)\|^2 \\ &= \|u^n - Ax^*\|^2 + 2\langle u^n - Ax^*, w^n - u^n\rangle + \|w^n - u^n\|^2 \\ &= \|u^n - Ax^*\|^2 + 2\langle A(t^n - x^*), w^n - u^n\rangle + \|w^n - u^n\|^2 \\ &\ge \|u^n - Ax^*\|^2 - 2\|A(t^n - x^*)\|\|w^n - u^n\| + \|w^n - u^n\|^2 \\ &\ge \|u^n - Ax^*\|^2 - 2\|A\|\|t^n - x^*\|\|w^n - u^n\|. \end{aligned} \tag{37}$$

Combining Lemma 5, (9) and (37) yields

$$\frac{1-\mu^2}{2}\|u^n - v^n\|^2 \le 2\|A\|\|t^n - x^*\|\|w^n - u^n\|, \quad \text{for all } n \ge n_0. \tag{38}$$

From (34), (38), and the boundedness of the sequence $\{t^n\}$, we obtain

$$\lim_{n\to\infty} \|u^n - v^n\| = 0. \tag{39}$$

We now prove that

$$\limsup_{n\to\infty} \langle F(x^*), x^* - s^n\rangle \le 0. \tag{40}$$

Select a subsequence $\{s^{n_k}\}$ of $\{s^n\}$ such that $\limsup_{n\to\infty} \langle F(x^*), x^* - s^n\rangle = \lim_{k\to\infty} \langle F(x^*), x^* - s^{n_k}\rangle$. Given that $\{s^{n_k}\}$ is bounded, we may assume that $\{s^{n_k}\}$ converges weakly to some $\overline{s} \in \mathcal{H}_1$.

Therefore

$$\limsup_{n\to\infty}\langle F(x^*), x^* - s^n\rangle = \lim_{k\to\infty}\langle F(x^*), x^* - s^{n_k}\rangle = \langle F(x^*), x^* - \overline{s}\rangle. \quad (41)$$

We deduce from $s^{n_k} \rightharpoonup \overline{s}$ and (35), (36) that $t^{n_k} \rightharpoonup \overline{s}$ and $y^{n_k} \rightharpoonup \overline{s}$. From (33), we have $\lim_{k\to\infty}\|y^{n_k} - z^{n_k}\| = 0$. Since $z^{n_k} = P_C(y^{n_k} - \lambda_{n_k}F_1(y^{n_k}))$, $y^{n_k} \rightharpoonup \overline{s}$, $\lambda_{n_k} \geq \min\left(\frac{\lambda}{L_1}, \lambda_0\right) > 0$. By Lemma 2, we get $\overline{s} \in \mathrm{Sol}(C, F_1)$.

From $t^{n_k} \rightharpoonup \overline{s}$, we get $u^{n_k} = A(t^{n_k}) \rightharpoonup A(\overline{s})$. This, together with (39), where $v^{n_k} = P_Q(u^{n_k} - \mu_{n_k}F_2(u^{n_k}))$ and $\mu_{n_k} \geq \min\left(\frac{\mu}{L_2}, \mu_0\right) > 0$, along with Lemma 2, implies that $A(\overline{s}) \in \mathrm{Sol}(Q, F_2)$.

With $\overline{s} \in \mathrm{Sol}(C, F_1)$ and $A(\overline{s}) \in \mathrm{Sol}(Q, F_2)$, we conclude that $\overline{s} \in \Omega_{\mathrm{SVIP}}$. Consequently, it follows from $x^* \in \mathrm{Sol}(\Omega_{\mathrm{SVIP}}, F)$ that $\langle F(x^*), \overline{s} - x^*\rangle \geq 0$, which together with (41) implies (40).

From the boundedness of $\{F(s^n)\}$, $\lim_{n\to\infty}\varepsilon_n = 0$ and (40), we have

$$\begin{aligned}\limsup_{n\to\infty}\langle F(x^*), x^* - x^{n+1}\rangle &= \limsup_{n\to\infty}\langle F(x^*), x^* - s^n + \varepsilon_n F(s^n)\rangle \\ &= \limsup_{n\to\infty}\left[\langle F(x^*), x^* - s^n\rangle + \varepsilon_n\langle F(x^*), F(s^n)\rangle\right] \\ &= \limsup_{n\to\infty}\langle F(x^*), x^* - s^n\rangle \leq 0. \quad (42)\end{aligned}$$

From Lemma 8, we get

$$\|x^{n+1} - x^*\|^2 \leq (1 - a_n)\|x^n - x^*\|^2 + a_n b_n, \quad \text{for all } n \geq n_0, \quad (43)$$

where $a_n = \frac{\varepsilon_n\tau}{\varepsilon}$ and

$$b_n = \frac{2\varepsilon\langle F(x^*), x^* - x^{n+1}\rangle}{\tau} + \frac{K\varepsilon}{\tau}\cdot\frac{\alpha_n}{\varepsilon_n}\|x^n - x^{n-1}\| + \frac{K\varepsilon}{\tau}\cdot\frac{\beta_n}{\varepsilon_n}\|x^{n-2} - x^{n-1}\|.$$

Given (19), (21), and (42), it follows that $\limsup_{n\to\infty} b_n \leq 0$. From $0 < \varepsilon_n < \varepsilon$ for all $n \geq n_0$ and $0 < \tau \leq 1$, we get $\left\{a_n = \frac{\varepsilon_n\tau}{\varepsilon}\right\}_{n\geq n_0} \subset (0,1)$. So, from (43), $\sum_{n=0}^{\infty}\varepsilon_n = \infty$, $\limsup_{n\to\infty} b_n \leq 0$ and Lemma 4, we have $\lim_{n\to\infty}\|x^n - x^*\|^2 = 0$, that is, $x^n \to x^*$ as $n \to \infty$.

