Iranian Journal of Numerical Analysis and Optimization
Vol. 15, No. 3, 2025, pp 1116–1144
https://doi.org/10.22067/ijnao.2025.92290.1601
https://ijnao.um.ac.ir/

Research Article

A new generalized model of cooperation of advertising companies based on differential games on networks

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Abstract

In this paper, we reconsider the sustainable cooperation of advertising companies problem on a network of companies and consumers. The aim of this paper is to investigate cooperation and profit distribution within networks involving companies and consumers with asymmetric roles and to compare two scenarios based on advertising efficiency. We extend a model

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Received 23 February 2025; revised 27 April 2025; accepted 20 May 2025

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How to cite this article
Jashnesade, M. Nikooeinejad, Z. and Loghmani, G.B. , A new generalized model of cooperation of advertising companies based on differential games on networks. *Iran. J. Numer. Anal. Optim.*, 2025; 15(3): 1116-1144. https://doi.org/10.22067/ijnao.2025.92290.1601

in the framework of cooperative differential games on network, which derive analytically the construction of the characteristic function. Unlike previous dynamic marketing models that all firms advertise for homogeneous consumers and they view companies as identical, in the proposed generalized model, we assume that companies and consumers are heterogeneous players in the network. The Shapley value, Core and τ-value are obtained analytically. A numerical simulation of the proposed model for a graph of advertising firms and consumers with different parameters is presented to obtain sensitivity analysis results. The results show that the optimal cooperative advertisement rate of firms are affected by the sensitivity parameters of consumers to companies. Furthermore, we demonstrate that the allocations of the companies in the cooperative game only for grand coalition will be greater than in the non-cooperative one. We find that the greater the coalition, the more gains from cooperative advertising. We also demonstrated that for the higher product price, the greater Shapley and τ-values attributed to the firms.

AMS subject classifications (2020): Primary 45D05; Secondary 42C10, 65G99.

Keywords: Differential game; Cooperative advertising; Shapley value; Network.

1 Introduction

In the current competitive landscape, neglecting cooperative advertising can result in missed strategic opportunities. Many businesses are finding it difficult to create supply chains that make the most money and keep customers happy. Recently, companies need to work together and agree on things like prices, how much to order, and advertising to succeed in a competitive global market. Instead of competing against each other, businesses are teaming up to make more money and reduce risks. Advertising is increasingly important for businesses to grow market share and revenue. A cooperative advertising strategy is essential for a win-win situation.

Cooperative advertising boosts exposure by allowing more ad outlets and partnering with a major company for increased visibility. Leading firms invest significantly in researching advanced advertising methodologies. Participation in cooperative advertising programs enables companies to leverage the experiences of more successful firms, contributing to their business growth. Companies use various media like TV, radio, print, Internet and social network for advertising. They also utilize local media and distribute fliers and often share advertising costs with retailers through cooperative advertising program (co-op), providing product images for ads. The cost-sharing depends on the relationship and brand prominence. Cooperative advertising is a crucial strategy for manufacturers to expand market capacity and maximize their self-interest, as per existing literature.

In this study, advertising companies are modeled as players within a game-theoretic framework. According to each company's network structure, we believe that the advantages of cooperative advertising for businesses in networks include growing the number of consumers and boosting profits through the synergy of joint advertising. In this research, ideas from graph theory are applied to construct a cooperative differential game for the network's advertising problem. Also, many cooperative game theory techniques are put out to ensure that the profits from the cooperative advertisements are distributed fairly among the participating companies. Accordingly, this study aims to address the following research questions:

- How can corporations determine the amount of money they make from social network advertising?
- How might more cooperative advertising income be equitably divided among participating agents?
- How will advertising affect the coalition's companies?

The remainder of the paper is arranged as follows. In Section 2, the relevant literature is examined. In addition to defining the mathematical model for cooperative advertising in networks, Section 3 offers a solution methodology. In Section 4, one numerical example is considered, and Section 5 explains the solution imputation and fair distribution of cooperative game models. Section 6 concludes with some recommendations and future research directions.

2 Literature review

Three main literature streams serve as the foundation for our work: Cooperative advertising, cooperative game theory applications in networks and cooperative advertising differential game. As a result, we go over these topics in the subsections that follows.

Several techniques based on cooperative game theory have been developed in recent years. In order to communicate the information to their network partners, Zinoviev and Duong [37] expanded the game theoretical model of information diffusion in a star-like network where personality traits are described through a feedback mechanism. In order to calculate the profit margin for their shops and customers, Aust [2] assessed the competition. Increased market competitiveness leads to higher profits and more effective advertising outcomes. In [33], authors investigated three cooperative scenarios for the horizontal firms: one in which a contractor is attempting to create a coalition, as well as cooperative and noncooperative scenarios. For every scenario, the best possible advertising and revenues were determined and contrasted.

