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Research Article

Mathematical modeling and optimal control strategies to limit cochineal infestation on cacti plants

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Abstract

This paper introduces a mathematical model, denoted as $SIRMC$, aimed at understanding the dynamics of cochineal infestation in cacti plants. The model incorporates two control strategies: biological control through *Hyperaspis trifurcata*, a natural predator of cochineal, and chemical control via insecticide spraying. The objective is to reduce the number of infected cacti while also achieving a balance between minimizing infection, maximizing recovery over time, and minimizing the costs associated with the control measures. The proposed framework effectively integrates these strategies to manage cochineal dynamics. Optimal control strategies are derived using Pontryagin's maximum principle, and numerical simulations conducted in MATLAB validate the theoretical results.

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1 Introduction

The cactus plant, often referred to as “the fruit of the poor” in some cultures, is quite popular in Morocco. It holds significant importance in both the culinary and natural heritage of many Moroccan regions. The fruit of the cactus is commonly used in cooking and is an integral part of the traditional diet in numerous households [1]. Additionally, the cactus is valued for its medicinal properties in folk and traditional medicine, as it is believed to offer various health benefits. Many also associate the cactus plant with important nutritional and medicinal values [15]. Cacti are among the most readily available and easily cared-for plants, making them a valuable resource in areas with limited resources.

The appearance of the cochineal insect in Morocco for the first time in late 2014, specifically in the village of Saniat Bergig in the Sidi Bennur province, led to significant losses, even causing the disappearance of the plant in certain regions of Morocco [16]. The cochineal insect negatively affects cactus plants primarily by extracting their sap. Through piercing and sucking mouthparts, the insect consumes plant tissues, leading to a substantial loss of sap and hindering the transport of essential nutrients and water within the plant. This depletion of sap results in stunted growth and a reduction in size, particularly affecting younger cactus plants. Observable signs of damage include changes in leaf color and a loss of turgidity, compromising the aesthetic appearance of the cactus [3]. Additionally, the overall health of the plant suffers, diminishing its ability to withstand environmental stressors such as drought or temperature fluctuations. Effectively controlling the cochineal insect is crucial for preserving the health of cactus plants, especially when infestations are widespread and threaten crop vitality.

Cultivating cactus plants presents significant economic potential for farmers due to the plant’s versatile applications [17]. Cactus oil, extracted for its moisturizing properties, is a key ingredient in beauty and skincare products, creating a lucrative market [14]. The cactus plant's resilience makes it well-suited for cultivation in arid regions where other crops struggle, providing a sustainable income source for farmers in such environments. Exporting cactus products to international markets, provided quality standards are met, opens opportunities for increased production and income. Additionally, adopting organic farming practices with cactus plants can enhance their value in niche markets, leading to higher prices and better overall income for farmers. In essence, cactus plant cultivation offers farmers a multifaceted opportunity to diversify their income streams and capitalize on the plant's marketable qualities across various industries [12].

Contrary to common belief, the cochineal insect is not native to Morocco. Originally from South America, the cochineal insect has spread to various parts of the world due to factors such as global trade and human movement [5].

The cochineal insect, scientifically known as *Dactylopius coccus*, holds significant historical importance for its role in producing a natural red dye. Its use as a dye dates back to ancient civilizations, where it was highly valued for coloring textiles and fabrics, contributing to various artistic and cultural expressions [4]. The crimson hue derived from cochineal had a notable economic impact, being one of the most valuable products globally during certain historical periods. It played a crucial role in international trade and the economies of specific countries. Beyond its economic value, the cochineal insect found applications in ancient medicine, potentially serving as a source of natural dyes with medicinal properties. The red dye's influence extended to cultural and artistic realms, shaping the colors and patterns in artworks and traditional clothing. The cochineal trade not only influenced economies but also encouraged exploration, contributing to the interconnectedness of the world. Despite its historical significance, the use of cochineal in dye production has declined with technological advancements and the rise of synthetic alternatives. Nevertheless, its historical impact is still evident in various aspects of human culture and history [5].

The cochineal insect poses a serious threat to cacti, adversely affecting crops and requiring efficient, eco-friendly control measures. Biological control using *Hyperaspis Trifurcata* offers an innovative and safe method to combat this pest without relying on harmful pesticides. As a natural predator, *Hyperaspis Trifurcata* feeds on the cochineal insect without harming agricultural plants, providing a dependable and safe solution for pest control. This method offers significant environmental benefits by reducing pesticide use, maintaining ecological balance, and mitigating the negative impacts on the environment and wildlife. Furthermore, it helps preserve plant health by limiting the spread of cochineal infestations, safeguarding agricultural crops, and minimizing yield losses. The integration of *Hyperaspis Trifurcata* aligns with sustainable agricultural practices, focusing on enhancing crop quality and naturally managing pest populations. However, successful implementation requires effective distribution, continuous monitoring, and proper deployment in agricultural settings, with ongoing impact assessments. In essence, *Hyperaspis Trifurcata* represents an eco-friendly solution for managing cochineal infestations and promoting sustainable agriculture, contributing to environmental and agricultural equilibrium.

A key contribution of this study is the introduction of a novel mathematical model, the $SIRMC$ model, designed to understand the dynamics of cochineal infestations in cactus plants. Additionally, our work explores two control strategies for managing the spread of the cochineal insect. The first involves biological control through *Hyperaspis Trifurcata*, a predator that feeds on the cochineal insect without harming the host plant. The second strategy involves insecticide spraying to chemically suppress infestations. The goal is to reduce the number of infected cacti while also achieving a balance between minimizing the infected and maximizing recovery over time.

This paper is structured as follows: Section 2 introduces a deterministic model for the cochineal, outlining its fundamental characteristics. Section 3 constructs a mathematical model that integrates an optimal control strategy for cochineal propagation, presenting results related to the existence of optimal control as defined by Pontryagin's maximum principle. Section 4 discusses a suitable numerical method and presents the corresponding simulation results. Finally, Section 5 concludes with a summary of the insights gained from this study.

2 Formulation of the mathematical model

This study presents the nonlinear mathematical model $SIRMC$, which was developed for the purpose of analyzing the control of the cochineal insect that destroys cactus plants. The following section will explain the five sections of the model in turn.

2.1 Description of $SIRMC$ model dynamic

Before introducing our model, it is essential to understand the life cycle and spread of the cochineal insect. These small, soft-bodied insects reproduce quickly, with females laying eggs that hatch into larvae covered in a white, waxy substance. This wax helps the larvae retain moisture and resist sun exposure. The waxy threads also allow the insects to be carried by the wind to nearby cacti, spreading infestations rapidly. Cochineal insects feed on the fluids of cacti, which can cause significant damage, often leading to the plant's death in severe cases. Farmers typically rely on cutting, burying, or burning infected plants to prevent further spread. However, the introduction of natural predators like Hyperaspis trifurcata, a species of lady beetle, offers a biological solution to this pest, reducing the need for more destructive methods.