Case 2. Suppose that for any integer m, there exists an integer n such that $n \geq m$ and $\|x^n - x^*\| \leq \|x^{n+1} - x^*\|$. In this situation, it follows from Lemma 3 that there exists a nondecreasing sequence $\{\tau(n)\}_{n\geq n_2}$ of $\mathbb{N}$ such that $\lim_{n\to\infty} \tau(n) = \infty$ and the following inequalities are true:

$$\|x^{\tau(n)} - x^*\| \le \|x^{\tau(n)+1} - x^*\|, \ \ \|x^n - x^*\| \le \|x^{\tau(n)+1} - x^*\|, \ \ \text{for all } n \ge n_2. \tag{44}$$

Choose $n_3 \ge n_2$ such that $\tau(n) \ge n_0$ for all $n \ge n_3$. From (23), (44) and (24), we get, for all $n \ge n_3$,

$$\begin{aligned} -\varepsilon_{\tau(n)} K_3 &\le \|y^{\tau(n)} - x^*\| - \|s^{\tau(n)} - x^*\| - \varepsilon_{\tau(n)} K_3 \\ &\le \|x^{\tau(n)} - x^*\| - \|s^{\tau(n)} - x^*\| \\ &\le \|x^{\tau(n)+1} - x^*\| - \|s^{\tau(n)} - x^*\| \\ &\le -\frac{\varepsilon_{\tau(n)}\tau}{\varepsilon}\|s^{\tau(n)} - x^*\| + \varepsilon_{\tau(n)}\|F(x^*)\|. \end{aligned}$$

Thus, from the boundedness of $\{s^n\}$ and $\lim_{n\to\infty} \varepsilon_n = 0$, we have

$$\begin{aligned} &\lim_{n\to\infty} \left(\|y^{\tau(n)} - x^*\| - \|s^{\tau(n)} - x^*\| - \varepsilon_{\tau(n)} K_3\right) = 0 \\ &\qquad \Rightarrow \lim_{n\to\infty} \left(\|y^{\tau(n)} - x^*\| - \|s^{\tau(n)} - x^*\|\right) = 0, \end{aligned} \tag{45}$$

$$\lim_{n\to\infty} (\|x^{\tau(n)} - x^*\| - \|s^{\tau(n)} - x^*\|) = 0. \tag{46}$$

From (45), (46), and the boundedness of $\{x^n\}$, $\{y^n\}$, $\{s^n\}$, we obtain

$$\lim_{n\to\infty} (\|y^{\tau(n)} - x^*\|^2 - \|s^{\tau(n)} - x^*\|^2) = 0, \quad \lim_{n\to\infty} (\|x^{\tau(n)} - x^*\|^2 - \|s^{\tau(n)} - x^*\|^2) = 0.$$

Applying a similar line of reasoning as in the first case, we can arrive at the conclusion that

$$\limsup_{n\to\infty} \langle F(x^*), x^* - s^{\tau(n)}\rangle \le 0.$$

Therefore, the boundedness of $\{F(s^n)\}$ and $\lim_{n\to\infty} \varepsilon_n = 0$ yield

$$\begin{aligned} \limsup_{n\to\infty} \langle F(x^*), x^* - x^{\tau(n)+1}\rangle &= \limsup_{n\to\infty} \left\langle F(x^*), x^* - s^{\tau(n)} + \varepsilon_{\tau(n)} F(s^{\tau(n)})\right\rangle \\ &= \limsup_{n\to\infty} \Big[\langle F(x^*), x^* - s^{\tau(n)}\rangle \\ &\quad + \varepsilon_{\tau(n)} \langle F(x^*), F(s^{\tau(n)})\rangle\Big] \\ &= \limsup_{n\to\infty} \langle F(x^*), x^* - s^{\tau(n)}\rangle \le 0. \end{aligned} \tag{47}$$

From Lemma 8 and (44), we have, for all $n \ge n_3$,

$$\begin{aligned}\|x^{\tau(n)+1}-x^*\|^2 &\leq \Big(1-\frac{\varepsilon_{\tau(n)}\tau}{\varepsilon}\Big)\|x^{\tau(n)}-x^*\|^2+2\varepsilon_{\tau(n)}\langle F(x^*),x^*-x^{\tau(n)+1}\rangle \\ &\quad + K\alpha_{\tau(n)}\|x^{\tau(n)}-x^{\tau(n)-1}\|+K\beta_{\tau(n)}\|x^{\tau(n)-2}-x^{\tau(n)-1}\| \\ &\leq \Big(1-\frac{\varepsilon_{\tau(n)}\tau}{\varepsilon}\Big)\|x^{\tau(n)+1}-x^*\|^2+2\varepsilon_{\tau(n)}\langle F(x^*),x^*-x^{\tau(n)+1}\rangle \\ &\quad + K\alpha_{\tau(n)}\|x^{\tau(n)}-x^{\tau(n)-1}\|+K\beta_{\tau(n)}\|x^{\tau(n)-2}-x^{\tau(n)-1}\|.\end{aligned}$$

In particular, since $\varepsilon_{\tau(n)}>0$, we have, for all $n\geq n_3$

$$\begin{aligned}\|x^{\tau(n)+1}-x^*\|^2 \leq \frac{2\varepsilon}{\tau}\langle F(x^*),x^*-x^{\tau(n)+1}\rangle + \frac{K\varepsilon}{\tau}\cdot\frac{\alpha_{\tau(n)}}{\varepsilon_{\tau(n)}}\|x^{\tau(n)}-x^{\tau(n)-1}\| \\ + \frac{K\varepsilon}{\tau}\cdot\frac{\beta_{\tau(n)}}{\varepsilon_{\tau(n)}}\|x^{\tau(n)-2}-x^{\tau(n)-1}\|.\end{aligned}$$

From (44) and the inequality given above, we derive, for all $n\geq n_3$,

$$\begin{aligned}\|x^{n}-x^*\|^2 \leq \frac{2\varepsilon}{\tau}\langle F(x^*),x^*-x^{\tau(n)+1}\rangle + \frac{K\varepsilon}{\tau}\cdot\frac{\alpha_{\tau(n)}}{\varepsilon_{\tau(n)}}\|x^{\tau(n)}-x^{\tau(n)-1}\| \\ + \frac{K\varepsilon}{\tau}\cdot\frac{\beta_{\tau(n)}}{\varepsilon_{\tau(n)}}\|x^{\tau(n)-2}-x^{\tau(n)-1}\|.\end{aligned} \tag{48}$$

By taking the limit in (48) as $n\to\infty$ and utilizing (47), (19), and (21), we deduce that

$$\limsup_{n\to\infty}\|x^n-x^*\|^2\leq 0,$$

which implies $x^n\to x^*$. □

From Algorithm 1, if we choose $\gamma_n=0$ and $\xi_n=0$ for all $n\geq 0$, it is evident that $\alpha_n=0$ and $\beta_n=0$ for all $n\geq 0$. In this case, Algorithm 1 reduces to the following algorithm. This algorithm, which we will refer to as Algorithm 2, closely resembles [25, Algorithm 3.1], as described in (8). The key difference between the two algorithms lies in the order in which the modified Tseng's extragradient method is applied in the two spaces $\mathcal{H}_1$ and $\mathcal{H}_2$.

