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Research Article

Combining the reproducing kernel method with Taylor series expansion to solve systems of nonlinear fractional Volterra integro-differential equations

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Abstract

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In this article, we present a novel approach for solving systems of nonlinear fractional Volterra integro-differential equations (NFVI-DEs) by reproducing the Hilbert kernel method. Kernel methods are powerful tools for addressing both linear and nonlinear problems. The reproducing kernel method stands out for its wide-ranging applications in solving complex scientific challenges. Our method combines the reproducing kernel method with a truncated Taylor series expansion, resulting in a more precise solution. This transformation converts the original NFVI-DEs into a system of nonlinear fractional differential equations. Our numerical results showcase this approach's effectiveness and align with theorems about error analysis.

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Keywords: Reproducing kernel method; Fractional Volterra integro-differential equations; Taylor series expansion; Error analysis.

1 Introduction

In the fields of physics, chemistry, biology, and other sciences, many phenomena can be accurately modeled by systems of nonlinear fractional-order Volterra integro-differential equations [1, 11, 13, 18]. Over the years, numerous scientists have attempted to solve these complex equations using various numerical methods, including the discrete Adomian decomposition method, perturbation-based approaches, the Chebyshev wavelet method, the Chebyshev spectral method, block-pulse functions, wavelet methods, the Legendre wavelet method, and multi-step collocation methods [8, 10, 16, 17, 22, 23, 25, 27].

Kernel methods are powerful techniques for solving linear and nonlinear problems. Notably, the reproducing kernel method (RKM) has many applications in tackling challenging scientific problems. Over the past decade, the RKM combined with the Gram-Schmidt orthogonalization process (G-SOP) has been widely used to solve systems of integral equations [15, 26]. However, recent trends have shifted toward using the RKM without the G-SOP due to its advantages, such as easier implementation, lower computational cost, and higher accuracy [19, 21]. Furthermore, a new RKM-based approach that omits the G-SOP has been developed to solve a wide range of equations, including linear and nonlinear differential equations, integral equations, and systems of fractional-order Volterra integro-differential equations [5, 6, 4].

The RKM relies on several key components: the space, points, inner product, bases, and the chosen solution method. By adjusting these components to suit the specific problem, one can solve complex problems effectively. However, certain problems cannot be resolved simply by modifying these components. In such cases, an innovative approach is required to enhance the numerical results. One such approach involves combining the RKM with a Taylor series expansion. Alvandi and Paripour [2, 3] successfully applied this combined method to solve linear and nonlinear Volterra integro-differential equations. By employing the Taylor series expansion, they transformed the integro-differential equations into a system of differential equations, yielding more accurate numerical solutions.

In this paper, we present a novel method for solving systems of nonlinear fractional Volterra integro-differential equations (NFVI-DEs). Our approach combines the RKM without the G-SOP with Taylor series expansion. Additionally, we address cases where the approximate solution exhibits significant errors without this combined approach. For such problems, we apply Volterra's integral to the nonlinear component and replace it with a Taylor series expansion. This substitution substantially improves numerical accuracy while avoiding the need for complete transformation into a system of nonlinear fractional differential equations (NFDEs). Furthermore, we compare our method with the wavelet method [23], with numerical results demonstrating the superior effectiveness of our approach.

This article is structured as follows: In section 2, we introduce the concept of space and then proceed to prove the basic theorem and lemmas. Next, we present a new algorithm that utilizes linear algebra techniques. In section 3, we evaluate the error of this method. In section 4, we provide four examples that have been solved using this method and demonstrate its efficiency in terms of numerical results compared to other methods. Finally, we conclude in the last section.

Consider the following systems of NFVI-DEs for $\tau \in [0, 1]$:

$$\begin{cases} L_{11}\gamma_1(\tau) + L_{12}\gamma_2(\tau) = g_1(\tau) - \lambda_1(\tau, \boldsymbol{\gamma}(\tau), \boldsymbol{\gamma}'(\tau), \int_a^\tau k_1(x, \boldsymbol{\gamma}(x), \boldsymbol{\gamma}'(x))\, dx), \\ \\ L_{21}\gamma_1(\tau) + L_{22}\gamma_2(\tau) = g_2(\tau) - \lambda_2(\tau, \boldsymbol{\gamma}(\tau), \boldsymbol{\gamma}'(\tau), \int_a^\tau k_2(x, \boldsymbol{\gamma}(x), \boldsymbol{\gamma}'(x))\, dx), \\ \\ \gamma_i(0) = \theta_i, \qquad i = 1, 2. \end{cases} \tag{1}$$

The operators $L_{i,j}$, $i, j = 1, 2$, and $\theta_d(\cdot, \cdot)$ are linear and nonlinear operators, respectively. Additionally, $g_d(\cdot)$ are predetermined functions for $d = 1,\ 2$, and $\boldsymbol{\gamma}(\cdot) = (\gamma_1(\cdot), \gamma_2(\cdot))^T$ are unknown vector functions to be determined.

In (1), we suppose

$$\begin{cases} L_{11}\gamma_1(\tau) = D^\alpha \gamma_1(\tau) - a_{11}\gamma_1(\tau) - \int_0^\tau k_{11}(\tau, x)\gamma_1(x)\, dx, \\ \\ L_{12}\gamma_2(\tau) = -a_{12}\gamma_2(\tau) - \int_0^\tau k_{12}(\tau, x)\gamma_2(x)\, dx, \\ \\ L_{21}\gamma_1(\tau) = -a_{21}\gamma_1(\tau) - \int_0^\tau k_{21}(\tau, x)\gamma_1(x)\, dx, \\ \\ L_{22}\gamma_2(\tau) = D^\beta(\tau)\gamma_2(\tau) - a_{22}\gamma_2(\tau) - \int_0^\tau k_{22}(\tau, x)\gamma_2(x)\, dx. \end{cases}$$

Suppose that $0 < \alpha, \beta \leq 1$, $D^\alpha \gamma_1(\tau)$ and $D^\beta \gamma_2(\tau)$ represent Caputo fractional derivatives. Additionally, $a_{ij}(\cdot)$ are given functions for $i, j = 1, 2$. In the nonlinear part of (1), we utilize a truncated Taylor series expansion centered at the point x within the interval $[0, 1]$ instead of using $\gamma(\tau)$ and $\gamma'(\tau)$. Therefore, we obtain the following:

$$\begin{cases} L_{11}\gamma_1(\tau) + L_{12}\gamma_2(\tau) = g_1(\tau) \\ \qquad -\lambda_1(\tau, \boldsymbol{\gamma}(\tau), \boldsymbol{\gamma}'(\tau), \int_a^\tau k_1(x, \sum\limits_{k=0}^{m} \frac{\boldsymbol{\gamma}^k(\tau)(x-\tau)^{(k)}}{k!} \\ \qquad , \sum\limits_{k=1}^{m} \frac{\boldsymbol{\gamma}^{(k)}(\tau)(x-\tau)^{k-1}}{(k-1)!})\, dx), \\ L_{21}\gamma_1(\tau) + L_{22}\gamma_2(\tau) = g_2(\tau) \\ \qquad -\lambda_2(\tau, \boldsymbol{\gamma}(\tau), \boldsymbol{\gamma}'(\tau), \int_a^\tau k_2(x, \sum\limits_{k=0}^{m} \frac{\boldsymbol{\gamma}^{(k)}(\tau)(x-\tau)^k}{k!} \\ \qquad , \sum\limits_{k=1}^{m} \frac{\boldsymbol{\gamma}^{(k)}(\tau)(x-\tau)^{k-1}}{(k-1)!})\, dx), \\ \gamma_i(0) = \theta_i, \qquad i = 1, 2, \end{cases} \tag{2}$$

where $\boldsymbol{\gamma}^{(0)}(\tau) = \boldsymbol{\gamma}(\tau)$ and $\int_a^\tau k_2(x, \sum_{k=0}^{m} \frac{\boldsymbol{\gamma}^{(k)}(\tau)(x-\tau)^k}{k!}, \sum_{k=1}^{m} \frac{\boldsymbol{\gamma}^{(k)}(\tau)(x-\tau)^{k-1}}{(k-1)!})\, dx)$ in term of $\boldsymbol{\gamma}(\tau)$ and its derivatives are computable. Therefore, let

$$\begin{cases} H_1(\tau, \boldsymbol{\gamma}(\tau), \boldsymbol{\gamma}'(\tau), \ldots, \boldsymbol{\gamma}^{(\boldsymbol{m})}(\tau)) \\ = \lambda_1(\tau, \boldsymbol{\gamma}(\tau), \boldsymbol{\gamma}'(\tau), \int_a^\tau k_1(x, \sum_{k=0}^{m} \frac{\boldsymbol{\gamma}^{k}(\tau)(x-\tau)^{(k)}}{k!}, \sum_{k=1}^{m} \frac{\boldsymbol{\gamma}^{(k)}(\tau)(x-\tau)^{k-1}}{(k-1)!})\, dx), \\ H_2(\tau, \boldsymbol{\gamma}(\tau), \boldsymbol{\gamma}'(\tau), \ldots, \boldsymbol{\gamma}^{(\boldsymbol{m})}(\tau)) \\ = \lambda_2(\tau, \boldsymbol{\gamma}(\tau), \boldsymbol{\gamma}'(\tau), \int_a^\tau k_2(x, \sum_{k=0}^{m} \frac{\boldsymbol{\gamma}^{(k)}(\tau)(x-\tau)^{k}}{k!}, \sum_{k=1}^{m} \frac{\boldsymbol{\gamma}^{(k)}(\tau)(x-\tau)^{k-1}}{(k-1)!})\, dx). \end{cases}$$

Eventually, we can write

$$\begin{cases} L_{11}\gamma_1(\tau) + L_{12}\gamma_2(\tau) = g_1(\tau) - H_1(\tau, \boldsymbol{\gamma}(\tau), \boldsymbol{\gamma}'(\tau), \ldots, \boldsymbol{\gamma}^{(\boldsymbol{m})}(\tau)), \\ L_{21}\gamma_1(\tau) + L_{22}\gamma_2(\tau) = g_2(\tau) - H_2(\tau, \boldsymbol{\gamma}(\tau), \boldsymbol{\gamma}'(\tau), \ldots, \boldsymbol{\gamma}^{(\boldsymbol{m})}(\tau)), \\ \gamma_i(0) = \theta_i, \qquad i = 1, 2. \end{cases} \tag{3}$$

Using matrix notation, we define the linear operator $\boldsymbol{L}$ as

$$\boldsymbol{L} = \begin{pmatrix} L_{11} & L_{12} \\ L_{21} & L_{22} \end{pmatrix},$$

and with $\boldsymbol{G} = (g_1, g_2)$, $\boldsymbol{H} = (H_1, H_2)$, so (2) can be written in the following form:

$$\begin{cases} \boldsymbol{L}(\boldsymbol{\gamma}(\tau)) = \boldsymbol{G}(\tau) - \boldsymbol{H}(\tau, \boldsymbol{\gamma}(\tau), \boldsymbol{\gamma}'(\tau), \ldots, \boldsymbol{\gamma}^{(\boldsymbol{m})}(\tau)), \qquad 0 < \tau \leq 1, \\ \gamma_i(0) = \theta_i,\ i = 1, 2. \end{cases} \tag{4}$$

In the nonlinear case, where $\boldsymbol{H} \neq 0$, we will examine (4) using the following iterative scheme:

$$\boldsymbol{L}(\boldsymbol{\gamma}_n(\tau)) = \boldsymbol{G}(\tau) - \boldsymbol{H}(\tau, \boldsymbol{\gamma}_{n-1}(\tau), \boldsymbol{\gamma}'_{n-1}(\tau), \ldots, \boldsymbol{\gamma}^{(m)}_{n-1}(\tau)), \qquad n = 2, 3, \ldots, \tag{5}$$

with $\boldsymbol{L}(\boldsymbol{\gamma}_1(\tau)) = \boldsymbol{G}(\tau)$; see [9] for more details.

Definition 1.1. [20] The Caputo fractional derivative operator of order $\alpha > 0$, is

$$D^\alpha u(\tau) = \frac{1}{\Gamma(z-\alpha)} \int_0^\tau (\tau - x)^{z-\alpha-1} u^{(z)}(x)\, dx,\ \tau > 0,$$

where $z - 1 < \alpha < z, z \in \mathbb{N}$.