Our proposed model, denoted as $SIRMC$, is designed to simulate the dynamics of this interaction, focusing on the spread of the cochineal insect and its control within cactus populations. The total cacti population is denoted by $N(t)$, which satisfies the equation:

$$N(t) = S(t) + I(t) + R(t).$$

The model is structured into five compartments:

Compartment S (Susceptible cacti): Represents healthy cacti that are vulnerable to cochineal infection.

Compartment I (Infectious cacti): Contains cacti that are currently infested by cochineal insects and can transmit the infestation to other plants.

Compartment R (Recovered cacti): Represents cacti that have recovered from the infestation, either naturally or due to external interventions.

Compartment M (Cochineal insect): This compartment models the population of cochineal insects responsible for spreading the infection among cacti.

Compartment C (Hyperaspis trifurcata): Represents the population of lady beetles that predate on cochineal insects, helping to control their spread and protect the cacti.

The $SIRMC$ model we propose is illustrated by the following diagram:

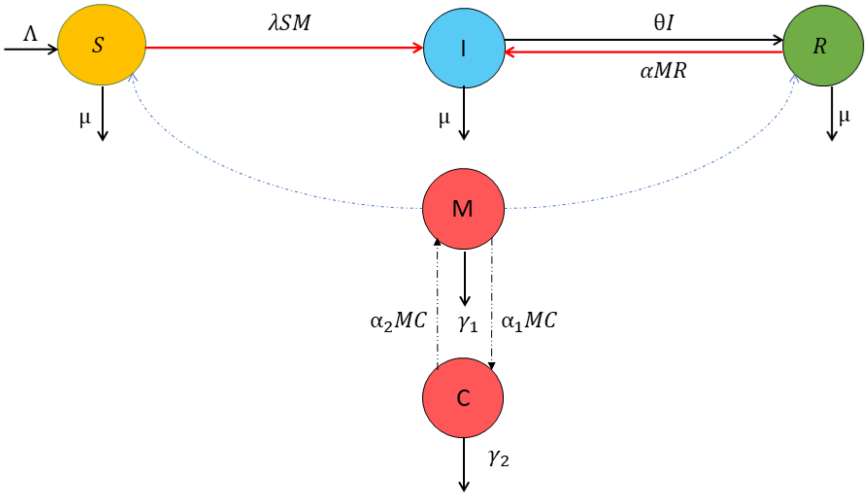

Figure 1: A diagram of the evolution of the transfer in the $SIRMC$ model depicts interactions between cactus populations, cochineal insects, and the control measure *Hyperaspis trifurcata*.

The transition rules between the groups are illustrated as follows:

The Susceptible Cacti Population:
Susceptible cacti are introduced into the system through natural recruitment at a rate Λ. However, they may become infested when they come into contact with cochineal insects at a rate λSM. Additionally, healthy cacti may die due to natural causes at a rate μS. The equation governing this transition is

$$\frac{dS}{dt} = \Lambda - \lambda SM - \mu S.$$

The Infected Cacti Population:
Once a healthy cactus becomes infested, it moves into the infected group. The number of infected cacti increases when healthy cacti acquire the infestation (λSM) or when recovered cacti are reinfected due to residual cochineal insects (αMR). Infected cacti may either recover at a rate θI or die naturally at a rate μI. This transition is expressed as

$$\frac{dI}{dt} = \lambda SM + \alpha MR - \theta I - \mu I.$$

The Recovered Cacti Population:
Cacti that recover from infestation enter the recovered group at a rate θI. However, some recovered cacti may be reinfected due to the presence of cochineal insects at a rate αMR, while others die naturally at a rate μR. The equation describing this transition is

$$\frac{dR}{dt} = \theta I - \alpha MR - \mu R.$$

The Cochineal Insect Population:
The cochineal insect population follows a logistic growth pattern, increasing at a rate $\beta_1 M(1 - M/K_1)$, where K_1 is the carrying capacity. However, their numbers decrease due to natural mortality at a rate $\gamma_1 M$ and predation by *Hyperaspis Trifurcata* at a rate $\alpha_1 MC$. The governing equation for this dynamic is

$$\frac{dM}{dt} = \beta_1 M \left(1 - \frac{M}{K_1}\right) - \gamma_1 M - \alpha_1 MC.$$

The Predator Population (*Hyperaspis Trifurcata*):

The predator species, *Hyperaspis Trifurcata*, follows a similar logistic growth pattern, increasing at a rate $\beta_2 C(1 - C/K_2)$, where K_2 is its carrying capacity. The predator also benefits from consuming cochineal insects, leading to additional reproduction at a rate $\alpha_2 MC$. However, its population declines due to natural mortality at a rate $\gamma_2 C$. This dynamic is represented by

$$\frac{dC}{dt} = \beta_2 C\left(1 - \frac{C}{K_2}\right) - \gamma_2 C + \alpha_2 MC.$$

The following system consists of nonlinear ordinary differential equations:

$$\begin{cases} \frac{dS}{dt} &= \Lambda - \lambda SM - \mu S, \\ \frac{dI}{dt} &= \lambda SM + \alpha MR - \theta I - \mu I, \\ \frac{dR}{dt} &= \theta I - \alpha MR - \mu R, \\ \frac{dM}{dt} &= \beta_1 M\left(1 - \frac{M}{K_1}\right) - \gamma_1 M - \alpha_1 M(t)C(t), \\ \frac{dC}{dt} &= \beta_2 C\left(1 - \frac{C}{K_2}\right) - \gamma_2 C + \alpha_2 M(t)C(t), \end{cases} \tag{1}$$

with the initial conditions: $S(0) \geq 0$, $I(0) \geq 0$, $R(0) \geq 0$, $M(0) \geq 0$, and $C(0) \geq 0$.

The parameters of model are defined in Table 1.

Table 1: Model parameters

Parameter	**Description**
Λ	The recruitment rate for the cacti plants.
β_1	The growth rate for the cochineal insect.
β_2	The growth rate for the Hyperaspis Trifurcata.
α	The reinfection rate of cacti plants after recovery from the cochineal insect.
α_1	The rate of Hyperaspis trifurcata encounters the cochineal insect and preys on it.
α_2	The reproduction rate of Hyperaspis trifurcata due to feeding on the cochineal insect.
θ	The recovery rate for cacti plants.
γ_1	The natural death rate for the cochineal insect.
γ_2	The natural death rate for Hyperaspis Trifurcata.
λ	λ is the rate of cochineal insect encounters with cacti.
μ	The cochineal insect mortality rate

2.2 Positivity of solutions

Theorem 1. If $S(0) \geq 0$, $I(0) \geq 0$, $R(0) \geq 0$, $M(0) \geq 0$ and $C(0) \geq 0$, then the solutions $S(t)$, $I(t)$, $R(t)$, $M(t)$, and $C(t)$ of system (1) are positive for all $t \geq 0$.