Assumption 2. Let the following hold

i) $F:\mathcal{H}\longrightarrow\mathcal{H}$ is strongly monotone and Lipschitz continuous on $\mathcal{H}$.

ii) $G:\mathcal{H}\longrightarrow\mathcal{H}$ is pseudomonotone on C, Lipschitz continuous on $\mathcal{H}$.

Algorithm 2

Step 0. Choose $\mu_0 > 0$, $\lambda_0 > 0$, $\mu \in (0,1)$, $\lambda \in (0,1)$, $\{\rho_n\} \subset [a,b] \subset (0,1)$, $\{\varepsilon_n\} \subset (0,1)$ such that $\lim_{n\to\infty} \varepsilon_n = 0$, $\sum_{n=0}^{\infty} \varepsilon_n = \infty$.

Step 1. Let $x^0 \in \mathcal{H}_1$. Set $n := 0$.

Step 2. Compute

$$y^n = P_C(x^n - \lambda_n F_1(x^n)),$$
$$z^n = y^n - \lambda_n(F_1(y^n) - F_1(x^n)),$$

where

$$\lambda_{n+1} = \begin{cases} \min\left\{\dfrac{\lambda\|x^n - y^n\|}{\|F_1(x^n) - F_1(y^n)\|}, \lambda_n\right\} & \text{if } F_1(x^n) \neq F_1(y^n), \\ \lambda_n & \text{if } F_1(x^n) = F_1(y^n). \end{cases}$$

Step 3. Compute $u^n = A(z^n)$ and

$$v^n = P_Q(u^n - \mu_n F_2(u^n)),$$
$$w^n = v^n - \mu_n(F_2(v^n) - F_2(u^n)),$$

where

$$\mu_{n+1} = \begin{cases} \min\left\{\dfrac{\mu\|u^n - v^n\|}{\|F_2(u^n) - F_2(v^n)\|}, \mu_n\right\} & \text{if } F_2(u^n) \neq F_2(v^n), \\ \mu_n & \text{if } F_2(u^n) = F_2(v^n). \end{cases}$$

Step 4. Compute

$$t^n = z^n + \delta_n A^*(w^n - u^n),$$

where the stepsize δ_n is chosen in such a way that

$$\delta_n = \begin{cases} \dfrac{\rho_n\|w^n - u^n\|^2}{\|A^*(w^n - u^n)\|^2} & \text{if } A^*(w^n - u^n) \neq 0, \\ 0 & \text{if } A^*(w^n - u^n) = 0. \end{cases}$$

Step 5. Compute

$$x^{n+1} = t^n - \varepsilon_n F(t^n).$$

Step 6. Set $n := n + 1$, and go to **Step 2**.

iii) $\limsup_{n\to\infty}\langle G(x^n), y - y^n\rangle \leq \langle G(\overline{x}), y - \overline{y}\rangle$ holds for any sequences $\{x^n\}$ and $\{y^n\}$ in $\mathcal{H}$ that converge weakly to $\overline{x}$ and $\overline{y}$, respectively.

When $F_2 = 0$ and $Q = \mathcal{H}_2$, the SVIP defined by (1) and (2) reduces to the VIP given by (1). Consequently, according to Algorithm 1 and Theorem 1 (where $\mathcal{H}_1 = \mathcal{H}$ and $F_1 = G$), we obtain the following result for solving the BVIP specified by (5). It is important to note that the proposed algorithm only requires a single projection onto the feasible set at each iteration and does not necessitate any knowledge of the Lipschitz constants for the mappings F and G, nor the modulus of strong monotonicity of F.

Algorithm 3

Step 0. Choose $\lambda_0 > 0$, $\lambda \in (0,1)$, $\{\gamma_n\} \subset [0,\infty)$, $\{\xi_n\} \subset [0,\infty)$, $\{\eta_n\} \subset (0,\infty)$, $\{\varepsilon_n\} \subset (0,1)$ such that $\lim_{n\to\infty} \frac{\eta_n}{\varepsilon_n} = 0$, $\lim_{n\to\infty} \varepsilon_n = 0$, $\sum_{n=0}^{\infty} \varepsilon_n = \infty$.

Step 1. Let $x^{-2}, x^{-1}, x^0 \in \mathcal{H}$. Set $n := 0$.

Step 2. Compute $y^n = x^n + \alpha_n(x^n - x^{n-1}) + \beta_n(x^{n-2} - x^{n-1})$, where

$$\alpha_n = \begin{cases} \min\left\{\dfrac{\eta_n}{\|x^n - x^{n-1}\|}, \gamma_n\right\} & \text{if } x^n \neq x^{n-1}, \\ \gamma_n & \text{if } x^n = x^{n-1}, \end{cases}$$

and

$$\beta_n = \begin{cases} \min\left\{\dfrac{\eta_n}{\|x^{n-2} - x^{n-1}\|}, \xi_n\right\} & \text{if } x^{n-2} \neq x^{n-1}, \\ \xi_n & \text{if } x^{n-2} = x^{n-1}. \end{cases}$$

Step 3. Compute

$$z^n = P_C(y^n - \lambda_n G(y^n)),$$
$$t^n = z^n - \lambda_n(G(z^n) - G(y^n)),$$

where

$$\lambda_{n+1} = \begin{cases} \min\left\{\dfrac{\lambda\|y^n - z^n\|}{\|G(y^n) - G(z^n)\|}, \lambda_n\right\} & \text{if } G(y^n) \neq G(z^n), \\ \lambda_n & \text{if } G(y^n) = G(z^n). \end{cases}$$

Step 6. Compute

$$x^{n+1} = t^n - \varepsilon_n F(t^n).$$

Step 7. Set $n := n+1$, and go to **Step 2**.

Corollary 1. Suppose that Assumption 2 holds. Then the sequence $\{x^n\}$ generated by Algorithm 3 converges strongly to the unique solution of the BVIP (5), provided the solution set $\mathrm{Sol}(C,G)$ of the VIP (6) is nonempty.

Assumption 3. Consider the functions f and g which satisfy the following conditions:

i) $f : \mathbb{R}^n \longrightarrow \mathbb{R}$ is continuously differentiable and strongly convex, and its gradient is Lipschitz continuous.

ii) $g : \mathbb{R}^n \longrightarrow \mathbb{R}$ is convex and continuously differentiable such that its gradient is Lipschitz continuous.