2 Main idea

In this section, we will introduce the space and then proceed to prove the basic theorem and lemmas. Additionally, we will present a new algorithm that utilizes linear algebra techniques. Now, we consider the Hilbert space

$$W_2^k[0,1] = \left\{x|x^{(k-1)} \text{ is absolutely continuous}, x^{(k)} \in L^2[0,1],\ x(0) = 0\right\},$$

with the inner product and norm as follows:

$$\langle x(\cdot), y(\cdot)\rangle_{W_2^k} = \sum_{i=0}^{k-1} x^{(i)}(0)\, y^{(i)}(0) + \int_0^1 x^{(k)}(\tau) y^{(k)}(\tau) d\tau,$$

$$\|x(\cdot)\|_{\mathcal{W}_2^k} = \sqrt{\langle x, x\rangle_{W_2^k}}, \quad x(\cdot), y(\cdot) \in W_2^k[0,1],$$

where k is a natural number. Also, we consider the Hilbert space

$$\boldsymbol{W}_2^6[0,1] = W_2^6[0,1] \oplus W_2^6[0,1],$$

with the inner product and norm

$$\langle \boldsymbol{x}, \boldsymbol{y}\rangle_{\boldsymbol{W}_2^6} = \langle x_1, y_1\rangle_{W_2^6} + \langle x_2, y_2\rangle_{W_2^6}, \qquad \|\boldsymbol{x}\|_{\boldsymbol{W}_2^6} = \left(\sum_{i=1}^{2} \|x_i\|_{W_2^6}^2\right)^{1/2},$$

where $\boldsymbol{x} = (x_1, x_2)^T$, $\boldsymbol{y} = (y_1, y_2)^T$, $x_i, y_i \in W_2^6[0,1]$.

Lemma 2.1. If $L_{i,j} : W_2^6[0,1] \to W_2^1[0,1]$ in (1) are bounded linear operators, then $\boldsymbol{L} : \boldsymbol{W}_2^6[0,1] \to \boldsymbol{W}_2^1[0,1]$ is a bounded linear operator, where

$$\boldsymbol{L} = \begin{pmatrix} L_{11} & L_{12} \\ L_{21} & L_{22} \end{pmatrix},$$

and the boundedness of L_{ij} implies that $\boldsymbol{L}$ is bounded, also the adjoint operator of $\boldsymbol{L}$ is

$$\boldsymbol{L}^* = \begin{pmatrix} L_{11}^* & L_{12}^* \\ L_{21}^* & L_{22}^* \end{pmatrix},$$

where L_{ij}^* is the adjoint operator of L_{ij}, [12]. Indeed, according to (5), we have

$$\boldsymbol{\gamma} \in \boldsymbol{W}_2^6[0,1], \qquad \boldsymbol{G} - \boldsymbol{H} \in \boldsymbol{W}_2^1[0,1].$$

Lemma 2.2. The spaces $W_2^4[0,1]$, $W_2^5[0,1]$, $W_2^6[0,1]$, and $W_2^1[0,1]$ are all reproducing kernel Hilbert spaces, with their respective reproducing kernels listed in Table 1.

Table 1: The reproducing kernels in the $W_2^k[0,1]$ space.

k	$W_2^k[0,1]$
1	$\boldsymbol{\kappa}_y(\tau) = \begin{cases} 1+y, & \tau \ge y, \\ 1+\tau, & \tau < y. \end{cases}$
4	$\boldsymbol{\kappa}_y(\tau) = \begin{cases} -(\tau^7/5040) + \tau y + (\tau^6 y)/720 + (\tau^2 y^2)/4 - (\tau^5 y^2)/240 + (\tau^3 y^3)/36 + (\tau^4 y^3)/144, & \tau \ge y, \\ y^7/5040 + 1/144\tau^3(4y^3+y^4) + 1/240\tau^2(60y^2 - y^5) + 1/720\tau(720y + y^6), & \tau < y. \end{cases}$
5	$\boldsymbol{\kappa}_y(\tau) = \begin{cases} \tau^9/362880 + \tau y - (\tau^8 y)/40320 + (\tau^2 y^2)/4 + (\tau^7 y^2)/10080 + (\tau^3 y^3)/36 - (\tau^6 y^3)/4320 & \\ \quad +(\tau^4 y^4)/576 + (\tau^5 y^4)/2880, & \tau \ge y, \\ y^9/362880 + (\tau^4(5y^4 + y^5))/2880 + (\tau^3(120y^3 - y^6))/4320 + (\tau^2(2520y^2 + y^7))/10080 & \\ \quad +(\tau(40320y - y^8))/40320, & \tau < y. \end{cases}$
6	$\boldsymbol{\kappa}_y(\tau) = \begin{cases} -(\tau^{11}/39916800) + \tau y + (\tau^{10} y)/3628800 + (\tau^2 y^2)/4 - (\tau^9 y^2)/725760 + (\tau^3 y^3)/36 + (\tau^8 y^3)/241920 & \\ \quad +(\tau^4 y^4)/576 - (\tau^7 y^4)/120960 + (\tau^5 y^5)/14400 + (\tau^6 y^5)/86400, & \tau \ge y, \\ -(y^{11}/39916800) + (\tau^5(6y^5 + y^6))/86400 + (\tau^4(210y^4 - y^7))/120960 + (\tau^3(6720y^3 + y^8))/241920 & \\ \quad +(\tau^2(181440y^2 - y^9))/725760 + (\tau(3628800y + y^{10}))/3628800, & \tau < y. \end{cases}$

Let $\{\tau_l\}_{l=1}^{\infty}$ be a node dense set on $[0,1]$. Then we can deduce that

$$\varphi_{lj}(\tau) = \tilde{\boldsymbol{\kappa}}_\tau(\tau_l)\overrightarrow{e_j} = \begin{cases} (\tilde{\boldsymbol{\kappa}}_\tau(\tau_l), 0)^T, & j = 1, \\ (0, \tilde{\boldsymbol{\kappa}}_\tau(\tau_l))^T, & j = 2, \end{cases} \tag{6}$$

where $\phi_{lj}(\tau)$ represents the reproducing kernels of $\boldsymbol{W}_2^1[0,1]$ and $\boldsymbol{W}_2^6[0,1]$, respectively, and is defined as $\boldsymbol{L}^*\varphi_{lj}(\tau)$. Here, $\overrightarrow{e_j}$ is a vector in $\mathbb{R}^2$ with a value of 1 in the jth coordinate and 0 in all other coordinates, as stated in [9]. It has been proven in [6] that

$$\langle \phi_{si}(\cdot), \phi_{lj}(\cdot) \rangle_{\boldsymbol{W}_2^{3,3}} = \begin{cases} 0, & i \neq j, \\ \|\boldsymbol{\kappa}_{\tau_s}\|^2, & s = l, \ i = j, \\ \boldsymbol{\kappa}_{\tau_s}(\tau_l), & s \neq l, \ i = j. \end{cases} \tag{7}$$

Theorem 2.1. [6] For $j = 1,2$ and $l = 1,2,\ldots,$

$$\phi_{lj}(\tau) = \boldsymbol{L}\boldsymbol{\kappa}_{\tau_l}(\tau)\overrightarrow{e_j}.$$

Lemma 2.3. For each fixed N, $\{\phi_{lj}(\tau)\}_{(1,1)}^{(N,2)}$ is linearly independent in $\boldsymbol{W}_2^6[0,1]$,

[14].

Theorem 2.2. If $\{\tau_s\}_{s=1}^{\infty}$ is dense on $[0,1]$ and the solution of (4) is unique, then this solution is

$$\boldsymbol{\gamma}(\tau)=\sum_{l=1}^{\infty}\sum_{j=1}^{2}c_{j,l}\phi_{lj}(\tau). \tag{8}$$

Proof. Substituting (8) into (4), then for $i=1$ or 2

$$\begin{aligned}\boldsymbol{L}\boldsymbol{\gamma}(\tau_s)=\langle\boldsymbol{L}\boldsymbol{\gamma}(\cdot),\varphi_{si}(\cdot)\rangle_{\boldsymbol{W}_2^1}&=\langle\boldsymbol{\gamma}(\cdot),\boldsymbol{L}^*\varphi_{si}(\cdot)\rangle_{\boldsymbol{W}_2^6}\\&=\left\langle\sum_{l=1}^{\infty}\sum_{j=1}^{2}c_{j,l}\phi_{lj}(\cdot),\phi_{si}(\cdot)\right\rangle_{\boldsymbol{W}_2^6}\\&=\sum_{l=1}^{\infty}\sum_{j=1}^{2}c_{j,l}\left\langle\phi_{lj}(\cdot),\phi_{si}(\cdot)\right\rangle_{\boldsymbol{W}_2^6}\\&=\sum_{l=1}^{\infty}\sum_{j=1}^{2}c_{j,l}\phi_{lj}(\tau_s)\\&=\boldsymbol{G}(\tau_s)-\boldsymbol{H}(\tau_s,\boldsymbol{\gamma}(\tau_s),\boldsymbol{\gamma}'(\tau_s),\dots,\boldsymbol{\gamma}^{(m)}(\tau_s)).\end{aligned}$$

In addition, we have

$$\begin{aligned}\boldsymbol{G}(\tau_s)-\boldsymbol{H}(\tau_s,\boldsymbol{\gamma}(\tau_s),\dots,\boldsymbol{\gamma}^{(m)}(\tau_s))&=\left\langle\boldsymbol{G}(\tau)-\boldsymbol{H}(\tau,\boldsymbol{\gamma}(\tau),\dots,\boldsymbol{\gamma}^{(m)}(\tau)),\varphi_{si}(\tau)\right\rangle_{\boldsymbol{W}_2^1}\\&=\left\langle\boldsymbol{G}(\tau)-\boldsymbol{H}(\tau,\boldsymbol{\gamma}(\tau),\dots,\boldsymbol{\gamma}^{(m)}(\tau)),\tilde{\boldsymbol{\kappa}}_\tau(\tau_s)\overrightarrow{e_i}\right\rangle_{\boldsymbol{W}_2^1}\\&=\left\langle\boldsymbol{L}\boldsymbol{\gamma}(\tau),\tilde{\boldsymbol{\kappa}}_\tau(\tau_s)\overrightarrow{e_i}\right\rangle_{\boldsymbol{W}_2^1}\\&=\left\langle\boldsymbol{\gamma}(\tau),\boldsymbol{L}^*\tilde{\boldsymbol{\kappa}}_\tau(\tau_s)\overrightarrow{e_i}\right\rangle_{\boldsymbol{W}_2^6}\\&=\langle\boldsymbol{\gamma}(\tau),\phi_{si}(\tau)\rangle_{\boldsymbol{W}_2^6}\\&=\boldsymbol{\gamma}(\tau_s)\\&=\sum_{l=1}^{\infty}\sum_{j=1}^{2}c_{j,l}\phi_{lj}(\tau_s).\end{aligned}$$

Therefore, $\boldsymbol{\gamma}(\cdot)$ is the solution to (4), where $c_{j,l}$ for $j=1,2$ and $l=1,\dots$ are the unknown coefficients to be determined. □

We denote the numerical solution of $\boldsymbol{\gamma}$ by

$$\boldsymbol{\gamma}_N(\tau)=\sum_{l=1}^{N}\sum_{j=1}^{2}c_{j,l}\phi_{lj}(\tau), \tag{9}$$

where $c_{j,l}$ are the unknown numbers to be determined, and N is the number of collocation points on $[0,1]$. In the following, we aim to obtain a matrix notation for the unknowns in (8) using the iterative scheme in (5) for the nonlinear case. Therefore, the numerical solution is as follows:

$$\boldsymbol{\gamma}_{n,N}(\tau) = \sum_{l=1}^{N} \sum_{j=1}^{2} c_{j,l,n} \phi_{lj}(\tau), \quad n = 2, 3, \ldots, \tag{10}$$

where n represents the iteration number, the coefficients $c_{j,l,n}$ are obtained as follows: By substituting (10) into (4) and for a sufficiently large value of N, we obtain the following:

$$\boldsymbol{L}\boldsymbol{\gamma}_{n,N}(\tau) = \boldsymbol{G}(\tau) - \boldsymbol{H}(\tau, \boldsymbol{\gamma}_{n-1,N}(\tau), \boldsymbol{\gamma}'_{n-1,N}(\tau), \ldots, \boldsymbol{\gamma}^{(m)}_{n-1,N}(\tau)).$$