Proof. It follows from the first equation of system (1) that

$$\frac{\mathrm{d}S}{\mathrm{d}t} = \Lambda - \lambda SM - \mu S \geq -\left(\lambda M + \mu\right) S. \tag{2}$$

Then, we have

$$\frac{\mathrm{d}S}{S} \geq -\left(\lambda M + \mu\right) \mathrm{d}t. \tag{3}$$

By integrating (2) from 0 to t, we obtain

$$S(t) \geq S_b(0) e^{-\int_0^t (\lambda M + \mu) ds}.$$

That implies

$$S(t) \geq 0 \quad \text{for all } t \geq 0.$$

Similarly, we prove that $I(t) \geq 0$, $R(t) \geq 0$, $M(t) \geq 0$ and $C(t) \geq 0$ for all $t \geq 0$.

□

2.3 Boundedness of the solutions.

Theorem 2. Let $T_1 = \max\{M(0), K_1\}$, $T_2 = \max\{C(0), K_2\}$. Then the set $\Gamma = \left\{(S, I) \in \mathbb{R}_+^2 : N(t) \leq \frac{\Lambda}{\mu}\right\} \times \{M \in \mathbb{R}_+ : M(t) \leq T_1\} \times \{C \in \mathbb{R}_+ : C(t) \leq T_2\}$ is positively invariant under system (1) with nonnegative initial conditions $S(0)$, $I(0)$, $R(0)$, $M(0)$, and $C(0)$.

Proof. From the initial equations of (1), we derive

$$\frac{dN(t)}{dt} = \Lambda - \mu N(t). \tag{4}$$

Then,

$$N(t) \leq N(0) e^{-\mu t} + \frac{\Lambda}{\mu}\left[1 - e^{-\mu t}\right]. \tag{5}$$

If we consider the limit $t \to \infty$, then $0 \leq N(t) \leq \frac{\Lambda}{\mu}$.

From the last equation of (1), we have

$$\frac{\mathrm{d}M}{\mathrm{d}t} \leq \beta M \left(1 - \frac{M}{K_1}\right). \tag{6}$$

Hence, employing a typical comparison approach yields $\limsup_{t\to\infty} M(t) \leq K_1$.

Similarly, we prove that $\limsup_{t\to\infty} C(t) \leq K_2$.

Finally, the set Γ is positivity invariant for the system (1). ☐

2.4 Existence of solutions

Theorem 3. The system (1) that satisfies a given initial condition $(S(0), I(0), R(0), M(0), C(0))$ has a unique solution.

Proof. The model (1) can be expressed in matrix form as follows: Let $X(t) = (S, I, R, M, C)^T$ and $F(X(t)) = \left(\frac{dS}{dt}, \frac{dI}{dt}, \frac{dR}{dt}, \frac{dM}{dt} \frac{dC}{dt}\right)^T$.

The model (1) can be rephrased as

$$F(X(t)) = AX + B(X(t)),$$

where

$$A = \begin{pmatrix} -\mu & 0 & 0 & 0 & 0 \\ 0 & -(\mu+\theta) & 0 & 0 & 0 \\ 0 & \theta & -\mu & 0 & 0 \\ 0 & 0 & 0 & \beta_1 - \gamma_1 & 0 \\ 0 & 0 & 0 & 0 & \beta_2 - \gamma_2 \end{pmatrix}$$

and

$$B(X(t)) = \begin{pmatrix} \Lambda - \lambda SM \\ \lambda SM + \alpha MR \\ -\alpha MR \\ -\frac{\beta_1 M^2}{K_1} - \alpha_1 MC \\ -\frac{\beta_2 C^2}{K_2} + \alpha_2 MC \end{pmatrix}.$$

Let X_1 and X_2 be solutions of model (1). Then

$$
\begin{aligned}
|B\left(X_1\right)-B\left(X_2\right)| \leq & 2 \mid \lambda(S_2M_2-S_1M_1)|+2 \mid \alpha(M_1R_1-M_2R_2)| \\
& +\mid \alpha_1(M_2C_2-M_1C_1)|+\mid \alpha_2(M_1C_1-M_2C_2)| \\
& +\mid \frac{\beta_1}{K_1}(M_2^2-M_1^2)|+\mid \frac{\beta_2}{K_2}(C_2^2-C_1^2)| \\
\leq & 2 \mid \lambda(S_2M_2-S_2M_1)|+2 \mid \lambda(S_2M_1-S_1M_1)| \\
& +2 \mid \alpha(M_1R_1-M_1R_2)|+2 \mid \alpha(M_1R_2-M_2R_2)| \\
& +\mid \alpha_1(M_2C_2-M_2C_1)|+\mid \alpha_1(M_2C_1-M_1C_1)| \\
& +\mid \alpha_2(M_1C_1-M_1C_2)|+\mid \alpha_2(M_1C_2-M_2C_2)| \\
& +\mid \frac{\beta_1}{K_1}(M_2^2-M_1^2)+\mid \frac{\beta_2}{K_2}(C_2^2-C_1^2)| \\
\leq & 2\lambda S_2 \mid M_2-M_1|+2\lambda M_1 \mid S_2-S_1| \\
& +2\alpha M_1 \mid R_1-R_2|+2\alpha R_2 \mid M_1-M_2|+\alpha_1 M_2 \mid C_2-C_1| \\
& +\alpha_1 C_1 \mid M_2-M_1|+\alpha_2 M_1 \mid C_1-C_2|+\alpha_2 C_2 \mid M_1-M_2| \\
& +\frac{\beta_1}{K_1} \mid M_2-M_1| \mid M_2+M_1|+\frac{\beta_2}{K_2} \mid C_2-C_1| \mid C_2+C_1| \\
\leq & \frac{2\lambda\Lambda}{\mu} \mid M_2-M_1|+2\lambda T_1 \mid S_2-S_1|+2\alpha T_1 \mid R_1-R_2| \\
& +\frac{2\alpha\Lambda}{\mu} \mid M_1-M_2|+\alpha_1 T_1 \mid C_2-C_1| \\
& +\alpha_1 T_2 \mid M_2-M_1|+\alpha_2 T_1 \mid C_1-C_2| \\
& +\alpha_2 T_2 \mid M_1-M_2|+\frac{2\beta_1 T_1}{K_1} \mid M_2-M_1|+\frac{2\beta_2 T_2}{K_2} \mid C_2-C_1| \\
\leq & \left(\frac{2\lambda\Lambda}{\mu}+\frac{2\alpha\Lambda}{\mu}+\frac{2\beta_1 T_1}{K_1}+(\alpha_1+\alpha_2)T_2\right) \mid M_1-M_2| \\
& +2\lambda T_1 \mid S_1-S_2|+2\alpha T_1 \mid R_1-R_2| \\
& +\left(\frac{2\beta_2}{K_2}+(\alpha_1+\alpha_2)T_1\right) \mid C_2-C_1| \\
\leq & N\left\|X_1-X_2\right\|,
\end{aligned}
$$