Assuming that all conditions stated in Assumption 3 are satisfied, we find that the gradient mapping $\nabla f : \mathbb{R}^n \longrightarrow \mathbb{R}^n$ is strongly monotone and Lipschitz continuous on $\mathbb{R}^n$. Similarly, $\nabla g : \mathbb{R}^n \longrightarrow \mathbb{R}^n$ is monotone and Lipschitz continuous on $\mathbb{R}^n$. By taking $F = \nabla f$, $G = \nabla g$, and $C = \mathbb{R}^n$ in Algorithm 3 and Corollary 1, we derive the following algorithm and corollary for the bilevel optimization problem:

$$\text{Find } x^* \in \Omega \text{ such that } f(x) \geq f(x^*), \quad \text{for all } x \in \Omega, \tag{49}$$

in which Ω represents the nonempty set of minimizers associated with the classical convex optimization problem $\min_{x \in \mathbb{R}^n} g(x)$.

Algorithm 4

Step 0. Choose $\lambda_0 > 0$, $\lambda \in (0,1)$, $\{\gamma_n\} \subset [0,\infty)$, $\{\xi_n\} \subset [0,\infty)$, $\{\eta_n\} \subset (0,\infty)$, $\{\varepsilon_n\} \subset (0,1)$ such that $\lim_{n\to\infty} \frac{\eta_n}{\varepsilon_n} = 0$, $\lim_{n\to\infty} \varepsilon_n = 0$, $\sum_{n=0}^{\infty} \varepsilon_n = \infty$.

Step 1. Let $x^{-2}, x^{-1}, x^0 \in \mathcal{H}_1$. Set $n := 0$.

Step 2. Compute $y^n = x^n + \alpha_n(x^n - x^{n-1}) + \beta_n(x^{n-2} - x^{n-1})$, where

$$\alpha_n = \begin{cases} \min\left\{\dfrac{\eta_n}{\|x^n - x^{n-1}\|}, \gamma_n\right\} & \text{if } x^n \neq x^{n-1}, \\ \gamma_n & \text{if } x^n = x^{n-1}, \end{cases}$$

and

$$\beta_n = \begin{cases} \min\left\{\dfrac{\eta_n}{\|x^{n-2} - x^{n-1}\|}, \xi_n\right\} & \text{if } x^{n-2} \neq x^{n-1}, \\ \xi_n & \text{if } x^{n-2} = x^{n-1}. \end{cases}$$

Step 3. Compute

$$z^n = y^n - \lambda_n \nabla g(y^n),$$
$$t^n = z^n - \lambda_n(\nabla g(z^n) - \nabla g(y^n)),$$

where

$$\lambda_{n+1} = \begin{cases} \min\left\{\dfrac{\lambda\|y^n - z^n\|}{\|\nabla g(y^n) - \nabla g(z^n)\|}, \lambda_n\right\} & \text{if } \nabla g(y^n) \neq \nabla g(z^n), \\ \lambda_n & \text{if } \nabla g(y^n) = \nabla g(z^n). \end{cases}$$

Step 6. Compute

$$x^{n+1} = t^n - \varepsilon_n \nabla f(t^n).$$

Step 7. Set $n := n + 1$, and go to **Step 2**.

Corollary 2. Assuming that Assumption 3 is satisfied. Then the sequence $\{x^n\}$ produced by Algorithm 4 converges strongly to the unique optimal solution of (49), given that the set Ω of all optimal solutions for the problem $\min_{x\in\mathbb{R}^n} g(x)$ is nonempty.

From Algorithm 1 and Theorem 1, by setting $F_1 = F_2 = 0$, we derive the following algorithm and corollary:

Algorithm 5

Step 0. Choose $\{\rho_n\} \subset [a, b] \subset (0, 1)$, $\{\gamma_n\} \subset [0, \infty)$, $\{\xi_n\} \subset [0, \infty)$, $\{\eta_n\} \subset (0, \infty)$, $\{\varepsilon_n\} \subset (0, 1)$ such that $\lim_{n\to\infty} \frac{\eta_n}{\varepsilon_n} = 0$, $\lim_{n\to\infty} \varepsilon_n = 0$, $\sum_{n=0}^{\infty} \varepsilon_n = \infty$.

Step 1. Let $x^{-2}, x^{-1}, x^0 \in \mathcal{H}_1$. Set $n := 0$.

Step 2. Compute $y^n = x^n + \alpha_n(x^n - x^{n-1}) + \beta_n(x^{n-2} - x^{n-1})$, where

$$\alpha_n = \begin{cases} \min\left\{\dfrac{\eta_n}{\|x^n - x^{n-1}\|}, \gamma_n\right\} & \text{if } x^n \neq x^{n-1}, \\ \gamma_n & \text{if } x^n = x^{n-1}, \end{cases}$$

and

$$\beta_n = \begin{cases} \min\left\{\dfrac{\eta_n}{\|x^{n-2} - x^{n-1}\|}, \xi_n\right\} & \text{if } x^{n-2} \neq x^{n-1}, \\ \xi_n & \text{if } x^{n-2} = x^{n-1}. \end{cases}$$

Step 3. Compute

$$\begin{cases} z^n = P_C(y^n), \quad u^n = A(z^n), \quad v^n = P_Q(u^n), \\ t^n = z^n + \delta_n A^*(v^n - u^n), \end{cases}$$

where the stepsize δ_n is chosen in such a way that

$$\delta_n = \begin{cases} \dfrac{\rho_n\|v^n - u^n\|^2}{\|A^*(v^n - u^n)\|^2} & \text{if } A^*(v^n - u^n) \neq 0, \\ 0 & \text{if } A^*(v^n - u^n) = 0. \end{cases}$$

Step 4. Compute

$$x^{n+1} = t^n - \varepsilon_n F(t^n).$$

Step 5. Set $n := n + 1$, and go to **Step 2**.

Corollary 3. Let C and Q be two nonempty closed convex subset of two real Hilbert spaces $\mathcal{H}_1$ and $\mathcal{H}_2$, respectively. Let $F : \mathcal{H}_1 \longrightarrow \mathcal{H}_1$ be a strongly monotone and Lipschitz continuous mapping. Then the sequence $\{x^n\}$ generated by Algorithm 5 converges strongly to $x^* \in \Gamma$, which is the unique solution of the VIP $\langle F(x^*), x - x^*\rangle \geq 0$, for all $x \in \Gamma$, provided the solution set $\Gamma = \{x^* \in C : Ax^* \in Q\}$ of the SFP is nonempty.