According to Theorem 2.2, we can write

$$\sum_{l=1}^{N} \sum_{j=1}^{2} c_{j,l,n} \phi_{lj}(\tau_s) = \boldsymbol{G}(\tau_s) - \boldsymbol{H}(\tau_s, \boldsymbol{\gamma}_{n-1,N}(\tau_s), \boldsymbol{\gamma}'_{n-1,N}(\tau_s), \ldots, \boldsymbol{\gamma}^{(m)}_{n-1,N}(\tau_s)), \tag{11}$$

where $s = 1, 2, \ldots, N$ is number of collocation points. Now, using Theorem 2.1 we have

$$\begin{aligned}
\sum_{l=1}^{N} \sum_{j=1}^{2} c_{j,l,n} \phi_{lj}(\tau_s) &= \sum_{l=1}^{N} c_{1,l,n} \phi_{l1}(\tau_s) + \sum_{l=1}^{N} c_{2,l,n} \phi_{l2}(\tau_s) \\
&= \sum_{l=1}^{N} c_{1,l,n} (\boldsymbol{L}\boldsymbol{\kappa}_{\tau_l}(\tau_s)\overrightarrow{e_1}) + \sum_{l=1}^{N} c_{2,l,n} (\boldsymbol{L}\boldsymbol{\kappa}_{\tau_l}(\tau_s)\overrightarrow{e_2}) \\
&= \boldsymbol{L} \sum_{l=1}^{N} c_{1,l,n} \boldsymbol{\kappa}_{\tau_l}(\tau_s)\overrightarrow{e_1} + \boldsymbol{L} \sum_{l=1}^{N} c_{2,l,n} \boldsymbol{\kappa}_{\tau_l}(\tau_s)\overrightarrow{e_2} \\
&= \begin{pmatrix} L_{11} & L_{12} \\ L_{21} & L_{22} \end{pmatrix} \begin{pmatrix} \sum_{l=1}^{N} c_{1,l,n} \boldsymbol{\kappa}_{\tau_l}(\tau_s) \\ 0 \end{pmatrix} \\
&\quad + \begin{pmatrix} L_{11} & L_{12} \\ L_{21} & L_{22} \end{pmatrix} \begin{pmatrix} 0 \\ \sum_{l=1}^{N} c_{2,l,n} \boldsymbol{\kappa}_{\tau_l}(\tau_s) \end{pmatrix} \\
&= \begin{pmatrix} L_{11} \sum_{l=1}^{N} c_{1,l,n} \boldsymbol{\kappa}_{\tau_l}(\tau_s) \\ L_{21} \sum_{l=1}^{N} c_{1,l,n} \boldsymbol{\kappa}_{\tau_l}(\tau_s) \end{pmatrix} + \begin{pmatrix} L_{12} \sum_{l=1}^{N} c_{2,l,n} \boldsymbol{\kappa}_{\tau_l}(\tau_s) \\ L_{22} \sum_{l=1}^{N} c_{2,l,n} \boldsymbol{\kappa}_{\tau_l}(\tau_s) \end{pmatrix}
\end{aligned}$$

$$= \begin{pmatrix} L_{11}\sum_{l=1}^{N} c_{1,l,n}\boldsymbol{\kappa}_{\tau_l}(\tau_s) + L_{12}\sum_{l=1}^{N} c_{2,l,n}\boldsymbol{\kappa}_{\tau_l}(\tau_s) \\ L_{21}\sum_{l=1}^{N} c_{1,l,n}\boldsymbol{\kappa}_{\tau_l}(\tau_s) + L_{22}\sum_{l=1}^{N} c_{2,l,n}\boldsymbol{\kappa}_{\tau_l}(\tau_s) \end{pmatrix}$$

$$= \begin{pmatrix} \sum_{l=1}^{N} L_{11}\boldsymbol{\kappa}_{\tau_l}(\tau_s) & \sum_{l=1}^{N} L_{12}\boldsymbol{\kappa}_{\tau_l}(\tau_s) \\ \sum_{l=1}^{N} L_{21}\boldsymbol{\kappa}_{\tau_l}(\tau_s) & \sum_{l=1}^{N} L_{22}\boldsymbol{\kappa}_{\tau_l}(\tau_s) \end{pmatrix} \begin{pmatrix} c_{1,l,n} \\ c_{2,l,n} \end{pmatrix}.$$

Also,

$$\boldsymbol{G}(\tau_s) - \boldsymbol{H}(\tau_s, \boldsymbol{\gamma}_{n-1,N}(\tau_s), \boldsymbol{\gamma}'_{n-1,N}(\tau_s), \dots, \boldsymbol{\gamma}^{(m)}_{n-1,N}(\tau_s))$$
$$= \begin{pmatrix} g_1(\tau_s) \\ g_2(\tau_s) \end{pmatrix} - \begin{pmatrix} H_1(\tau_s, \boldsymbol{\gamma}_{n-1,N}(\tau_s), \boldsymbol{\gamma}'_{n-1,N}(\tau_s), \dots, \boldsymbol{\gamma}^{(m)}_{n-1,N}(\tau_s)) \\ H_2(\tau_s, \boldsymbol{\gamma}_{n-1,N}(\tau_s), \boldsymbol{\gamma}'_{n-1,N}(\tau_s), \dots, \boldsymbol{\gamma}^{(m)}_{n-1,N}(\tau_s)) \end{pmatrix}.$$

So, according to (11) we can deduce that

$$\begin{pmatrix} \sum_{l=1}^{N} L_{11}\boldsymbol{\kappa}_{\tau_l}(\tau_s) & \sum_{l=1}^{N} L_{12}\boldsymbol{\kappa}_{\tau_l}(\tau_s) \\ \sum_{l=1}^{N} L_{21}\boldsymbol{\kappa}_{\tau_l}(\tau_s) & \sum_{l=1}^{N} L_{22}\boldsymbol{\kappa}_{\tau_l}(\tau_s) \end{pmatrix} \begin{pmatrix} c_{1,l,n} \\ c_{2,l,n} \end{pmatrix}$$
$$= \begin{pmatrix} g_1(\tau_s) \\ g_2(\tau_s) \end{pmatrix} - \begin{pmatrix} H_1(\tau_s, \boldsymbol{\gamma}_{n-1,N}(\tau_s), \boldsymbol{\gamma}'_{n-1,N}(\tau_s), \dots, \boldsymbol{\gamma}^{(m)}_{n-1,N}(\tau_s)) \\ H_2(\tau_s, \boldsymbol{\gamma}_{n-1,N}(\tau_s), \boldsymbol{\gamma}'_{n-1,N}(\tau_s), \dots, \boldsymbol{\gamma}^{(m)}_{n-1,N}(\tau_s)) \end{pmatrix}.$$

Then

$$\boldsymbol{A} = \begin{pmatrix} \sum_{l=1}^{N} L_{11}\boldsymbol{\kappa}_{\tau_l}(\tau_s) & \sum_{l=1}^{N} L_{12}\boldsymbol{\kappa}_{\tau_l}(\tau_s) \\ \sum_{l=1}^{N} L_{21}\boldsymbol{\kappa}_{\tau_l}(\tau_s) & \sum_{l=1}^{N} L_{22}\boldsymbol{\kappa}_{\tau_l}(\tau_s) \end{pmatrix}_{2N\times 2N},$$

$$\mathcal{C} = \begin{pmatrix} c_{1,l,n} \\ c_{2,l,n} \end{pmatrix}_{2N\times 1},$$

$$\boldsymbol{M} = \begin{pmatrix} g_1(\tau_s) \\ g_2(\tau_s) \end{pmatrix}_{2N\times 1} - \begin{pmatrix} H_1(\tau_s, \boldsymbol{\gamma}_{n-1,N}(\tau_s), \boldsymbol{\gamma}'_{n-1,N}(\tau_s), \dots, \boldsymbol{\gamma}^{(m)}_{n-1,N}(\tau_s)) \\ H_2(\tau_s, \boldsymbol{\gamma}_{n-1,N}(\tau_s), \boldsymbol{\gamma}'_{n-1,N}(\tau_s), \dots, \boldsymbol{\gamma}^{(m)}_{n-1,N}(\tau_s)) \end{pmatrix}_{2N\times 1}.$$

Therefore, we can write

$$\boldsymbol{A}\,\mathcal{C} = \boldsymbol{M}.$$

Finally, according to Lemma 2.3 $\boldsymbol{A}^{-1}$ exists and

$$\mathcal{C} = \boldsymbol{A}^{-1}\,\boldsymbol{M}.$$

3 Error estimation

Lemma 3.1. Let $S = \left\{\boldsymbol{\gamma}(\cdot) = (\gamma_1(\cdot), \gamma_2(\cdot)) \mid \|\boldsymbol{\gamma}\|_{\boldsymbol{W}_2^6} \leq \delta\right\}$. Then S is a compact set in the space $C^2[0,1]$, where δ is a constant [24].

Lemma 3.2. Assuming that in system (4), the norm of $\boldsymbol{\gamma}$ in $\boldsymbol{W}_2^6$ is bounded, $\{\tau_s\}_{s=1}^{\infty}$ is a dense set on $[0,1]$, $\boldsymbol{L}(\boldsymbol{\gamma}(\cdot))$ is a continuous function of $\boldsymbol{\gamma}(\cdot)$ that is also invertible, and $\boldsymbol{H}(., \boldsymbol{\gamma}(\cdot), \boldsymbol{\gamma}'(\cdot), \ldots, \boldsymbol{\gamma}^{(m)}(\cdot))$ is a continuous function of $\boldsymbol{\gamma}(\cdot)$, then both the analytical solution $\boldsymbol{\gamma}(\cdot)$ and the numerical solution $\boldsymbol{\gamma}_{n,N}(\cdot)$ for (4) exist, [24].

Theorem 3.1. If $\boldsymbol{\gamma}(\cdot) = (\gamma_1(\cdot), \gamma_2(\cdot)) \in \boldsymbol{W}_2^6[0,1]$ is the solution of (4), then the numerical solution $\boldsymbol{\gamma}_{n,N}(\cdot) = (\gamma_{1,n,N}(\cdot), \gamma_{2,n,N}(\cdot))$ converges uniformly to $\boldsymbol{\gamma}(\cdot)$.

Proof. By subtracting the two equations in (3), we can obtain the following form:

$$D^{\alpha}\gamma_1(\tau) - \bar{a}(\tau)\gamma_1(\tau) - \int_0^{\tau} k_{11}(\tau, x)\gamma_1(x)\, dx = \bar{g}(\tau) - \bar{H}(\tau, \boldsymbol{\gamma}(\tau), \boldsymbol{\gamma}'(\tau), \ldots, \boldsymbol{\gamma}^{(m)}(\tau)), \tag{12}$$

where $\bar{a}(\cdot), \bar{g}(\cdot)$ and $\bar{H}(\cdot)$ are known functions and (12) is a nonlinear equation in the reproducing kernel space $W_2^6[0,1]$. Furthermore, according to Lemma 3.2, $\gamma_{1,n,N}(\cdot)$ is a numerical solution of $\gamma_1(\cdot)$. Hence,

$$\begin{aligned} |\gamma_1(\tau) - \gamma_{1,n,N}(\tau)| &= |\langle \gamma_1 - \gamma_{1,n,N}, \boldsymbol{\kappa}_{\tau}\rangle_{W_2^6}| \leq \|\gamma_1 - \gamma_{1,n,N}\|_{W_2^6} \|\boldsymbol{\kappa}_{\tau}\|_{W_2^6} \\ &\leq Q_1 \|\gamma_1 - \gamma_{1,n,N}\|_{W_2^6}, \end{aligned}$$

where Q_1 is constant. Similarly, we have

$$|\gamma_2(\tau) - \gamma_{2,n,N}(\tau)| \leq Q_2 \|\gamma_2 - \gamma_{2,n,N}\|_{W_2^6}.$$

□

Theorem 3.2. If $\gamma_{1,n,N}(\tau) \xrightarrow{\|\cdot\|_{W_2^k}} \gamma_1(\tau)$ and $\tau_s \to y(s \to \infty)$, then

$$\bar{H}(\tau_s, \boldsymbol{\gamma}_N(\tau_s), \boldsymbol{\gamma}'_N(\tau_s), \ldots, \boldsymbol{\gamma}_N^{(m)}(\tau_s)) \to \bar{H}(y, \boldsymbol{\gamma}(y), \boldsymbol{\gamma}'(y), \ldots, \boldsymbol{\gamma}^{(m)}(y))(s \to \infty).$$

Proof. See [3]. □

Theorem 3.3. Let $\boldsymbol{\gamma}(\cdot) = (\gamma_1(\cdot), \gamma_2(\cdot))$ and $\boldsymbol{\gamma}_{n,N}(\cdot) = (\gamma_{1,n,N}(\cdot), \gamma_{2,n,N}(\cdot))$ be the analytical and numerical solution of (4), respectively. If $\boldsymbol{\gamma} \in C^6[0,1]$, $\boldsymbol{\gamma}_{n,N} \in \boldsymbol{W}_2^6[0,1]$ and $\left\| \gamma_{i,n,N}^{(6)} \right\|_\infty \leq M_i$, $i = 1, 2$, then for $j = 1, 2$,

$$\|\gamma_j - \gamma_{j,n,N}\|_\infty \leq C_j h^6,$$

where C_j is a constant.