where

$$
N=\max\left(\frac{2\lambda\Lambda}{\mu}+\frac{2\alpha\Lambda}{\mu}+\frac{2\beta_1 T_1}{K_1}+(\alpha_1+\alpha_2)T_2, 2\lambda T_1, 2\alpha T_1, \frac{2\beta_2}{K_2}+(\alpha_1+\alpha_2)T_1, \|A\|\right).
$$

Therefore,

$$
\left\|F\left(X_1\right)-F\left(X_2\right)\right\| \leq N\left\|X_1-X_2\right\|.
$$

Thus, it follows that the function F is uniformly Lipschitz continuous, and the restriction on $S(t) \geq 0$, $I(t) \geq 0$, $R(t) \geq 0$, $M(t) \geq 0$ and $C(t) \geq 0$ in $\mathbb{R}^5_+$. Therefore, a solution of the model (1) exists [2].

$\square$

3 The optimal control problem

Given the ongoing threat of cochineal infestations and their severe economic impact on cactus production, farmers need a cost-effective strategy to control the pest's spread within a specific timeframe. To address this, we develop an optimal control problem that focuses on minimizing the number of infected plants while also achieving a balance between minimizing infection and maximizing recovery over time. A key aspect of our approach is the natural control provided by *Hyperaspis trifurcata*, a predatory beetle that feeds on cochineal insects. By incorporating this biological control agent into the model, we emphasize the beetle's role in naturally reducing cochineal infestations. *Hyperaspis trifurcata* offers a sustainable and environmentally friendly alternative to chemical pesticides, as it directly targets the cochineal population, helping to curb its spread.

The system of equations (1) is adjusted to include two control variables, $u_1(t)$ and $u_2(t)$ for $t \in [t_0, t_f]$.

$$\begin{cases} \frac{dS}{dt} &= \Lambda - \lambda S(t)M(t) - \mu S(t), \\ \frac{dI}{dt} &= \lambda S(t)M(t) + \alpha M(t)R(t) - \theta I(t) - \mu I(t) - u_1(t)I(t), \\ \frac{dR}{dt} &= \theta I(t) - \alpha M(t)R(t) - \mu R(t) + u_1(t)I(t), \\ \frac{dM}{dt} &= \beta_1 M(t)\left(1 - \frac{M(t)}{K_1}\right) - \gamma_1 M(t) - \alpha_1 M(t)C(t) - u_2(t)\sigma_1 M(t)C(t), \\ \frac{dC}{dt} &= \beta_2 C(t)\left(1 - \frac{C(t)}{K_2}\right) - \gamma_2 C(t) + \alpha_2 M(t)C(t) + u_2(t)\sigma_2 M(t)C(t), \end{cases} \tag{7}$$

with the initial conditions $S(0) \geq 0$, $I(0) \geq 0$, $R(0) \geq 0$, $M(0) \geq 0$, and $C(0) \geq 0$.

The control $u_1(t)$ represents the application of insecticide to combat cochineal, while the control $u_2(t)$ denotes the use of Hyperaspis trifurcata, a predator that feeds on cochineal.

The problem is to minimize the objective functional:

$$\begin{aligned} J(u_1,u_2) =& I(t_f)+M(t_f)-R(t_f) \\ & + \int_{t_0}^{t_f} \left[I(t)+M(t)-R(t)+\frac{C_1}{2}\left(u_1(t)\right)^2+\frac{C_2}{2}\left(u_2(t)\right)^2 \right] dt, \end{aligned} \tag{8}$$

where $C_1 > 0$ and $C_2 > 0$, are chosen to assign the relative importance of $u_1(t)$ and $u_2(t)$ at any given time t, with t_f representing the final time.

In other words, our goal is to find the optimal controls u_1^* and u_2^* such that

$$J(u_1^*,u_2^*) = \min_{(u_1,u_2)\in U} J(u_1,u_2),$$

where U is the set of admissible controls defined by

$$U = \left\{(u_1(t),u_2(t)) : 0 \leq u_1(t) \leq 1\, 0 \leq u_2(t) \leq 1,\ /\ t\in[t_0,t_f]\right\}.$$

3.1 Existence of optimal controls

In this part, we present the theorem which proves the existence of an optimal control (u_1^*,u_2^*) minimizing the cost function J.

Theorem 4. There exists an optimal control $(u_1^*,u_2^*)\in U$ such that

$$J(u_1^*,u_2^*) = \min_{(u_1,u_2)\in U} J(u_1,u_2).$$

Proof. To use the existence result in [6], we must check the following properties:

(A_1): The set of controls and the corresponding state variables is nonempty.
(A_2): The control set U is convex and closed.
(A_3): The right-hand side of the state system is bounded by a linear function in the state and control variables.

(A_4): The integral $L(I,R,M,u_1,u_2)$ of the objective functional is convex on U, and there exist constants $\varkappa_1 > 0$, $\varkappa_2 > 0$, and $\varepsilon > 1$ such that

$$L(I,R,M,u_1,u_2) \geq -\varkappa_1 + \varkappa_2 \left(|u_1|^2 + |u_2|^2\right)^{\frac{\varepsilon}{2}} .$$

The first condition (A_1) is verified using the result in [11]. The set U is convex and closed by the definition. Thus the condition (A_2). Our state system is linear in u_1 and u_2. Moreover, the solutions of the system are bounded as proved in model (1), hence the condition (A_3). Also, we have the last needed condition (A_4),

$$L(I,R,M,u_1,u_2) \geq -\varkappa_1 + \varkappa_2 \left(|u_1|^2 + |u_2|^2\right)^{\frac{\varepsilon}{2}} ,$$

where $\varkappa_1 = 2\sup_{t\in[t_0,t_f]} (I(t),R(t),M(t))$, $\varkappa_2 = \inf(\dfrac{C_1}{2}, \dfrac{C_2}{2})$, and $\varepsilon = 2$, since $C_1 > 0$ and $C_1 > 0$.