We now apply Corollary 3 with $F(x) = x$ for all $x \in \mathcal{H}_1$. It is clear that the identity mapping $F : \mathcal{H}_1 \longrightarrow \mathcal{H}_1$ is 1-Lipschitz continuous and 1-strongly monotone on $\mathcal{H}_1$. This leads us to the following result:

Corollary 4. Let C and Q be two nonempty closed convex subsets of two real Hilbert spaces $\mathcal{H}_1$ and $\mathcal{H}_2$, respectively. The sequence $\{x^n\}$ generated by Algorithm 5, in which step 4 specifies $x^{n+1} = (1 - \varepsilon_n)t^n$, converges strongly to the minimum-norm solution of the SFP, assuming that the solution set $\Gamma = \{x^* \in C : Ax^* \in Q\}$ is nonempty.

Next, we will analyze how Corollary 4 can be utilized in discrete optimal control problems.

Let A_i and B_i be real matrices of size $q \times q$ and $q \times p$, respectively, for $i = 0, 1, \ldots, N - 1$. We are examining a linear discrete optimal control problem

$$\begin{cases} x_{i+1} = A_{i+1}x_i + B_{i+1}u_i, & \\ u_i \in C_i, & i = 0, 1, \ldots, N-1, \\ x_0 = 0, & x_N \in \mathcal{Q}, \\ J(x, u) := \|u_0\|^2 + \|u_1\|^2 + \cdots + \|u_{N-1}\|^2 \longrightarrow \min\limits_{u_i}, & \end{cases} \tag{50}$$

where $C_i \subset \mathbb{R}^p$ for $i = 0, 1, \ldots, N - 1$, and $\mathcal{Q} \subset \mathbb{R}^q$ are nonempty closed convex subsets that define the control and state constraints, respectively.

Establish a matrix of dimension $q \times Np$

$$\mathcal{A} = [\mathcal{D}_0 \;\; \mathcal{D}_1 \;\; \ldots \;\; \mathcal{D}_{N-1}],$$

where $\mathcal{D}_i := A_N A_{N-1} \ldots A_{i+2} B_{i+1}, i = 0, 1, \ldots, N - 2$, and $\mathcal{D}_{N-1} = B_N$.

Let $u := (u_0, u_1, \ldots, u_{N-1})$, $\|u\|^2 := \|u_0\|^2 + \|u_1\|^2 + \cdots + \|u_{N-1}\|^2$ and $\mathcal{C} := C_0 \times C_1 \times \cdots \times C_{N-1}$. Then, (50) transforms into finding the minimum-norm solution of the following SFP:

$$\text{Find } u \in \mathcal{C} \text{ such that } \mathcal{A}u \in \mathcal{Q}.$$

Thus, we can utilize Algorithm 5, where step 4 is given by $x^{n+1} = (1 - \varepsilon_n)t^n$, to solve the problem.

4 Applications in production and consumption systems

A variation of the SVIP, defined by (1) and (2) and referred to as the SMNP, arises when each F_i is the identity mapping on $\mathcal{H}_i$ for all $i = 1, 2$. In the SMNP framework, the goal is to determine a solution $x^* \in C$ that minimizes its norm while ensuring that its image $y^* = Ax^*$ belongs to Q and also has the smallest possible norm. Mathematically, this is expressed as follows:

$$\text{Find } x^* \in C : \|x^*\| \leq \|x\| \quad \text{for all } x \in C$$

subject to the condition:

$$y^* = Ax^* \in Q : \|y^*\| \leq \|y\| \quad \text{for all } y \in Q.$$

In many practical applications, particularly in supply chain management and production planning, there is a need to achieve efficiency in both production and distribution. In this context, we consider a system where production and consumption are intrinsically linked via a linear transformation. Let $x \in \mathbb{R}^N$ represent the production vector, quantifying the goods produced, and let $y \in \mathbb{R}^M$ denote the consumption vector, representing the goods delivered to the market. The connection between production and consumption is modeled by the matrix $A \in \mathbb{R}^{M \times N}$, such that $y = Ax$. The production process is constrained by various operational factors, including capacity and resource limitations. These are encapsulated in the feasible set $C \subset \mathbb{R}^N$. For instance, one may define

$$C = \{x \in \mathbb{R}^N_+ : Bx \leq b\},$$

where the matrix B and the vector b represent production constraints such as available resources or maximum production capacities. On the other hand, the consumption or distribution process must satisfy market demand or quality requirements, which are modeled by the feasible set $Q \subset \mathbb{R}^M$. One common formulation is

$$Q = \{y \in \mathbb{R}^M_+ : y \geq d\},$$

with d being the vector of minimum demand requirements ensuring that the market receives at least the prescribed quantities.

In the production set C, selecting x^* with the smallest norm is crucial because it ensures that among all feasible production plans, x^* consumes the least resources or incurs the lowest production cost. This minimality directly translates into enhanced efficiency in production. Similarly, in the consumption set Q, requiring that $y^* = Ax^*$ has the smallest norm means that the corresponding distribution of goods is accomplished with minimal overhead or waste. This condition is essential for achieving an efficient distribution process. Thus, the overall objective is to select a production plan $x^* \in C$ that minimizes the production norm:

$$\|x^*\| \leq \|x\| \quad \text{for all } x \in C,$$

thereby reducing production costs, resource usage, or energy consumption. Simultaneously, the corresponding consumption vector $y^* = Ax^*$ must belong to Q and minimize the consumption norm:

$$\|y^*\| \leq \|y\| \quad \text{for all } y \in Q.$$

The SMNP model provides an integrated framework for addressing the challenges of simultaneously optimizing production and distribution. By merging the operational constraints of production with the market's consumption requirements and enforcing minimal norm conditions, the SMNP formulation successfully reduces costs while enhancing overall supply chain efficiency.

5 Numerical illustration

In this section, we present numerical experiments to assess the performance of the proposed algorithms and provide results from various comparisons. All Python code was executed on a 2017 MacBook Pro featuring a 2.3 GHz Intel Core i5 processor, an Intel Iris Plus Graphics 640 GPU with 1536 MB of memory, and 8 GB of 2133 MHz LPDDR3 RAM. The experiments were conducted using Python version 3.11.