Proof. See [7]. □

Remark 3.1. The accuracy of the RKM method is affected by the choice of space. Therefore, it is crucial to carefully select the appropriate space based on the specific problem at hand. It is important to note that changing the space can also affect the convergence order. For instance, if we choose $\boldsymbol{W}_2^5[0,1]$ for a particular problem, then the convergence order will be Ch^5.

Remark 3.2. According to Lemma 2.3, if $\boldsymbol{A}^{-1}$ exists, then the solution of (4) also exists and is unique. Additionally, we can conclude that the present method is stable in $\boldsymbol{W}_2^6[0,1]$.

Remark 3.3. The formula for convergence order is as follows:

$$C.F_i = log_2 \frac{||\gamma_i - \gamma_{i,n,N}||_\infty}{||\gamma_i - \gamma_{i,n,2N}||_\infty},$$

where $i = 1, 2$.

4 Numerical results

In this section, we will demonstrate the application of the present method in solving four examples of NFVI-DEs. Additionally, we will compare the effectiveness of the present method with the method presented in [23]. In the first example, we will introduce the present method without using the Taylor series expansion and showcase its effectiveness in solving certain problems. However, we will also demonstrate that this method is not effective for solving problems where the Volterra integral is applied to nonlinear components. The following examples have been solved using Mathematica 12 software.

Example 4.1. [23] Consider the NFVI-DEs:

$$\begin{cases} D^{\alpha}\gamma_1(\tau) - \frac{1}{3}\gamma_1(\tau)\gamma_2(\tau) - \frac{1}{2}\gamma_2^2(\tau) - 2\gamma_2(\tau) + \int_0^{\tau}[\gamma_1(x) - \gamma_2(x)]dx = g_1(\tau), \\ D^{\beta}\gamma_2(\tau) - \frac{1}{3}\gamma_1(\tau)\gamma_2(\tau) + \gamma_1(\tau) + \int_0^{\tau}[\gamma_1(x) - 2\gamma_2(x)]dx = g_2(\tau), \\ \gamma_1(0) = 0,\ \gamma_2(0) = 0, \end{cases}$$

where

$$0 < \alpha, \beta \leq 1,$$

and the analytical solution for $\alpha = \beta = 1$ is

$$\boldsymbol{\gamma}(\tau) = (\tau^2,\ \tau).$$

We solved this example in the $\boldsymbol{W}_2^5[0,1]$ space using $\tau_l = \frac{l}{N+1}$ points. In this example, the Volterra integral is not applied to the nonlinear component, so there is no need to use the Taylor series expansion. As a result, (1) can be rewritten as follows:

$$\begin{cases} L_{11}\gamma_1(\tau) + L_{12}\gamma_2(\tau) = g_1(\tau) - \lambda_1(\tau, \boldsymbol{\gamma}(\tau)), \\ L_{21}\gamma_1(\tau) + L_{22}\gamma_2(\tau) = g_2(\tau) - \lambda_2(\tau, \boldsymbol{\gamma}(\tau)), \\ \gamma_i(0) = \theta_i,\ i = 1, 2. \end{cases} \tag{13}$$

Therefore, the matrix form of (13) is

$$\begin{cases} \boldsymbol{L}(\boldsymbol{\gamma}(\tau)) = \boldsymbol{G}(\tau) - \boldsymbol{\lambda}(\tau, \boldsymbol{\gamma}(\tau)), & 0 < \tau \leq 1, \\ \gamma_i(0) = \theta_i,\ i = 1, 2, \end{cases} \tag{14}$$

where $\boldsymbol{G} = (g_1, g_2)$ and $\boldsymbol{\lambda} = (\lambda_1, \lambda_2)$. Finally, the unknown coefficients can be determined using the following equation:

$$\begin{pmatrix} \sum_{l=1}^{N} L_{11}\boldsymbol{\kappa}_{\tau_l}(\tau_s) & \sum_{l=1}^{N} L_{12}\boldsymbol{\kappa}_{\tau_l}(\tau_s) \\ \sum_{l=1}^{N} L_{21}\boldsymbol{\kappa}_{\tau_l}(\tau_s) & \sum_{l=1}^{N} L_{22}\boldsymbol{\kappa}_{\tau_l}(\tau_s) \end{pmatrix} \begin{pmatrix} c_{1,l,n} \\ c_{2,l,n} \end{pmatrix} = \begin{pmatrix} g_1(\tau_s) \\ g_2(\tau_s) \end{pmatrix} - \begin{pmatrix} \lambda_1(\tau_s, \boldsymbol{\gamma}_{n-1,N}(\tau_s)) \\ \lambda_2(\tau_s, \boldsymbol{\gamma}_{n-1,N}(\tau_s)) \end{pmatrix}.$$

Next, we compared this method with the method proposed in [23], which is based on absolute error and numerical solution. The results are presented in Tables 2, 3, and 4 and Figures 1, 2, and 3. The convergence order is also shown in Table 5. These numerical results demonstrate the efficiency of this method without the use of Taylor series expansion. However, it should be noted that this method may not be as efficient in problems where the Volterra integral is applied to nonlinear components.

Table 2: Error comparison in Example 4.1 for $\alpha = \beta = 1$.

τ	M in [23] $\lvert\gamma_1(\tau) - \gamma_{1,64}(\tau)\rvert$	M in [23] $\lvert\gamma_2(\tau) - \gamma_{2,64}(\tau)\rvert$	$\boldsymbol{W}_2^5[0,1]$ $\lvert\gamma_1(\tau) - \gamma_{1,10,24}(\tau)\rvert$	$\boldsymbol{W}_2^5[0,1]$ $\lvert\gamma_2(\tau) - \gamma_{2,10,24}(\tau)\rvert$
0	1.04×10^{-5}	2.97×10^{-9}	0	0
0.1	1.22×10^{-4}	3.04×10^{-9}	2.19×10^{-11}	2.48×10^{-12}
0.2	3.24×10^{-5}	4.31×10^{-6}	2.10×10^{-11}	4.89×10^{-12}
0.3	1.07×10^{-4}	9.06×10^{-6}	1.94×10^{-11}	7.41×10^{-12}
0.4	1.03×10^{-5}	1.41×10^{-5}	1.70×10^{-11}	1.00×10^{-11}
0.5	1.85×10^{-5}	1.95×10^{-5}	1.39×10^{-11}	1.27×10^{-11}
0.6	2.48×10^{-6}	3.06×10^{-5}	9.90×10^{-12}	1.55×10^{-11}
0.7	7.08×10^{-6}	3.62×10^{-5}	5.00×10^{-12}	1.83×10^{-11}
0.8	6.01×10^{-6}	4.17×10^{-5}	8.83×10^{-13}	2.13×10^{-11}
0.9	2.94×10^{-6}	4.70×10^{-5}	8.68×10^{-12}	2.46×10^{-11}
1	1.97×10^{-4}	5.20×10^{-5}	2.68×10^{-11}	3.22×10^{-11}

Table 3: Comparison the numerical solutions $\gamma_1(\cdot)$ for different value of α in Example 4.1, when $n = 10,\ N = 16$.

	$\alpha = 0.7$		$\alpha = 0.8$		$\alpha = 0.9$		$\alpha = 1$	
τ	M in [23]	$\boldsymbol{W}_2^5[0,1]$	M in [23]	$\boldsymbol{W}_2^5[0,1]$	M in [23]	$\boldsymbol{W}_2^5[0,1]$	M in [23]	$\boldsymbol{W}_2^5[0,1]$
0	−.0037573	0	−0.0038780	0	−0.0029321	0	−0.0019508	0
0.1	0.0622484	0.0597477	0.0351106	0.0334030	0.0193767	0.0183878	0.0105184	0.0100000
0.2	0.1589985	0.1616368	0.1045106	0.1035549	0.0667343	0.0648684	0.0417159	0.0400000
0.3	0.2689171	0.2834621	0.1938697	0.1986078	0.1349966	0.1350721	0.0916416	0.0900000
0.4	0.3869518	0.4164330	0.2991345	0.3128956	0.2222319	0.2266003	0.1602961	0.1600000
0.5	1.0196899	0.5547315	0.8345105	0.4423304	0.6537138	0.3376242	0.2503604	0.2500000
0.6	0.6369025	0.6939084	0.5492630	0.5834873	0.4526805	0.4665406	0.3600394	0.3600000
0.7	0.7667133	0.8303810	0.6901640	0.7332801	0.5932708	0.6118313	0.4911326	0.4900000
0.8	0.8998356	0.9612530	0.8394357	0.8888207	0.7480303	0.7719908	0.6409619	0.6400000
0.9	1.0371178	1.0842653	0.9968868	1.0473609	0.9165373	0.9454807	0.8095297	0.8100000
1	1.1798913	1.1978056	1.1626464	1.2062802	1.0984737	1.1306994	0.9968391	1.0000000

Table 4: Comparison the numerical solutions $\gamma_2(\cdot)$ for different values of β in Example 4.1, when $n = 10,\ N = 16$.

	$\alpha = 0.7$		$\alpha = 0.8$		$\alpha = 0.9$		$\alpha = 1$	
τ	M in [23]	$\boldsymbol{W}_2^5[0,1]$	M in [23]	$\boldsymbol{W}_2^5[0,1]$	M in [23]	$\boldsymbol{W}_2^5[0,1]$	M in [23]	$\boldsymbol{W}_2^5[0,1]$
0	0.355758	0	0.0170683	0	0.0061149	0	$7.21E-07$	0
0.1	0.2153236	0.2046448	0.1689943	0.1624234	0.1306502	0.1277562	0.9999993	0.1000000
0.2	0.3372475	0.3323294	0.2886770	0.2850171	0.2419557	0.2401117	0.1998547	0.2000000
0.3	0.4360154	0.4330255	0.3939580	0.3913621	0.3468649	0.3454015	0.2997735	0.3000000
0.4	0.5189111	0.5172701	0.4886216	0.4869908	0.4468382	0.4457686	0.3996882	0.4000000
0.5	1.1814930	0.5898213	1.1502742	0.5744086	1.0855705	0.5421777	0.4998003	0.5000000
0.6	0.6528540	0.6535326	0.6534302	0.6550868	0.6342915	0.6351784	0.5995105	0.6000000
0.7	0.7104426	0.7104326	0.7271108	0.7300247	0.7230138	0.7251225	0.6994215	0.7000000
0.8	0.7651144	0.7621614	0.7971754	0.7999801	0.8093225	0.8122505	0.7993340	0.8000000
0.9	0.8181686	0.8101849	0.8644942	0.8655891	0.8935603	0.8967336	0.8992493	0.9000000
1	0.8708896	0.8559100	0.9300103	0.9274389	0.9761266	0.9786977	0.9991684	1.0000000

Example 4.2. [23] Consider the NFVI-DEs:

Table 5: Convergence order in Example 4.1.