We conclude that there exists an optimal control $(u_1^*, u_2^*) \in U$ such that

$$J(u_1^*, u_2^*) = \min_{(u_1,u_2)\in U} J(u_1,u_2).$$

□

3.2 Characterization of the optimal controls

In this section, we utilize Pontryagin's principle [13]. The central concept is to introduce the adjoint function, which connects the system of differential equations to the objective functional. This connection leads to the formulation of the Hamiltonian. By applying this principle, the task of determining a control that optimizes the objective functional with a specified initial condition is transformed into the problem of finding a control that optimizes the Hamiltonian pointwise.

To derive the optimal control conditions, we apply Pontryagin's maximum principle such that the Hamiltonian H at time t is defined by

$$H(t) = I(t) + M(t) - R(t) + \frac{C_1}{2}\left(u_1(t)\right)^2 + \frac{C_2}{2}\left(u_2(t)\right)^2 + \sum_{i=1}^{5} \lambda_i h_i, \tag{9}$$

where h_i is the right side of the system of differential equations (7) of ith state variable.

Theorem 5. Given the optimal controls (u_1^*, u_2^*) and solutions S^*, I^*, R^*, M^* and C^* of the corresponding state system (7), there exist adjoint functions $\lambda_1,\ \lambda_2, \lambda_3, \lambda_3$, and λ_5 satisfying

$$\begin{cases} \lambda_1' = -\frac{dH}{dS} = \lambda_1(\lambda M(t) + \mu) - \lambda_2 \lambda M(t), \\ \lambda_2' = -\frac{dH}{dI} = -1 + \lambda_2\left(\theta + \mu + u_1(t)\right) - \lambda_3\left(\theta + u_1(t)\right), \\ \lambda_3' = -\frac{dH}{dR} = 1 - \lambda_2 \alpha M(t) + \lambda_3\left(\alpha M(t) + \mu\right), \\ \lambda_4' = -\frac{dH}{dM} = -1 + \lambda_1 \lambda S(t) - \lambda_2\left(\lambda S(t) + \alpha R(t)\right) + \lambda_3 \alpha R(t), \\ \qquad -\lambda_4\left(\beta_1\left(1 - \frac{2M(t)}{K_1}\right) - \gamma_1 - \alpha_1 C(t) - u_2(t)\sigma_1 C(t)\right) \\ \qquad -\lambda_5\left(\alpha_2 C(t) + u_2(t)\sigma_2 C(t)\right), \\ \lambda_5' = -\frac{dH}{dC} = \lambda_4\left(\alpha_1 M(t) + u_2(t)\sigma_1 M(t)\right) \\ \qquad -\lambda_5\left(\beta_2\left(1 - \frac{2C(t)}{K_2}\right) - \gamma_2 + \alpha_2 M(t) + u_2(t)\sigma_2 M(t)\right), \end{cases} \tag{10}$$

Such that the transversality conditions at time t_f are

$$\begin{cases} \lambda_1(t_f) = 0, \\ \lambda_2(t_f) = 1, \\ \lambda_3(t_f) = -1, \\ \lambda_4(t_f) = 1, \\ \lambda_5(t_f) = 0. \end{cases} \tag{11}$$

In addition to that we have, for $t \in [t_0, t_f]$, optimal controls $u_1^*(t)$ and $u_2^*(t)$ are given by

$$\begin{cases} u_1^*(t) = \min\left(1, \max\left(0,\ \frac{1}{C_1}\left(\lambda_2 - \lambda_3\right) I(t)\right)\right), \\ u_2^*(t) = \min\left(1, \max\left(0, \frac{1}{C_2}\left(\sigma_1 \lambda_4 - \sigma_2 \lambda_5\right) C(t) M(t)\right)\right). \end{cases} \tag{12}$$

Proof. The Hamiltonian H is defined as follows:

$$H(t)=I(t)+M(t)-R(t)+\frac{C_1}{2}\left(u_1(t)\right)^2+\frac{C_2}{2}\left(u_2(t)\right)^2+\sum_{i=1}^{5}\lambda_i h_i,$$

where

$$\begin{cases} h_1=\Lambda-\lambda S(t)M(t)-\mu S(t),\\ h_2=\lambda S(t)M(t)+\alpha M(t)R(t)-\theta I(t)-\mu I(t)-u_1(t)I(t),\\ h_3=\theta I(t)-\alpha M(t)R(t)-\mu R(t)+u_1(t)I(t),\\ h_4=\beta_1 M(t)\left(1-\frac{M(t)}{K_1}\right)-\gamma_1 M(t)-\alpha_1 M(t)C(t)-u_2(t)\sigma_1 M(t)C(t),\\ h_5=\beta_2 C(t)\left(1-\frac{C(t)}{K_2}\right)-\gamma_2 C(t)+\alpha_2 M(t)C(t)+u_2(t)\sigma_2 M(t)C(t). \end{cases}$$

For $t\in[t_0,t_f]$, the adjoint equations and transversality conditions can be obtained by using Pontryagin's maximum principle [13] such that

$$\begin{cases} \lambda_1'=-\frac{dH}{dS}=\lambda_1(\lambda M(t)+\mu)-\lambda_2\lambda M(t),\\ \lambda_2'=-\frac{dH}{dI}=-1+\lambda_2\left(\theta+\mu+u_1(t)\right)-\lambda_3\left(\theta+u_1(t)\right),\\ \lambda_3'=-\frac{dH}{dR}=1-\lambda_2\alpha M(t)+\lambda_3\left(\alpha M(t)+\mu\right),\\ \lambda_4'=-\frac{dH}{dM}=-1+\lambda_1\lambda S(t)-\lambda_2\left(\lambda S(t)+\alpha R(t)\right)+\lambda_3\alpha R(t),\\ \qquad -\lambda_4\left(\beta_1\left(1-\frac{2M(t)}{K_1}\right)-\gamma_1-\alpha_1 C(t)-u_2(t)\sigma_1 C(t)\right)\\ \qquad -\lambda_5\left(\alpha_2 C(t)+u_2(t)\sigma_2 C(t)\right)\\ \lambda_5'=-\frac{dH}{dC}=\lambda_4\left(\alpha_1 M(t)+u_2(t)\sigma_1 M(t)\right)\\ \qquad -\lambda_5\left(\beta_2\left(1-\frac{2C(t)}{K_2}\right)-\gamma_2+\alpha_2 M(t)+u_2(t)\sigma_2 M(t)\right). \end{cases}$$

For $t\in[t_0,t_f]$, the optimal controls u_1^* and u_2^* can be solved from the optimality condition we have

$$\frac{dH}{du_1}=C_1u_1(t)-\lambda_2 I(t)+\lambda_3 I(t)=0.$$

So

$$u_1(t)=\quad\frac{1}{C_1}\left(\lambda_2-\lambda_3\right)I(t),$$

we have

$$\frac{dH}{du_2}=C_2u_2(t)-\lambda_4\sigma_1 M(t)C(t)+\lambda_5\sigma_2 M(t)C(t)=0.$$

So

$$u_2(t)=\frac{1}{C_2}\left(\sigma_1\lambda_4-\sigma_2\lambda_5\right)M(t)C(t).$$

By the bounds in U of the controls, it is convenient to obtain u_1^* and u_2^* in the form of (12).