Example 1. (see [25, Example 4.1]). Let $\mathbb{R}^K$ be equipped with the standard norm $\|x\| = \sqrt{x_1^2 + x_2^2 + \cdots + x_K^2}$ for all $x = (x_1, x_2, \ldots, x_K)^T \in \mathbb{R}^K$. Let

$A(x) = (x_1 + x_3 + x_4, x_2 + x_3 - x_4)^T$ for all $x = (x_1, x_2, x_3, x_4)^T \in \mathbb{R}^4$. This shows that A is a bounded linear operator from $\mathbb{R}^4$ into $\mathbb{R}^2$.

Now, define the set

$$C = \{(x_1, x_2, x_3, x_4)^T \in \mathbb{R}^4 : x_1 - 3x_2 - 2x_3 + x_4 \geq -2\},$$

and let the mapping $F_1 : \mathbb{R}^4 \longrightarrow \mathbb{R}^4$ be defined by $F_1(x) = (\sin \|x\| + 4)b^0$ for all $x \in \mathbb{R}^4$, where $b^0 = (1, -3, -2, 1)^T \in \mathbb{R}^4$. It is easy to verify that F_1 is pseudomonotone and Lipschitz continuous on $\mathbb{R}^4$.

Now, let $Q = \{(u_1, u_2)^T \in \mathbb{R}^2 : u_1 - 2u_2 \geq -1\}$, and define another mapping $F_2 : \mathbb{R}^2 \longrightarrow \mathbb{R}^2$ by $F_2(u) = (\sin \|u\| + 2)c^0$ for all $u \in \mathbb{R}^2$, where $c^0 = (1, -2)^T \in \mathbb{R}^2$. Similarly, F_2 is pseudomonotone and Lipschitz continuous on $\mathbb{R}^2$.

Consider the mapping $F : \mathbb{R}^4 \longrightarrow \mathbb{R}^4$ defined by $F(x) = 2x + a^0$ for all $x \in \mathbb{R}^4$, where $a^0 = (-2, 0, 4, -6)^T \in \mathbb{R}^4$. It is straightforward to verify that F is strongly monotone and Lipschitz continuous on $\mathbb{R}^4$. In [25], the authors demonstrated that the unique solution to the BSVIP (4) is given by $x^* = \left(\frac{4}{27}, \frac{44}{27}, -\frac{11}{9}, \frac{8}{27}\right)^T$.

Table 1: A comparison between Algorithm 1 and [25, Algorithm 3.1] with different tolerances ε and the stopping criterion $\|x^n - x^*\| \leq \varepsilon$

	$\varepsilon = 10^{-3}$	
	Iter(n)	CPU time(s)
Algorithm 1	9945	1.7804
[25, Algorithm 3.1]	14611	2.4233
	$\varepsilon = 10^{-4}$	
	Iter(n)	CPU time(s)
Algorithm 1	99490	19.1084
[25, Algorithm 3.1]	146159	24.8806

We will now assess the performance of Algorithm 1 in comparison to [25, Algorithm 3.1], as outlined in [25]. Both algorithms use the termination criterion $\|x^n - x^*\| \leq \varepsilon$ and start with the same initial point, x^0, where its components are randomly generated within the closed interval $[-10, 10]$. Additionally, for Algorithm 1, the components of the initial points x^{-2} and x^{-1} are also randomly selected from the same interval. The parameters for each algorithm are specified as follows:

- Algorithm 1: $\lambda_0 = 3$, $\mu_0 = 2$, $\lambda = 0.3$, $\mu = 0.4$, $\gamma_n = 0.1$, $\xi_n = 0.2$, $\rho_n = 0.99$, $\eta_n = \frac{1}{(n+2)^{1.01}}$ and $\varepsilon_n = \frac{1}{n+2}$.
- [25, Algorithm 3.1]: $\lambda_0 = 3$, $\mu_0 = 2$, $\lambda = 0.3$, $\mu = 0.4$ and $\varepsilon_n = \frac{1}{n+2}$.

The results presented in Table 1 indicate that Algorithm 1 outperforms [25, Algorithm 3.1] in terms of both runtime and iteration count.

Example 2. Let $a^0 = (1, -6, -3, 2, -3, 6, -1, -2)^T \in \mathbb{R}^8$, and consider the set C defined as $C = \{x = (x_1, x_2, \ldots, x_8)^T \in \mathbb{R}^8 : \langle a^0, x\rangle \geq -2\}$. Now, let us define a mapping $G : \mathbb{R}^8 \longrightarrow \mathbb{R}^8$ by $G(x) = (\sin\|x\| + 4)a^0$ for all $x \in \mathbb{R}^8$. It can be easily verified that G is pseudomonotone on $\mathbb{R}^8$ and Lipschitz continuous on $\mathbb{R}^8$. Furthermore, it is evident that the solution set $\mathrm{Sol}(C, G)$ of the VIP $VIP(C, G)$ is given by

$$\mathrm{Sol}(C, G) = \{x = (x_1, x_2, \ldots, x_8)^T \in \mathbb{R}^8 : \langle a^0, x\rangle = -2\}.$$

Let us consider the mapping $F : \mathbb{R}^8 \longrightarrow \mathbb{R}^8$ defined as $F(x) = x$ for all $x \in \mathbb{R}^8$. This mapping F is strongly monotone with $\eta = 1$ and Lipschitz continuous with $L = 1$ on $\mathbb{R}^8$. In this context, problem (5) transforms into finding the minimum-norm solution of the $VIP(C, G)$. The resulting minimum-norm solution x^* for the $VIP(C, G)$ is $x^* = P_{\mathrm{Sol}(C,G)}(0) = (-0.02, 0.12, 0.06, -0.04, 0.06, -0.12, 0.02, 0.04)^T$.