	$\alpha=\beta=1,\ n=10$, in $\boldsymbol{W}_2^5[0,1]$				
N	$\left\Vert\gamma_1-\gamma_{1,n,N}\right\Vert_\infty$	$C.F_1$	$\left\Vert\gamma_2-\gamma_{2,n,N}\right\Vert_\infty$	$C.F_2$	$Cpu\ time(sec)$
4	2.42×10^{-6}	–	3.12×10^{-6}	–	2
8	2.74×10^{-8}	6.46	3.44×10^{-8}	6.34	4
16	3.36×10^{-10}	6.50	4.12×10^{-10}	6.38	8

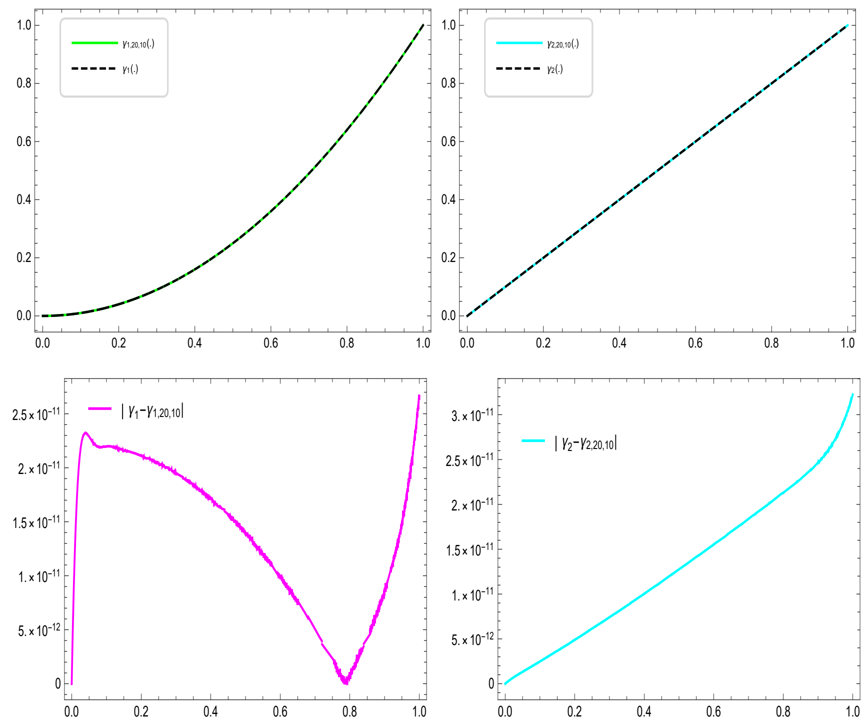

Figure 1: Numerical solution and absolute error without using the Taylor series expansion in Example 4.1.

$$
\begin{cases}
D^\alpha\gamma_1(\tau)-\gamma_1^2(\tau)-\gamma_2^2(\tau)+\int_0^\tau\gamma_1(x)\,dx=g_1(\tau),\\
D^\beta\gamma_2(\tau)+\frac{1}{2}\gamma_2^2(\tau)+\gamma_1(\tau)+\int_0^\tau\gamma_1(x)\gamma_2(x)\,dx=g_2(\tau),\\
\gamma_1(0)=0,\ \gamma_2(0)=1,
\end{cases}
$$

where

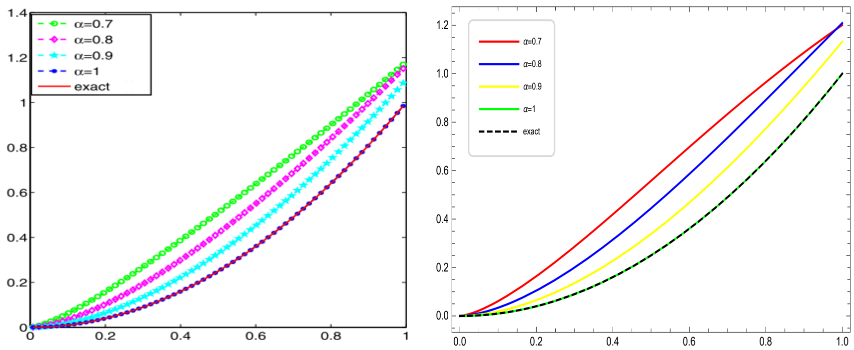

Figure 2: Comparison of the numerical solutions without using the Taylor series expansion in Example 4.1 for $\alpha = 0.65, 0.75, 0.85, 1$. (Left: $\gamma_{1,64}(\cdot)$, M in [23]; Right: $\gamma_{1,20,24}(\cdot)$, Present method).

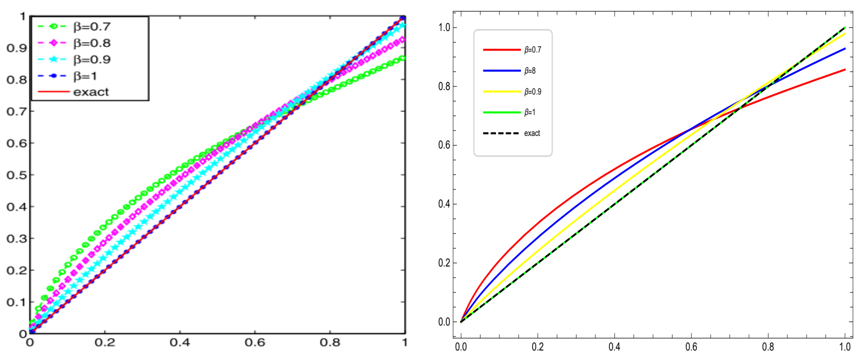

Figure 3: Comparison of the numerical solutions without using Taylor series expansion in Example 4.1 for $\beta = 0.65, 0.75, 0.85, 1$. (Left: $\gamma_{2,64}(\cdot)$, M in [23]; Right: $\gamma_{2,20,24}(\cdot)$, Present method).

$$0 < \alpha, \beta \leq 1,$$

and the analytical solution for $\alpha = \beta = 1$ is

$$\boldsymbol{\gamma}(\tau) = (sin(\tau), cos(\tau)).$$

First, we solved this example in the $\boldsymbol{W}_2^6[0,1]$ space without using the Taylor expansion, which utilizes $\tau_l = \frac{l}{N+1}$ points. The numerical solution and absolute error are shown in Figure 4. However, this method is not effective.

To improve the results, we compared the Present method, which uses the Taylor expansion, with the method proposed in [23]. This comparison was based on the absolute error and numerical solution, as shown in Table 6 and Figures 5, 6, and 7. The convergence order is also shown in Table 7. Finally, to solve this example using (2) and (3) instead of $\gamma_1(\cdot)$ and $\gamma_2(\cdot)$, we utilized the truncated Taylor series expansion around the point x in the interval $[0,1]$ for $m=3$. This allowed us to obtain the nonlinear parts of $H_2(\tau,\boldsymbol{\gamma}(\cdot),\boldsymbol{\gamma}'(\cdot),\boldsymbol{\gamma}''(\cdot),\boldsymbol{\gamma}^{(3)}(\cdot))$ as follows:

$$\begin{aligned}
&\frac{1}{2}\gamma_2^2(\tau)+\tau\gamma_1(\tau)\gamma_2(\tau)-\frac{1}{2}\tau^2\gamma_2(\tau)\gamma_1'(\tau)-\frac{1}{2}\tau^2\gamma_1(\tau)\gamma_2'(\tau)+\frac{1}{3}\tau^3\gamma_1(\tau)\gamma_2(\tau)\\
&+\frac{1}{6}\tau^3\gamma_2(\tau)\gamma_1''(\tau)-\frac{1}{8}\tau^4\gamma_2(\tau)\gamma_1''(\tau)+\frac{1}{6}\tau^3\gamma_1(\tau)\gamma_2''(\tau)-\frac{1}{8}\tau^4\gamma_1'(\tau)\gamma_2''(\tau)\\
&+\frac{1}{20}\tau^5\gamma_1''(\tau)\gamma_2''(\tau)-\frac{1}{24}\tau^4\gamma_2(\tau)\gamma_1^{(3)}(\tau)+\frac{1}{30}\tau^5\gamma_2'(\tau)\gamma_1^{(3)}(\tau)-\frac{1}{72}\tau^6\gamma_2''(\tau)\gamma_1^{(3)}(\tau)\\
&-\frac{1}{24}\tau^4\gamma_1(\tau)\gamma_2^{(3)}(\tau)+\frac{1}{30}\tau^5\gamma_1'(\tau)\gamma_2^{(3)}(\tau)-\frac{1}{72}\tau^6\gamma_1''(\tau)\gamma_2^{(3)}(\tau)+\frac{1}{252}\tau^7\gamma_1^{(3)}(\tau)\gamma_2^{(3)}(\tau).
\end{aligned}$$

Example 4.3. Consider the NFVI-DEs:

$$\begin{cases} D^{\alpha}\gamma_1(\tau)-\int_0^{\tau}\gamma_1^2(x)\,dx=g_1(\tau),\\ D^{\beta}\gamma_2(\tau)+\int_0^{\tau}(\gamma_1^2(x)+\gamma_2^2(x))\,dx=g_2(\tau),\\ \gamma_1(0)=0,\ \gamma_2(0)=0,\end{cases}$$

where

$$0<\alpha,\beta\le 1,$$

and the analytical solution for $\alpha=\beta=1$ is

$$\boldsymbol{\gamma}(\tau)=(\tau sin(\tau),\,1-cos(\tau)).$$

To solve this example, we first need to move the nonlinear part to the right side of the equation. Then, we can create a coefficient matrix using the linear part. This will give us the following equation:

$$\begin{cases} D^{\alpha}\gamma_1(\tau)=g_1(\tau)+\int_0^{\tau}\gamma_1^2(x)\,dx,\\ D^{\beta}\gamma_2(\tau)=g_2(\tau)-\int_0^{\tau}(\gamma_1^2(x)+\gamma_2^2(x))\,dx,\\ \gamma_1(0)=0,\ \gamma_2(0)=0.\end{cases} \tag{15}$$

In the first equation of (15), the linear operator is not applied to $\gamma_2(\cdot)$. As a result, the coefficient matrix does not depend on $\gamma_2(\cdot)$, and we have substituted it with an $N\times N$ zero matrix. Similarly, in the second equation of

Table 6: Error comparison in Example 4.2 for $\alpha = \beta = 1$.

τ	M in [23] $\lvert\gamma_1(\tau) - \gamma_{1,32}(\tau)\rvert$	M in [23] $\lvert\gamma_2(\tau) - \gamma_{1,32}(\tau)\rvert$	$\boldsymbol{W}_2^6[0,1]$ $\lvert\gamma_1(\tau) - \gamma_{1,10,10}(\tau)\rvert$	$\boldsymbol{W}_2^6[0,1]$ $\lvert\gamma_2(\tau) - \gamma_{2,10,10}(\tau)\rvert$	τ	M in [23] $\lvert\gamma_1(\tau) - \gamma_{1,32}(\tau)\rvert$	M in [23] $\lvert\gamma_2(\tau) - \gamma_{1,32}(\tau)\rvert$	$\boldsymbol{W}_2^6[0,1]$ $\lvert\gamma_1(\tau) - \gamma_{1,10,10}(\tau)\rvert$	$\boldsymbol{W}_2^6[0,1]$ $\lvert\gamma_2(\tau) - \gamma_{2,10,10}(\tau)\rvert$
0.015625	5.68×10^{-6}	1.21×10^{-4}	3.42×10^{-9}	9.26×10^{-9}	0.515625	1.67×10^{-4}	3.05×10^{-5}	1.19×10^{-8}	6.54×10^{-9}
0.046875	1.70×10^{-5}	1.18×10^{-4}	5.51×10^{-9}	1.70×10^{-8}	0.546875	1.76×10^{-4}	2.18×10^{-5}	1.13×10^{-8}	5.72×10^{-9}
0.078125	2.81×10^{-5}	1.15×10^{-4}	6.07×10^{-9}	1.81×10^{-8}	0.578125	1.85×10^{-4}	1.27×10^{-5}	1.02×10^{-8}	4.58×10^{-9}
0.109375	3.91×10^{-5}	1.11×10^{-4}	7.14×10^{-9}	1.73×10^{-8}	0.609375	1.94×10^{-4}	3.26×10^{-6}	8.75×10^{-9}	2.86×10^{-9}
0.140625	4.99×10^{-5}	1.07×10^{-4}	8.60×10^{-9}	1.64×10^{-8}	0.640625	2.02×10^{-4}	6.51×10^{-6}	7.03×10^{-9}	1.19×10^{-11}
0.171875	6.05×10^{-5}	1.03×10^{-4}	9.83×10^{-9}	1.55×10^{-8}	0.671875	2.10×10^{-4}	1.66×10^{-5}	5.21×10^{-9}	5.10×10^{-9}
0.203125	7.10×10^{-5}	9.79×10^{-5}	1.05×10^{-8}	1.47×10^{-8}	0.703125	2.19×10^{-4}	2.71×10^{-5}	3.05×10^{-9}	1.45×10^{-8}
0.234375	8.14×10^{-5}	8.14×10^{-5}	1.08×10^{-8}	1.38×10^{-8}	0.734375	2.27×10^{-4}	3.78×10^{-5}	1.20×10^{-10}	3.18×10^{-8}
0.265625	9.15×10^{-5}	8.73×10^{-5}	1.12×10^{-8}	1.29×10^{-8}	0.765625	2.35×10^{-4}	4.89×10^{-5}	3.91×10^{-9}	6.21×10^{-8}
0.296875	1.02×10^{-4}	8.15×10^{-5}	1.17×10^{-8}	1.21×10^{-8}	0.796875	2.43×10^{-4}	6.04×10^{-5}	8.78×10^{-9}	1.13×10^{-7}
0.328125	1.11×10^{-4}	7.53×10^{-5}	1.24×10^{-8}	1.14×10^{-8}	0.828125	2.51×10^{-4}	7.21×10^{-5}	1.34×10^{-8}	1.96×10^{-7}
0.359375	1.21×10^{-4}	6.87×10^{-5}	1.28×10^{-8}	1.06×10^{-8}	0.859375	2.59×10^{-4}	8.42×10^{-5}	1.62×10^{-8}	3.28×10^{-7}
0.390625	1.31×10^{-4}	6.18×10^{-5}	1.29×10^{-8}	9.75×10^{-9}	0.890625	2.67×10^{-4}	9.66×10^{-5}	1.58×10^{-8}	5.32×10^{-7}
0.421875	1.40×10^{-4}	5.45×10^{-5}	1.26×10^{-8}	8.84×10^{-9}	0.921875	2.75×10^{-4}	1.09×10^{-4}	1.26×10^{-8}	8.44×10^{-7}
0.453125	1.49×10^{-4}	4.69×10^{-5}	1.24×10^{-8}	7.99×10^{-9}	0.953125	2.83×10^{-4}	1.22×10^{-4}	1.08×10^{-8}	1.31×10^{-6}
0.484375	1.58×10^{-4}	3.88×10^{-5}	1.21×10^{-8}	7.25×10^{-9}	0.984375	2.92×10^{-4}	1.36×10^{-4}	2.16×10^{-8}	2.00×10^{-6}