$\square$

4 Numerical simulations

This section begins by introducing an iterative method for numerically solving the optimality system, followed by a presentation of the numerical results obtained using MATLAB.

4.1 Discretization and control algorithm

The numerical algorithm presented below uses a semi-implicit finite difference method to discretize the time interval $[t_0, t_f]$ at the points $t_i = t_0 + ih$ $(i = 0, 1, \ldots, n)$, where h is the time step such that $t_n = t_f$ [7]. The state variables $S(t), I(t), R(t), M(t), C(t)$, and the adjoint variables $\lambda_1, \lambda_2, \lambda_3, \lambda_4, \lambda_5$, along with the control variables u_1 and u_2, are defined at the nodal points as $S_i, I_i, R_i, M_i, C_i, \lambda_1^i, \lambda_2^i, \lambda_3^i, \lambda_4^i, \lambda_5^i, u_1^i, u_2^i$.

We proceed with the discretization using a combination of forward and backward difference approximations as follows: The method, developed by [8] and presented in [9] and [10], is then read as

$$\begin{aligned}
\frac{S_{i+1} - S_i}{h} &= \Lambda - \lambda S_{i+1} M_{i+1} - \mu S_{i+1}, \\
\frac{I_{i+1} - I_i}{h} &= \lambda S_{i+1} M_{i+1} + \alpha M_{i+1} R_{i+1} - \theta I_{i+1} - \mu I_{i+1} - u_1^i I_{i+1}, \\
\frac{R_{i+1} - R_i}{h} &= \theta I_{i+1} - \alpha M_{i+1} R_{i+1} - \mu R_{i+1} + u_1^i I_{i+1}, \\
\frac{M_{i+1} - M_i}{h} &= \beta_1 M_{i+1} \left(1 - \frac{M_{i+1}}{K_1}\right) - \gamma_1 M_{i+1} - \alpha_1 M_{i+1} C_{i+1} - u_2^i \sigma_1 M_{i+1} C_{i+1}, \\
\frac{C_{i+1} - C_i}{h} &= \beta_2 C_{i+1} \left(1 - \frac{C_{i+1}}{K_2}\right) - \gamma_2 C_{i+1} + \alpha_2 M_{i+1} C_{i+1} + u_2^i \sigma_2 M_{i+1} C_{i+1}.
\end{aligned}$$

Using a similar approach, we approximate the time derivative of the adjoint variables by applying a first-order backward difference and then use the corresponding scheme as follows:

$$
\begin{aligned}
\frac{\lambda_1^{n-i}-\lambda_1^{n-i-1}}{h} &= \left(\lambda_1^{n-i-1}-\lambda_3^{n-i}\right)(\lambda M_{i+1}+\mu)-\lambda_2^{n-i}\lambda M_{i+1},\\
\frac{\lambda_2^{n-i}-\lambda_2^{n-i-1}}{h} &= -1+\left(\lambda_1^{n-i-1}-\lambda_2^{n-i-1}\right)(\theta+\mu+u_1^i)-\lambda_3^{n-i}(\theta+u_1^i),\\
\frac{\lambda_3^{n-i}-\lambda_3^{n-i-1}}{h} &= 1-\lambda_2^{n-i-1}\alpha M_{i+1}+\lambda_3^{n-i-1}(\alpha M_{i+1}+\mu),\\
\frac{\lambda_4^{n-i}-\lambda_4^{n-i-1}}{h} &= -1+\lambda_1^{n-i-1}\lambda S_{i+1}-\lambda_2^{n-i-1}(\lambda S_{i+1}+\alpha R_{i+1})+\lambda_3^{n-i-1}\alpha R_{i+1}\\
&\quad -\lambda_4^{n-i-1}\left(\beta_1\left(1-\frac{2M_{i+1}}{K_1}\right)-\gamma_1-\alpha_1 C_{i+1}-u_2^i\sigma_1 C_{i+1}\right)\\
&\quad -\lambda_5^{n-i-1}\left(\alpha_2 C_{i+1}+u_2^i\sigma_2 C_{i+1}\right),\\
\frac{\lambda_5^{n-i}-\lambda_5^{n-i-1}}{h} &= \lambda_4^{n-i-1}\left(\alpha_1 M_{i+1}+u_2^i\sigma_1 M_{i+1}\right)\\
&\quad -\lambda_5^{n-i-1}\left(\beta_2\left(1-\frac{2C_{i+1}}{K_2}\right)-\gamma_2+\alpha_2 M_{i+1}+u_2^i\sigma_2 M_{i+1}\right).
\end{aligned}
$$

The control variables are updated as follows:

Algorithm 2

Step 1:

$$
\begin{aligned}
&S(0)=S_0,\quad I(0)=I_0,\quad R(0)=R_0,\quad M(0)=M_0,\quad C(0)=C_0,\\
&\lambda_1(t_f)=0,\quad \lambda_2(t_f)=1,\quad \lambda_3(t_f)=-1,\quad \lambda_4(t_f)=1,\quad \lambda_5(t_f)=0,\\
&u_1(0)=0,\quad u_2(0)=0.
\end{aligned}
$$

Step 2:

For $i=0,\dots,n-1$, do:

$$
\begin{aligned}
S_{i+1} &= \frac{S_i+h\Lambda}{1+h(\lambda M_{i+1}+\mu)},\\
I_{i+1} &= \frac{I_i+h(\lambda S_{i+1}M_{i+1}+\alpha M_{i+1}R_{i+1})}{1+h(\theta+\mu+u_1^i)},\\
R_{i+1} &= \frac{R_i+h(\theta I_{i+1}-\alpha M_{i+1}R_{i+1}+u_1^i I_{i+1})}{1+h\mu},\\
M_{i+1} &= \frac{M_i+h\beta_1 M_{i+1}\left(1-\frac{M_{i+1}}{K_1}\right)}{1+h(\gamma_1+\alpha_1 C_{i+1}+u_2^i\sigma_1 C_{i+1})},
\end{aligned}
$$

$$C_{i+1} = \frac{C_i + h(\beta_2 C_{i+1}\left(1 - \frac{C_{i+1}}{K_2}\right) + \alpha_2 M_{i+1} C_{i+1} + u_2^i \sigma_2 M_{i+1} C_{i+1})}{1 + h\gamma_2}.$$