We are set to compare the performance of Algorithm 3 with [25, Algorithm 3.6], as presented in [25], for solving the BVIP problem (5). Both algorithms start with the same initial point, x^0, whose components are randomly generated within the closed interval $[-10, 10]$, and both use the termination criterion $\|x^n - x^*\| \leq \varepsilon$. Additionally, for Algorithm 3, the components of the initial points x^{-2} and x^{-1} are also randomly chosen from the same closed interval $[-10, 10]$. The parameter settings for these methods are as follows:

Table 2: A comparison between Algorithm 3 and [25, Algorithm 3.6] with different tolerances ε and the stopping criterion $\|x^n - x^*\| \leq \varepsilon$

	$\varepsilon = 10^{-3}$	
	Iter(n)	CPU time(s)
Algorithm 3	1343	0.1411
[25, Algorithm 3.6]	13359	1.1035
	$\varepsilon = 10^{-4}$	
	Iter(n)	CPU time(s)
Algorithm 3	12969	1.1198
[25, Algorithm 3.6]	133599	10.9758

- Algorithm 3: $\lambda_0 = 3$, $\lambda = 0.6$, $\gamma_n = 10^4$, $\xi_n = 10^{-2}$, $\eta_n = \dfrac{1}{(n+2)^{1.01}}$ and $\varepsilon_n = \dfrac{1}{n+2}$.
- [25, Algorithm 3.6]: $\lambda_0 = 3$, $\lambda = 0.6$ and $\varepsilon_n = \dfrac{1}{n+2}$.

The results shown in Table 2 suggest that Algorithm 3 demonstrates superior performance when compared to [25, Algorithm 3.6].

Example 3. Let $\mathcal{H}_1 = \mathbb{R}^K$ and let $\mathcal{H}_2 = \mathbb{R}^L$, where $K = 200$ and $L = 150$. We consider the SFP with the sets $C = \{x \in \mathbb{R}^K : \langle c, x\rangle \geq 0\}$, $Q = \{y \in \mathbb{R}^L : \langle q, y\rangle \geq 0\}$ and the bounded linear operator $A : \mathbb{R}^K \longrightarrow \mathbb{R}^L$ defined by $A(x) = Mx$ for all $x \in \mathbb{R}^K$, where M is an $L \times K$ real matrix. We generate the elements of M randomly within the closed interval $[-10, 10]$, and the coordinates of c and q within the closed interval $[2, 10]$. It is straightforward to observe that $0 \in C$ and $A(0) = 0 \in Q$. Therefore, $0 \in \Gamma = \{x^* \in C : Ax^* \in Q\}$. Thus, the minimum-norm solution x^* of the SFP is $x^* = 0$.

We aim to compare the performance of Algorithm 5, where F is the identity mapping, with the algorithm described in [18, Corollary 3.2] for solving the minimum-norm solution of the SFP. Both algorithms begin with the same initial point, x^0, whose components are randomly generated within the closed interval $[-10, 10]$. They also both use the same stopping criterion,

$\|x^n - x^*\| \leq \varepsilon$ and the same $\varepsilon_n = \dfrac{1}{n+2}$ (in [18, Corollary 3.2], this is denoted as α_n). Additionally, in Algorithm 5, the components of the initial points x^{-2} and x^{-1} are also randomly selected from the same closed interval $[-10, 10]$. The parameter values in Algorithm 5 are chosen as $\gamma_n = 10^6$, $\xi_n = 10^{-4}$, $\rho_n = 0.99$, and $\eta_n = \dfrac{1}{(n+2)^{1.01}}$.

Table 3: A comparison between Algorithm 5, where F is the identity mapping, and the algorithm described in [18, Corollary 3.2], with different tolerances ε and the stopping criterion $\|x^n - x^*\| \leq \varepsilon$

	$\varepsilon = 10^{-3}$	
	Iter(n)	CPU time(s)
Algorithm 5	189	0.0148
Algorithm in [18, Corollary 3.2]	79652	4.9645
	$\varepsilon = 10^{-4}$	
	Iter(n)	CPU time(s)
Algorithm 5	2498	0.1653
Algorithm in [18, Corollary 3.2]	776266	45.2463

Table 3 illustrates that our Algorithm 5 significantly outperforms the algorithm in [18, Corollary 3.2] in terms of both iteration count and CPU time.

6 Conclusions

This paper presented an iterative algorithm for addressing BSVIPs. We established that the iterative sequence strongly converges to the unique solution of the BSVIP without needing to compute or estimate the norm of a bounded linear operator. Moreover, the algorithm can be implemented without requiring any calculations or estimations of the Lipschitz and strongly monotone constants of the mappings involved. We also applied this algorithm to specific cases, including the bilevel VIPs, the bilevel optimization problems, and strongly monotone VIPs with split feasibility constraints. Finally, we provided an application of the SMNP in production and consumption systems and presented several numerical experiments to demonstrate the implementability of the proposed algorithms.

As a potential direction for future research, it would be interesting to investigate the extension of our results to Banach spaces. This generalization may present new challenges, particularly in handling the lack of Hilbert space structure, but it could also broaden the applicability of our approach to a wider class of problems.

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References

[1] Anh, P.K., Anh, T.V., and Muu, L.D. *On bilevel split pseudomonotone variational inequality problems with applications*, Acta Math. Vietnam, 42 (2017), 413–429.

[2] Anh, P.N., Kim, J.K., and Muu, L.D. *An extragradient algorithm for solving bilevel pseudomonotone variational inequalities*, J. Glob. Optim., 52 (2012), 627–639.

[3] Anh, T.V. *A parallel method for variational inequalities with the multiple-sets split feasibility problem constraints*, J. Fixed Point Theory Appl., 19 (2017), 2681–2696.

[4] Anh, T.V. *A strongly convergent subgradient extragradient-Halpern method for solving a class of bilevel pseudomonotone variational inequalities*, Vietnam J. Math., 45 (2017), 317–332.

[5] Buong, N. *Iterative algorithms for the multiple-sets split feasibility problem in Hilbert spaces*, Numer. Algorithms, 76 (2017), 783–798.

[6] Ceng, L.C., Ansari, Q.H., and Yao, J.C. *Some iterative methods for finding fixed points and for solving constrained convex minimization problems*, Nonlinear Anal., 74 (2011), 5286–5302.

[7] Ceng, L.C., Ansari, Q.H., and Yao, J.C. *An extragradient method for solving split feasibility and fixed point problems*, Comput. Math. Appl., 64 (2012), 633–642.

[8] Ceng, L.C., Ansari, Q.H., and Yao, J.C. *Relaxed extragradient methods for finding minimum-norm solutions of the split feasibility problem*, Nonlinear Anal., 75 (2012), 2116–2125.

[9] Ceng, L.C., Coroian, I., Qin, X., and Yao, J.C. *A general viscosity implicit iterative algorithm for split variational inclusions with hierarchical variational inequality constraints*, Fixed Point Theory, 20 (2019), 469–482.

[10] Ceng, L.C., Liou, Y.C., and Sahu, D.R. *Multi-step hybrid steepest-descent methods for split feasibility problems with hierarchical variational inequality problem constraints*, J. Nonlinear Sci. Appl., 9 (2016), 4148–4166.