Table 7: Convergence order in Example 4.2.

	$\alpha=\beta=1,\ n=10$, in $\boldsymbol{W}_2^6[0,1]$				
N	$\left\|\gamma_1-\gamma_{1,n,N}\right\|_\infty$	$C.F_1$	$\left\|\gamma_2-\gamma_{2,n,N}\right\|_\infty$	$C.F_2$	$Cpu\ time(sec)$
2	6.79×10^{-3}	–	1.71×10^{-2}	–	3
4	4.64×10^{-5}	7.19	1.52×10^{-4}	6.81	5
8	2.75×10^{-7}	7.39	2.05×10^{-6}	6.21	9

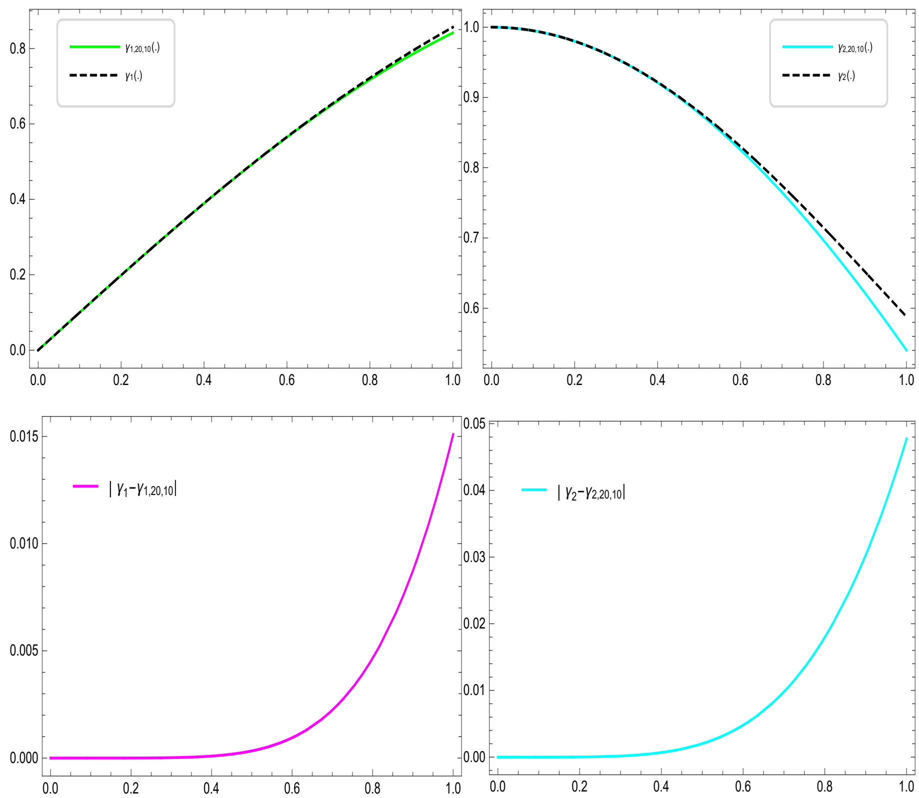

Figure 4: Numerical solution and absolute error without using the Taylor series expansion in Example 4.2.

(15), the coefficient matrix related to $\gamma_1(\cdot)$ is not used, and we have replaced it with an $N\times N$ zero matrix.

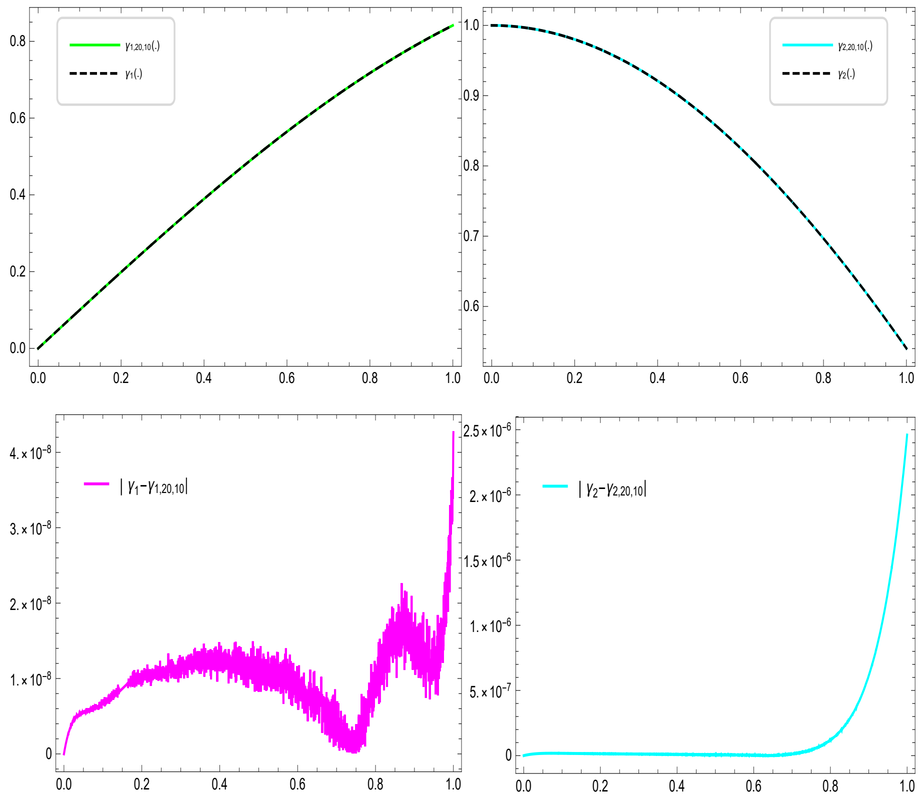

Figure 5: Numerical solution and absolute error using the Present method with the Taylor series expansion in Example 4.2.

First, we solved this example in the $\boldsymbol{W}_2^6[0,1]$ space without using the Taylor expansion, which utilizes $\tau_l = \frac{l}{N+1}$ points. The numerical solution and absolute error are shown in Figure 8. However, this method is not effective. To improve the results, we compared the Present method, which uses the Taylor expansion, with the method proposed in [23]. This comparison was based on the absolute error and numerical solution, as shown in Table 8 and Figures 9 and 10. The convergence order is also shown in Table 9. Finally, to solve this example according to (2) and (3) instead of $\gamma_1(\cdot)$ and $\gamma_2(\cdot)$, we use the truncated Taylor series expansion around the point x in the interval $[0,1]$ for $m=3$. As a result, the nonlinear parts of H_1 as,

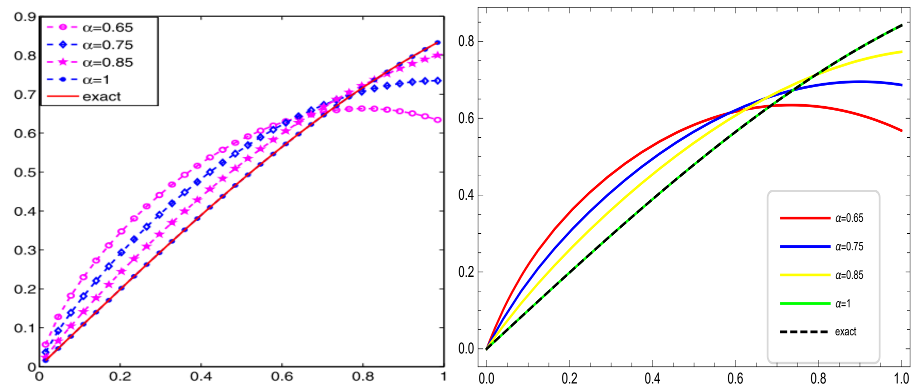

Figure 6: Comparison of the numerical solutions using the Present method with the Taylor series expansion in Example 4.2 for $\alpha = 0.65, 0.75, 0.85, 1$. (Left: $\gamma_{1,32}(\cdot)$, M in [23]; Right: $\gamma_{1,20,10}(\cdot)$, Present method).

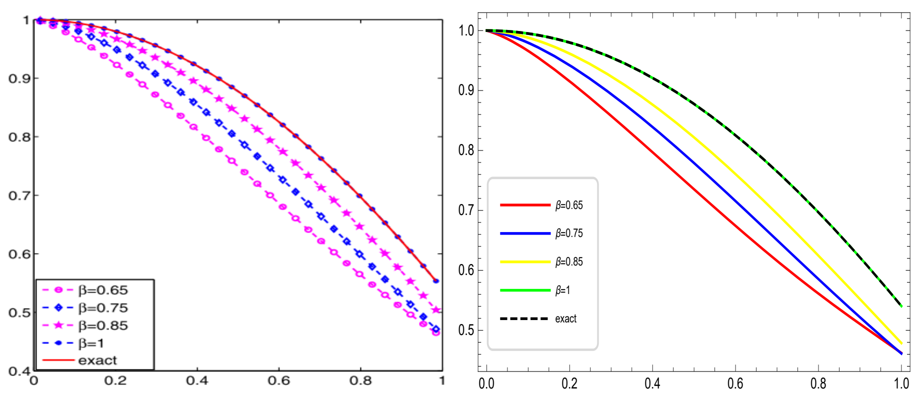

Figure 7: Comparison of the numerical solutions using the Present method with the Taylor series expansion in Example 4.2 for $\beta = 0.65, 0.75, 0.85, 1$. (Left: $\gamma_{2,32}(\cdot)$, M in [23]; Right: $\gamma_{2,20,10}(\cdot)$, Present method).