$$\begin{aligned}
\lambda_1^{n-i-1} &= \frac{\lambda_1^{n-i} + h\left((\lambda_1^{n-i-1} - \lambda_3^{n-i})(\lambda M_{i+1} + \mu) - \lambda_2^{n-i}\lambda M_{i+1}\right)}{1 + h(\lambda M_{i+1} + \mu)},\\
\lambda_2^{n-i-1} &= \frac{\lambda_2^{n-i} + h\left(-1 + (\lambda_1^{n-i-1} - \lambda_2^{n-i-1})(\theta + \mu + u_1^i) - \lambda_3^{n-i}(\theta + u_1^i)\right)}{1 + h(\theta + \mu + u_1^i)},\\
\lambda_3^{n-i-1} &= \frac{\lambda_3^{n-i} + h\left(1 - \lambda_2^{n-i-1}\alpha M_{i+1} + \lambda_3^{n-i-1}(\alpha M_{i+1} + \mu)\right)}{1 + h(\alpha M_{i+1} + \mu)},\\
\lambda_4^{n-i-1} &= \frac{\lambda_4^{n-i} + h\big(-1 + \lambda_1^{n-i-1}\lambda S_{i+1} - \lambda_2^{n-i-1}(\lambda S_{i+1} + \alpha R_{i+1}) + \lambda_3^{n-i-1}\alpha R_{i+1}\big)}{1 + h(\beta_1(1 - \frac{2M_{i+1}}{K_1}) - \gamma_1 - \alpha_1 C_{i+1} - u_2^i \sigma_1 C_{i+1})},\\
\lambda_5^{n-i-1} &= \frac{\lambda_5^{n-i} + h\big(\lambda_4^{n-i-1}(\alpha_1 M_{i+1} + u_2^i \sigma_1 M_{i+1})\big)}{1 + h(\beta_2(1 - \frac{2C_{i+1}}{K_2}) - \gamma_2 + \alpha_2 M_{i+1} + u_2^i \sigma_2 M_{i+1})}.
\end{aligned}$$

$$\begin{aligned}
T^{i+1} &= \frac{(\lambda_1^{n-i-1} - \lambda_3^{n-i-1})S_{i+1}}{A},\\
u^{i+1} &= \min(0.9, \max(0, T^{i+1})).
\end{aligned}$$

End for.

Step 3:

For $i = 0, \dots, n$, write:

$$S^*(t_i) = S_i, \quad I^*(t_i) = I_i, \quad R^*(t_i) = R_i, \quad M^*(t_i) = M_i, \quad C^*(t_i) = C_i, \quad u^*(t_i) = u^i.$$

End for.

4.2 Numerical results

In this subsection, we present the results obtained by solving the optimality system. For our control problem, we define conditions for the state variables and terminal conditions for the adjoint variables. The optimality system is essentially a two-point boundary value problem, with conditions at the initial time step $i = t_0$ and the final time step $i = t_f$. To solve this system, we initially solve the state model, followed by solving the adjoint system in reverse order. In the first iteration, we start with an initial guess for the control variables and update them based on a characterization of the optimal controls before moving on to the next iteration. This process is repeated until the iterates converge. To achieve this, we created a MATLAB code utilizing the following parameters. Given the lack of real-world data, the parameter values were chosen hypothetically. The plots illustrating susceptible, infected, and recovered individuals—both with and without control measures—are generated based on these parameter values: $\Lambda = 1000$, $\lambda = 0.0005$, $\beta_1 = 0.0001$, $\beta_1 = 0.000001$, $\mu = 0.00009$, $\theta = 0.0002$, $\alpha = 0.003$, $\alpha_1 = 0.001$, $\alpha_2 = 0.001$, $\gamma_1 = 0.008$, $\gamma_2 = 0.001$, $\sigma = 1000$, $\sigma_1 = 0.2$, $\sigma_2 = 0.1$. When analyzing the graphs, please be aware that solid lines represent individuals without control measures, whereas dashed lines indicate those with control measures.

4.3 Control Strategy 1: Impacts of Insecticide Application on Cochineal

The goal of this approach is to minimize the function (8), with a primary focus on reducing the cochineal population through insecticide spraying. Figure 2 illustrates the effects of this spraying on the cacti plants.

In Figure 2 (2.1), it is clear that in the absence of control measures, the number of infected cacti steadily increases, reaching approximately 1.5×10^5 within the first two months. However, when control measures are applied, the number of infected cacti begins to decrease from day one of implementation, eventually dropping to around 1.2×10^5.

In Figure 2 (2.2), the recovered cacti rises to about 2×10^4 with the application of the control strategy, compared to 0.1×10^4 birds when control measures are not implemented.

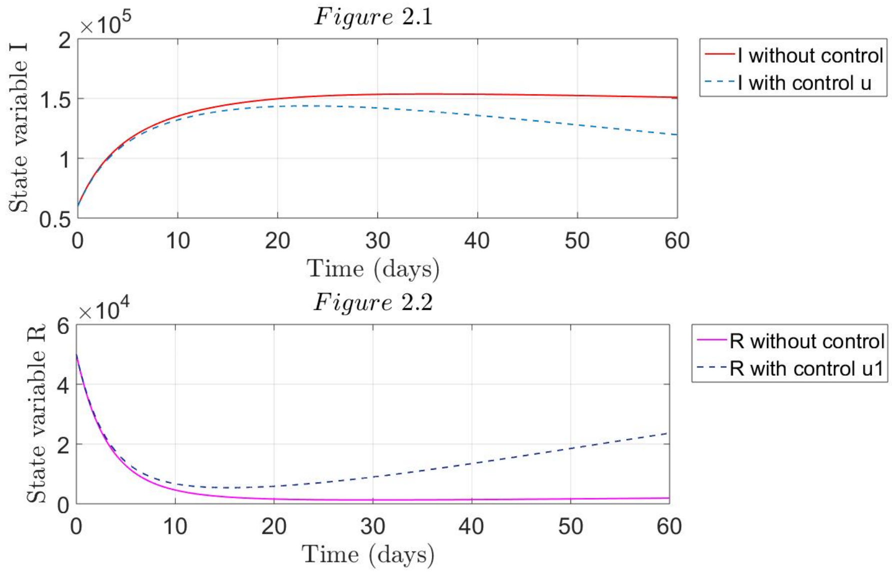

Figure 2: Dynamics with control u_1.

4.4 Control Strategy 2: Use of Hyperaspis Trifurcata that feeds on the cochineal

The main objective of treating cacti infested with cochineal, within the context of a strategy, is to minimize the function (8) while maintaining other control measures at zero. Figure 3 illustrates the disease dynamics, taking into account the presence or absence of this control measure.

In Figure 3 (3.1), it is clear that without any control measures, the cochineal population steadily increases, peaking at around 1,000 during the first two months. However, with the implementation of controls, the cochineal population begins to decline from the first day and decreases to approximately 500 within two months.