[11] Ceng, L.C., Wong, N.C., and Yao, J.C. *Hybrid extragradient methods for finding minimum-norm solutions of split feasibility problems*, J. Nonlinear Convex Anal., 16 (2015), 1965–1983.

[12] Censor, Y., Bortfeld, T., Martin, B., et al. *A unified approach for inversion problems in intensity-modulated radiation therapy*, Phys. Med. Biol., 51 (2006), 2353–2365.

[13] Censor, Y., and Elfving, T. *A multiprojection algorithm using Bregman projections in a product space*, Numer. Algorithms, 8 (1994), 221–239.

[14] Censor, Y., Elfving, T., Kopf, N. and Bortfeld, T. *The multiple-sets split feasibility problem and its applications for inverse problems*, Inverse Prob., 21 (2005), 2071–2084.

[15] Censor, Y., Gibali, A., and Reich, S. *Algorithms for the split variational inequality problem*, Numer. Algorithms, 59 (2012), 301–323.

[16] Censor, Y., and Segal, A. *Iterative projection methods in biomedical inverse problems*, in: Censor, Y., Jiang, M., and Louis, A.K., eds., *Mathematical methods in biomedical imaging and intensity-modulated therapy*, Edizioni della Norale, Pisa, Italy, IMRT, (2008) 65–96.

[17] Combettes, P.L. and Hirstoaga, S.A. *Equilibrium Programming in Hilbert Spaces*, J. Nonlinear Convex Anal., 6 (2005), 117–136.

[18] Cuong, T.L., Anh, T.V., and Van, L.H.M. *A self-adaptive step size algorithm for solving variational inequalities with the split feasibility problem with multiple output sets constraints*, Numer. Funct. Anal. Optim., 43 (2022), 1009–1026.

[19] Daniele, P., Giannessi, F., and Maugeri, A. *Equilibrium problems and variational models*, Dordrecht, Kluwer Academic, 2003.

[20] Dempe, S. *Foundations of bilevel programming*, Dordrecht, Kluwer Academic Press, 2002.

[21] Facchinei, F., and Pang, J.S. *Finite-dimensional variational inequalities and complementarity problems*, Berlin, Springer, 2002.

[22] Giannessi, F., Maugeri, A., and Pardalos, P.M. *Equilibrium problems: nonsmooth optimization and variational inequality models*, Dordrecht, Kluwer Academic, 2004.

[23] He, S., Zhao, Z., and Luo, B. *A relaxed self-adaptive CQ algorithm for the multiple-sets split feasibility problem*, Optimization, 64 (2015), 1907–1918.

[24] Huy, P.V., Hien, N.D., and Anh, T.V. *A strongly convergent modified Halpern subgradient extragradient method for solving the split variational inequality problem*, Vietnam J. Math., 48 (2020), 187–204.

[25] Huy, P.V., Van, L.H.M., Hien, N.D., and Anh, T.V. *Modified Tseng's extragradient methods with self-adaptive step size for solving bilevel split variational inequality problems*, Optimization, 71 (2022), 1721–1748.

[26] Iyiola, O. S., and Shehu, Y. *Convergence results of two-step inertial proximal point algorithm*, Appl. Numer. Math., 182 (2022), 57–75.

[27] Izuchukwu, C., Aphane, M., and Aremu, K. O. *Two-step inertial forward–reflected–anchored–backward splitting algorithm for solving monotone inclusion problems*, Comput. Appl. Math., 42 (2023), 351.

[28] Kinderlehrer, D., and Stampacchia, G. *An introduction to variational inequalities and their applications*, New York, Academic,; 1980.

[29] Konnov, I.V. *Combined Relaxation Methods for Variational Inequalities*, Springer, Berlin (2000).

[30] Korpelevich, G.M. *The extragradient method for finding saddle points and other problems*, Ekonomikai Matematicheskie Metody, 12 (1976), 747–756.

[31] Maingé, P.E. *A hybrid extragradient-viscosity method for monotone operators and fixed point problems*, SIAM J. Control Optim., 47 (2008), 1499–1515.

[32] Okeke, C. C., Jolaoso, L. O., and Shehu, Y. *Inertial accelerated algorithms for solving split feasibility with multiple output sets in Hilbert spaces*, Int. J. Nonlinear Sci. Numer. Simul., 24 (2021), 769–790.

[33] Polyak, B. T. *Some methods of speeding up the convergence of iteration methods*, USSR Comput. Math. Math. Phys., 4 (1964), 1–17.

[34] Raeisi, M., Zamani Eskandani, G., and Eslamian, M. *A general algorithm for multiple-sets split feasibility problem involving resolvents and Bregman mappings*, Optimization, 67 (2018), 309–327.

[35] Rudin, W. *Functional Analysis*, 2nd ed., McGraw-Hill, New York, 1991.

[36] Shehu, Y. *Strong convergence theorem for multiple sets split feasibility problems in Banach spaces*, Numerical Funct. Anal. Optim., 37 (2016), 1021–1036.

[37] Solodov, M.V. *An explicit descent method for bilevel convex optimization*, J. Convex Anal., 14 (2007), 227–237.

[38] Thong, D.V., and Hieu, D.V. *A strong convergence of modified subgradient extragradient method for solving bilevel pseudomonotone variational inequality problems*, Optimization, 69 (2020), 1313–1334.

[39] Tseng, P. *A modified forward–backward splitting method for maximal monotone mappings*, SIAM J. Control Optim., 38 (2000), 431–446.

[40] Uzor, V., Alakoya, T., and Mewomo, O. T. *On split monotone variational inclusion problem with multiple output sets with fixed point constraints*, Comput. Methods Appl. Math., 23 (2023), 729–749.

[41] Xu, H.K. *Iterative algorithms for nonlinear operators*, J. London Math. Soc., 66 (2002), 240–256.

[42] Yao, Y., Marino, G., and Muglia, L. *A modified Korpelevich's method convergent to the minimum-norm solution of a variational inequality*, Optimization, 63 (2014), 559–569.

[43] Yao, Y., Postolache, M., and Zhu, Z. *Gradient methods with selection technique for the multiple-sets split feasibility problem*, Optimization, 69 (2020), 269–281.

[44] Zegeye, H., Shahzad, N., and Yao, Y. *Minimum-norm solution of variational inequality and fixed point problem in Banach spaces*, Optimization, 64 (2015), 453–471.