$$
\begin{aligned}
&\tau\gamma_1(\tau)^2 - \tau^2\gamma_1(\tau)\gamma_1'(\tau) + \frac{1}{3}\tau^3\gamma_1'(\tau)^2 + \frac{1}{3}\tau^3\gamma_1(\tau)\gamma_1''(\tau)^2 - \frac{1}{4}\tau^4\gamma_1'(\tau)\gamma_1''(\tau)\\
&+ \frac{1}{20}\tau^5\gamma_1''(\tau)^2 - \frac{1}{12}\tau^4\gamma_1(\tau)\gamma_1^{(3)}(\tau) + \frac{1}{15}\tau^5\gamma_1'(\tau)\gamma_1^{(3)}(\tau)\\
&- \frac{1}{36}\tau^6\gamma_1''(\tau)\gamma_1^{(3)}(\tau) + \frac{1}{252}\tau^7\gamma_1^{(3)}(\tau)^2,
\end{aligned}
$$

and H_2 as,

$$\tau\gamma_1(\tau)^2 - \tau^2\gamma_1(\tau)\gamma_1'(\tau) + \frac{1}{3}\tau^3\gamma_1'(\tau)^2 + \frac{1}{3}\tau^3\gamma_1(\tau)\gamma_1''(\tau)^2 - \frac{1}{4}\tau^4\gamma_1'(\tau)\gamma_1''(\tau)$$
$$+ \frac{1}{20}\tau^5\gamma_1''(\tau)^2 - \frac{1}{12}\tau^4\gamma_1(\tau)\gamma_1^{(3)}(\tau) + \frac{1}{15}\tau^5\gamma_1'(\tau)\gamma_1^{(3)}(\tau)$$
$$- \frac{1}{36}\tau^6\gamma_1''(\tau)\gamma_1^{(3)}(\tau) + \frac{1}{252}\tau^7\gamma_1^{(3)}(\tau)^2$$
$$+ \tau\gamma_2(\tau)^2 - \tau^2\gamma_2(\tau)\gamma_2'(\tau) + \frac{1}{3}\tau^3\gamma_2'(\tau)^2 + \frac{1}{3}\tau^3\gamma_2(\tau)\gamma_2''(\tau)^2$$
$$- \frac{1}{4}\tau^4\gamma_2'(\tau)\gamma_2''(\tau) + \frac{1}{20}\tau^5\gamma_2''(\tau)^2 - \frac{1}{12}\tau^4\gamma_2(\tau)\gamma_2^{(3)}(\tau)$$
$$+ \frac{1}{15}\tau^5\gamma_2'(\tau)\gamma_2^{(3)}(\tau) - \frac{1}{36}\tau^6\gamma_2''(\tau)\gamma_2^{(3)}(\tau) + \frac{1}{252}\tau^7\gamma_2^{(3)}(\tau)^2.$$

Table 8: Error in Example 4.3 for $\alpha = \beta = 1$.

τ	$\boldsymbol{W}_2^6[0,1]$ $\lvert\gamma_1(\tau) - \gamma_{1,20,10}(\tau)\rvert$	$\boldsymbol{W}_2^6[0,1]$ $\lvert\gamma_2(\tau) - \gamma_{2,20,10}(\tau)\rvert$	τ	$\boldsymbol{W}_2^6[0,1]$ $\lvert\gamma_1(\tau) - \gamma_{1,20,10}(\tau)\rvert$	$\boldsymbol{W}_2^6[0,1]$ $\lvert\gamma_2(\tau) - \gamma_{2,20,10}(\tau)\rvert$
0.015625	9.80×10^{-8}	1.13×10^{-7}	0.515625	2.35×10^{-7}	2.54×10^{-7}
0.046875	2.10×10^{-7}	2.29×10^{-7}	0.546875	2.34×10^{-7}	2.53×10^{-7}
0.078125	2.46×10^{-7}	2.63×10^{-7}	0.578125	2.37×10^{-7}	2.55×10^{-7}
0.109375	2.46×10^{-7}	2.62×10^{-7}	0.609375	2.42×10^{-7}	2.57×10^{-7}
0.140625	2.35×10^{-7}	2.54×10^{-7}	0.640625	2.45×10^{-7}	2.59×10^{-7}
0.171875	2.29×10^{-7}	2.50×10^{-7}	0.671875	2.44×10^{-7}	2.58×10^{-7}
0.203125	2.30×10^{-7}	2.51×10^{-7}	0.703125	2.38×10^{-7}	2.54×10^{-7}
0.234375	2.34×10^{-7}	2.53×10^{-7}	0.734375	2.32×10^{-7}	2.50×10^{-7}
0.265625	2.37×10^{-7}	2.55×10^{-7}	0.765625	2.33×10^{-7}	2.49×10^{-7}
0.296875	2.36×10^{-7}	2.54×10^{-7}	0.796875	2.45×10^{-7}	2.53×10^{-7}
0.328125	2.33×10^{-7}	2.53×10^{-7}	0.828125	2.67×10^{-7}	1.61×10^{-7}
0.359375	2.32×10^{-7}	2.52×10^{-7}	0.859375	2.95×10^{-7}	2.60×10^{-7}
0.390625	2.33×10^{-7}	2.53×10^{-7}	0.890625	3.16×10^{-7}	2.21×10^{-7}
0.421875	2.36×10^{-7}	2.54×10^{-7}	0.921875	3.21×10^{-7}	9.44×10^{-8}
0.453125	2.38×10^{-7}	2.55×10^{-7}	0.953125	3.05×10^{-7}	2.00×10^{-7}
0.484375	2.37×10^{-7}	2.55×10^{-7}	0.984375	2.85×10^{-7}	7.78×10^{-7}

Table 9: Convergence order in Example 4.3.

	$\alpha = \beta = 1,\ n = 20$, in $\boldsymbol{W}_2^6[0,1]$				
N	$\lVert\gamma_1 - \gamma_{1,n,N}\rVert_\infty$	$C.F_1$	$\lVert\gamma_2 - \gamma_{2,n,N}\rVert_\infty$	$C.F_2$	$Cpu\ time(sec)$
2	6.64×10^{-2}	–	1.72×10^{-2}	–	1
4	7.99×10^{-4}	6.37	2.29×10^{-4}	10.62	3
8	5.06×10^{-7}	6.23	2.78×10^{-6}	6.36	6

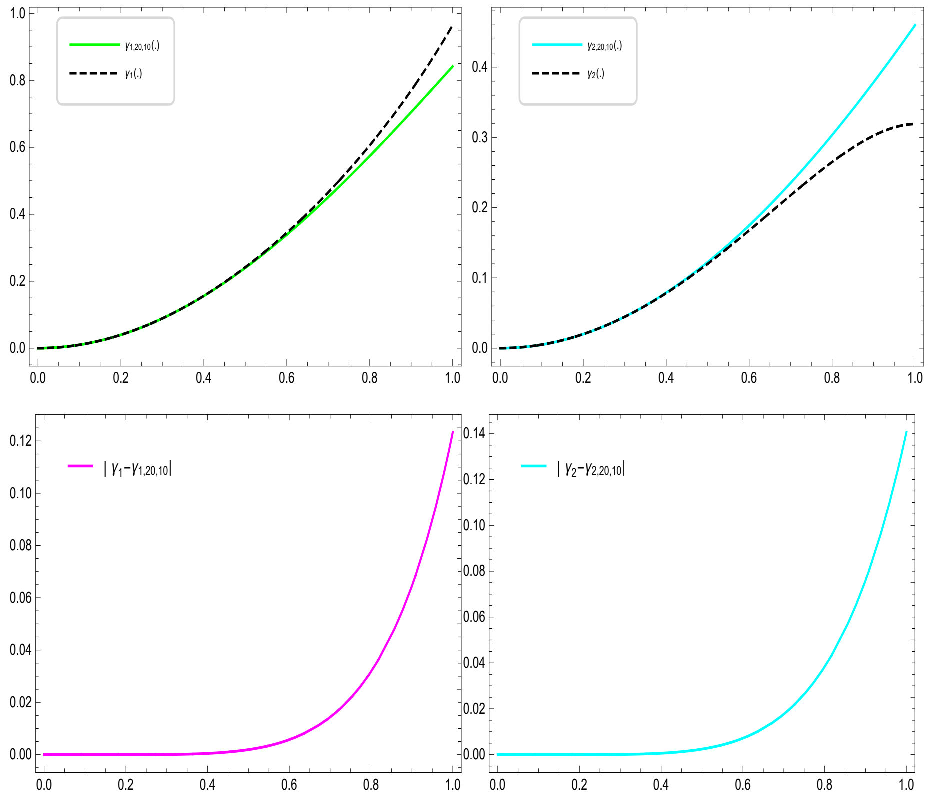

Figure 8: Numerical solution and absolute error without using the Taylor series expansion in Example 4.3.

Example 4.4. Consider the NFVI-DEs:

$$\begin{cases} D^{\alpha}\gamma_1(\tau) - \int_0^{\tau} \gamma_1^2(x)\,dx = g_1(\tau), \\ D^{\beta}\gamma_2(\tau) - \int_0^{\tau} \gamma_2^2\,dx = g_2(\tau), \\ \gamma_1(0) = 0,\ \gamma_2(0) = 0, \end{cases}$$

where

$$0 < \alpha, \beta \leq 1,$$

and the analytical solution for $\alpha = \beta = 1$ is

$$\boldsymbol{\gamma}(\tau) = (\tau + \frac{\tau^3}{2}, \tau - \frac{\tau^3}{2}).$$

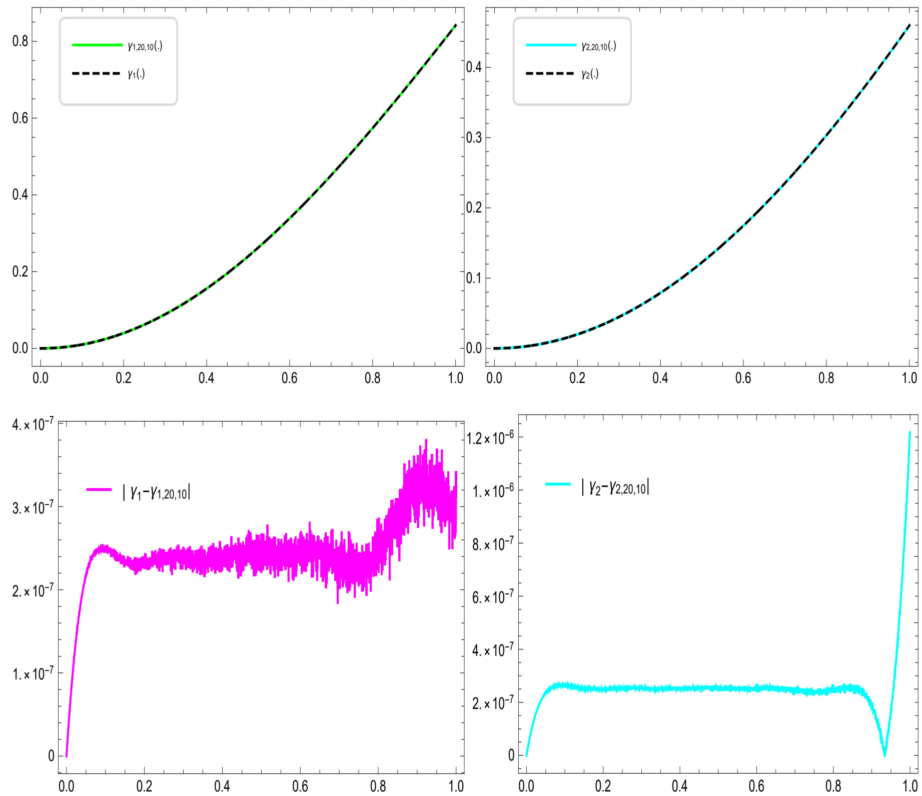

Figure 9: Numerical solution and absolute error with using the Taylor series expansion in Example 4.3.