In Figure 3 (3.2), the number of recovered cacti increases to approximately 4×10^4 with the use of the control strategy, whereas only 0.1×10^4 are recovered when no control measures are in effect.

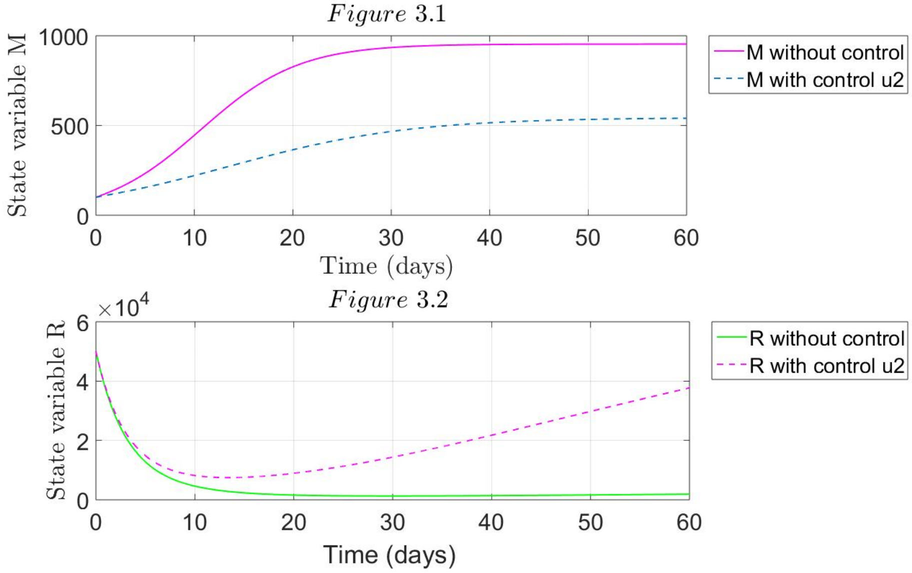

Figure 3: Dynamics with control u_2.

4.5 Control Strategy 3: insecticide spraying and Hyperaspis Trifurcata

This strategy aims to minimize the objective function (8) by implementing both control measures. Figure 4 illustrates the disease progression with both controls in effect, compared to the scenario where no control measures are utilized to manage the disease.

In Figure 4 (4.1),the impact of insecticide as a control measure, along with Use of an insect that feeds on the cochineal, on curbing the propagation of the cochineal are clearly illustrated. It is evident that in the absence of control measures, the number of infected cacti increases, peaking at around 1.5×10^5 during the first two months. In contrast, with the implementation of control measures, the infected cacti decreases to approximately 0.2×10^5 within two months.

In Figure 4 (4.2), it is observed that in the absence of control measures, the number of the cochineal rises, peaking at around 1000 the first month. In contrast, when control measures are put in place, the number of the cochineal

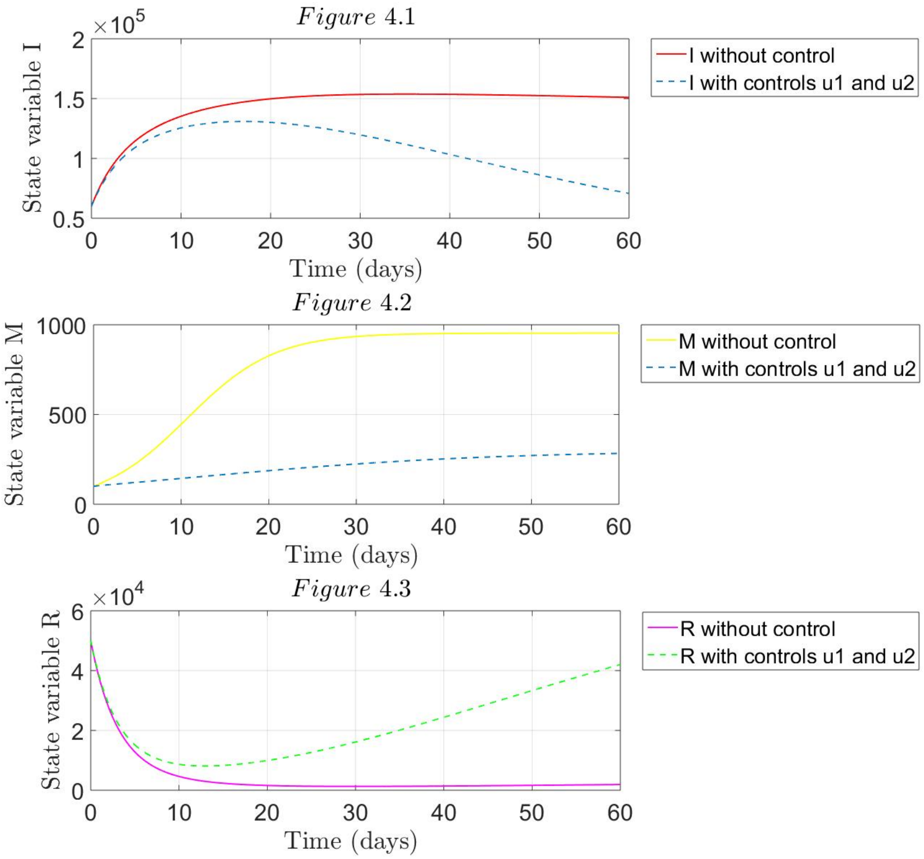

Figure 4: Dynamics with control u_1 and u_2.

consistently decreases, ultimately reaching 250 within two months.

In Figure 4 (4.3), the number of recovered cacti rises within the first week of implementing the control strategy, ultimately reaching about 4.5×10^4 over the course of two months. In contrast, when no control measures are in effect, the number of recovered cacti decreases to nearly zero.

5 Conclusion

In this paper, we introduced a novel model designed to improve the understanding of cochineal dynamics in cactus plants. Our goal is to develop an optimal control strategy based on the $SIRMC$ model that minimizes the number of infected cacti while also achieving a balance between minimizing infection and maximizing recovery. We compared scenarios with and without control measures, demonstrating that the implementation of control strategies substantially decreases the number of infected plants. To offer a thorough overview of cochineal dynamics, we presented figures that display the counts of infected, recovered, and cochineal in both scenarios (u_1 and u_2), highlighting the differences with and without control measures, as illustrated in Figures 1, 2, 3 and 4. Our results indicate the effectiveness of control measures in reducing the spread of cochineal in cacti.

By incorporating the $SIRMC$ model with optimal control strategies, we underscore the potential to decrease disease prevalence and improve recovery rates. These findings highlight the importance of proactive intervention strategies in cactus fields, offering valuable insights for farmers.

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