First, we solved this example in the $\boldsymbol{W}_2^4[0,1]$ space without using the Taylor expansion, which utilizes $\tau_l = \frac{l}{2N+1}$ points. The numerical solution and absolute error are shown in Figure 11. However, this method is not effective. To improve the results, we compared the Present method, which uses the Taylor expansion, with the method proposed in [23]. This comparison was based on the absolute error and numerical solution, as shown in Table 10 and Figures 12 and 13. The convergence order is also shown in Table 11. Finally, to solve this example according to (2) and (3) instead of $\gamma_1(\cdot)$ and $\gamma_2(\cdot)$, we use the truncated Taylor series expansion around the point x in the interval $[0,1]$ for $m=3$. As a result, the nonlinear parts of H_1 as

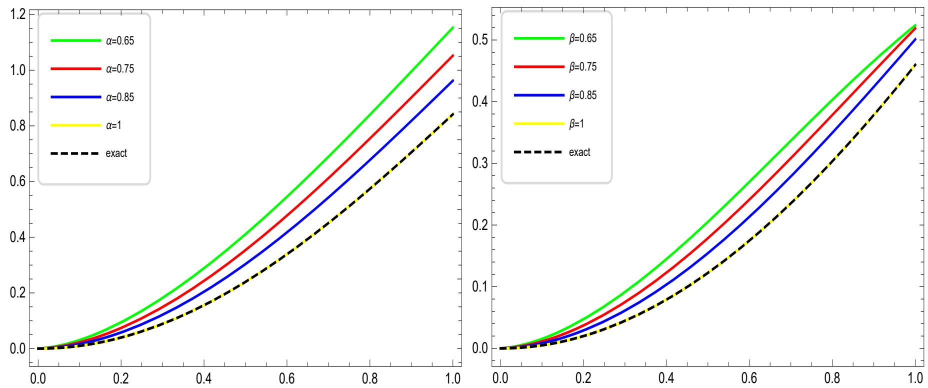

Figure 10: Comparison of the numerical solutions using the Present method with the Taylor series expansion in Example 4.3 for different values α and β. (Left: $\gamma_{1,20,10}(\cdot)$; Right: $\gamma_{2,20,10}(\cdot)$).

$$\tau\gamma_1(\tau)^2 - \tau^2\gamma_1(\tau)\gamma_1'(\tau) + \frac{1}{3}\tau^3\gamma_1'(\tau)^2 + \frac{1}{3}\tau^3\gamma_1(\tau)\gamma_1''(\tau)^2 - \frac{1}{4}\tau^4\gamma_1'(\tau)\gamma_1''(\tau) + \frac{1}{20}\tau^5\gamma_1''(\tau)^2,$$

and H_2 as

$$\tau\gamma_2(\tau)^2 - \tau^2\gamma_2(\tau)\gamma_2'(\tau) + \frac{1}{3}\tau^3\gamma_2'(\tau)^2 + \frac{1}{3}\tau^3\gamma_2(\tau)\gamma_2''(\tau)^2 - \frac{1}{4}\tau^4\gamma_2'(\tau)\gamma_2''(\tau) + \frac{1}{20}\tau^5\gamma_2''(\tau)^2.$$

Table 10: Error in Example 4.4 for $\alpha = \beta = 1$.

τ	$\boldsymbol{W}_2^4[0,1]$ $\lvert\gamma_1(\tau)-\gamma_{1,20,16}(\tau)\rvert$	$\boldsymbol{W}_2^4[0,1]$ $\lvert\gamma_2(\tau)-\gamma_{2,20,16}(\tau)\rvert$	τ	$\boldsymbol{W}_2^4[0,1]$ $\lvert\gamma_1(\tau)-\gamma_{1,20,16}(\tau)\rvert$	$\boldsymbol{W}_2^4[0,1]$ $\lvert\gamma_2(\tau)-\gamma_{2,20,16}(\tau)\rvert$
0.015625	2.82×10^{-7}	2.82×10^{-7}	0.515625	3.28×10^{-7}	2.99×10^{-7}
0.046875	3.14×10^{-7}	3.14×10^{-7}	0.546875	3.31×10^{-7}	2.97×10^{-7}
0.078125	3.12×10^{-7}	3.12×10^{-7}	0.578125	3.35×10^{-7}	2.94×10^{-7}
0.109375	3.12×10^{-7}	3.12×10^{-7}	0.609375	3.39×10^{-7}	2.91×10^{-7}
0.140625	3.12×10^{-7}	3.12×10^{-7}	0.640625	3.44×10^{-7}	2.88×10^{-7}
0.171875	3.13×10^{-7}	3.12×10^{-7}	0.671875	3.49×10^{-7}	2.85×10^{-7}
0.203125	3.13×10^{-7}	3.11×10^{-7}	0.703125	3.55×10^{-7}	2.82×10^{-7}
0.234375	3.13×10^{-7}	3.11×10^{-7}	0.734375	3.61×10^{-7}	2.78×10^{-7}
0.265625	3.14×10^{-7}	3.10×10^{-7}	0.765625	3.69×10^{-7}	2.74×10^{-7}
0.296875	3.15×10^{-7}	3.09×10^{-7}	0.796875	3.76×10^{-7}	2.70×10^{-7}
0.328125	3.16×10^{-7}	3.09×10^{-7}	0.828125	3.85×10^{-7}	2.65×10^{-7}
0.359375	3.17×10^{-7}	3.07×10^{-7}	0.859375	3.95×10^{-7}	2.60×10^{-7}
0.390625	3.19×10^{-7}	3.06×10^{-7}	0.890625	4.05×10^{-7}	2.55×10^{-7}
0.421875	3.20×10^{-7}	3.05×10^{-7}	0.921875	4.16×10^{-7}	2.50×10^{-8}
0.453125	3.22×10^{-7}	3.03×10^{-7}	0.953125	4.28×10^{-7}	2.44×10^{-7}
0.484375	3.25×10^{-7}	3.01×10^{-7}	0.984375	4.41×10^{-7}	2.38×10^{-7}

Table 11: Convergence order in Example 4.4.

	$\alpha=\beta=1,\ n=20$, in $\boldsymbol{W}_2^4[0,1]$				
N	$\left\|\gamma_1-\gamma_{1,n,N}\right\|_\infty$	$C.F_1$	$\left\|\gamma_2-\gamma_{2,n,N}\right\|_\infty$	$C.F_2$	$Cpu\ time(sec)$
4	2.32×10^{-4}	$-$	1.85×10^{-4}	$-$	1
8	6.28×10^{-6}	5.20	4.45×10^{-6}	3.80	4
16	4.48×10^{-7}	5.37	3.25×10^{-7}	3.77	8

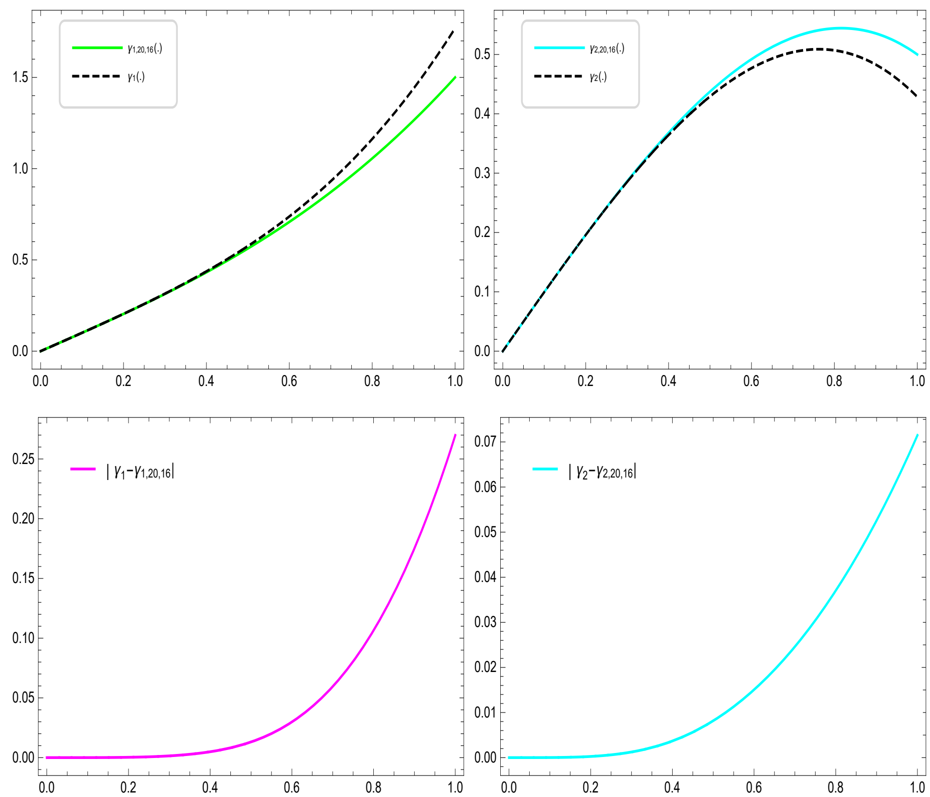

Figure 11: Numerical solution and absolute error without using the Taylor series expansion in Example 4.4.

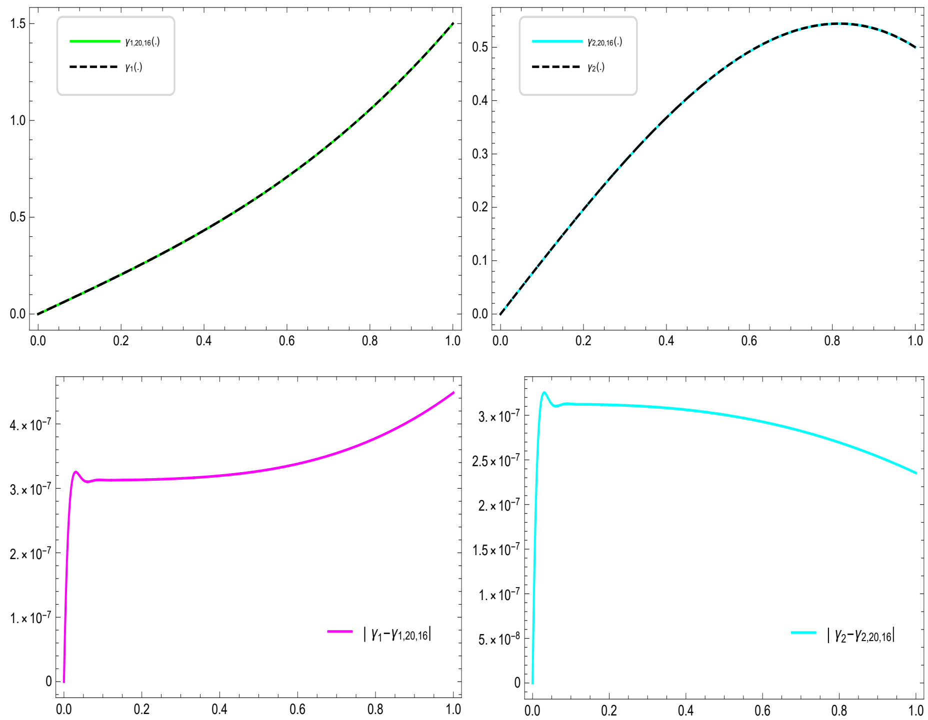

Figure 12: Numerical solution and absolute error with using the Taylor series expansion in Example 4.4.

5 Conclusions

In this article, we proposed a novel approach for solving systems of NFVI-DEs by combining the RKM without the G-SOP with the Taylor series expansion. In Example 4.1, we presented an alternative method that does not employ Taylor series expansion and demonstrated its effectiveness for certain systems of integro-differential equations. However, we found that this method proves less effective for problems involving Volterra's integral applied to nonlinear components. Even modifications to the space and points within this method fail to yield improved results. Consequently, to solve such problems effectively, we must integrate the RKM with Taylor series expansion. Our numerical results validate the efficacy of this combined approach.

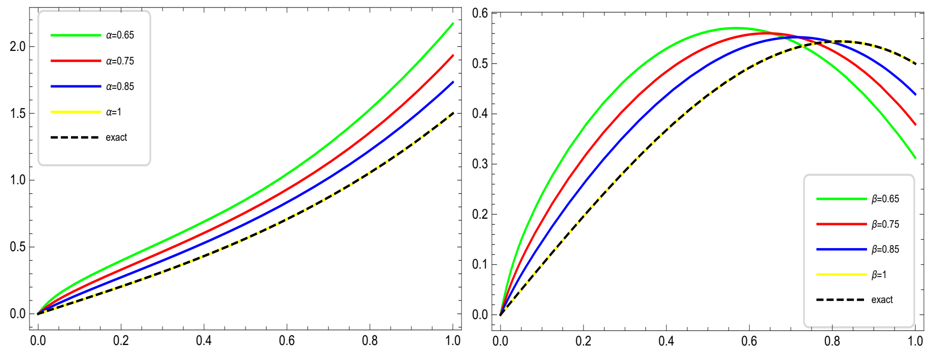

Figure 13: Comparison of the numerical solutions using the Present method with the Taylor series expansion in Example 4.4 for for different values α and β. (Left: $\gamma_{1,20,16}(\cdot)$; Right: $\gamma_{2,20,16}(\cdot)$).

Declarations

Author Contributions: I confirm that all authors listed on the title page have contributed significantly to the work, have read the manuscript, attest to the validity and legitimacy of the data and its interpretation, and agree to its submission.

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Data availability: Our manuscript has no associated data.

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