The majority of research on cooperative advertising in the literature has concentrated on supply chain advertising campaigns. Jørgensen and Zaccour [19] presented a comprehensive review of cooperative advertising in marketing channels using game-theoretic approaches. Their survey was structured into two parts. The first part focused on simple supply chains consisting of a single supplier and a single retailer, while the second part examined more complex channels involving multiple suppliers and/or retailers. They noted that many findings from static models could be extended to dynamic settings and that static models tended to be homogeneous in terms of assumptions and structure. In contrast, models that incorporated horizontal interactions within or across supply chain levels were relatively scarce. The study also showed that firms' participation in cooperative advertising programs is influenced by both intra-brand and inter-brand competition, and that such participation do not always align with the firms' best interests. Sarkar, Omair, and Kim [25] proposed a model for co-op advertising in supply chains, which considers fluctuating demand based on selling price and advertising expenses. The ideal outcomes improved revenue across the supply chain, overcoming risks and improving economic analysis and feasibility. In [35], authors studied cooperative advertising in a supply chain involving manufacturers and retailers, examining how power and information structure affect price and advertising decisions. The study found that a dominating member can enhance supply chain performance. In [34], the authors studied how digital platforms and participants balance value generation and appropriation in a Stackelberg game. They emphasized the necessity of cooperative partnerships in strategic alliances.The corona virus epidemic has generated new marketing methods, influencing client purchase behaviors. Ghosh, Seikh, and Chakrabortty [13] examined how cooperative advertising and online services affect decision-making in dual channel supply chains. The model revealed the most profitable channel methods, highlighting the benefits of cooperative advertising but downsides for traditional shops. Jørgensen, Sigué, and Zaccour [18] examined how a manufacturer and exclusive retailer's advertising affects sales in a two-member channel. The study analyzed four scenarios: no support, support for both forms of advertising, and partial support for one type. Supporting both categories was the most profitable for both sides, followed by moderate support. No assistance was the least profitable option. The purpose was to maximize the manufacturer's profits. The study found that supporting both forms of advertising benefited both members the best. Theoretical analyses provide insight into management.

The Nash bargaining model is used to analyze profit-sharing methods and manufacturer pool rates for cooperative advertising. Chutani and Sethi [6] proposed a Stackelberg differential game model in which the manufacturer allocates advertising shares to N shops based on their subsidy value, and retailers engage in a Nash differential game to optimize their advertising efforts. In [9], authors employed a game-theoretic model to investigate cooperative advertising in a supply chain, discovering that retailers' concerns regarding Nash bargaining fairness increase local advertising spending, thereby enhancing supply chain efficiency.

Cooperative game theory has been frequently used to assess the collaboration between companies in real-world networks. Studies have shown that cooperative game theory can estimate cost savings and propose fair allocations. Frisk et al. [11] investigated cooperation among forest enterprises in

Sweden's wood sector and offered strategies to disperse savings among participants. Lozano et al. [21] discovered that shippers in a logistics network collaborate by integrating transportation requirements. Razmi, Hassani, and Hafezalkotob [23] discovered that horizontal collaboration in natural gas distribution can lead to cost savings and equitable allocation.

In logistic networks, suppliers may collaborate to make better use of their vehicles. A mathematical model was created to evaluate the advantages of collaboration among distribution businesses, and several cooperative game theory methodologies were compared and examined. Wang et al. [30] investigated a multiple-center truck routing issue in which logistics service providers collaborated in a network, providing cooperative game theory strategies for profit allocation.

Cooperation can also increase flow and reduce costs in maximum flow problems. Reyes [24] showed that logistic companies can increase flow through cooperation, and Hafezalkotob and Makui [15] suggested a mathematical programming approach based on cooperative game theory for player collaboration in logistic networks. It covers coalitional games, cooperative game theory, and its practical applications in communication and wireless networks.

It classifies them into canonical, formation, and graph games, providing a comprehensive treatment for communications and network engineers. Gharehbolagh et al. [12] developed mathematical models to improve reliability and cost in logistics networks. Zhao et al. [36] examined how to handle a large number of community energy consumers (CEPs) in a distribution network. The researchers developed a hybrid game-based optimal operating model that incorporates Stackelberg and cooperative games. It also includes cloud energy storage for economic efficiency.

Differential games are used to handle problems involving several decision-makers making decisions continuously, managing complicated systems, making dynamic decisions. Researchers can refer to [4] for further study. Sorger [27] analyzed a modified version of non-cooperative advertising differential game of Case. Sorger derived Nash equilibria for open-loop controls as well as for feedback strategies for finite and infinite planning horizons. Dockner [7] introduced the differential games for modeling economic and management issues involving strategic decision making and explored applications in capital accumulation, industrial organization, and more. He, Prasad, and Sethi [17] analyzed co-op advertising as a stochastic Stackelberg differential game, identifying optimal policies for both sides and comparing it to a vertically integrated channel. They also proposed a framework for coordination. Erickson [10] introduced an oligopoly model to determine advertising strategies in an oligopoly. Unit contribution and advertising effectiveness boosted own advertising and sales, while discount and decay rates had negative effects. In asymmetric oligopolies, these factors influenced rivals' advertising and sales differently.

Tur and Petrosyan [29] investigated cooperative differential games on networks, with a focus on optimality principles, characteristic function development, and Shapley value computation. They presented their findings using a differential marketing game. Liu and Wu [20] investigated the shared manufacturing model, a sustainable way to production that involves a producer and a platform with government subsidies. The study analyzed price, collaborative advertising, and decision-making for centralized, decentralized, and bilateral cost-sharing contracts. Centralized decision-making results in cheaper prices, more promotional effort, and higher profits. Despite improved decentralized decision-making, total revenues remained below centralized levels. Wu and Liu [31] examined four models based on differential games to examine pricing and advertising efficiency in shared manufacturing: Classic cooperation, cost-sharing contract, revenue-sharing contract, and bilateral cost-sharing contract. It included suggestions and numerical examples. Du et al. [8] examined the effectiveness of advertising in promoting water-saving products in a two-level supply chain. Contracts for cooperative, noncooperative, and cooperative cooperation were the three scenarios that were distinguished. The outcomes show pareto optimality for market demand and product goodwill. Han et al. [16] studied a dynamic Stackelberg game between a company and retailer to analyze their advertising decisions. The study found that while national advertising can improve advertising behaviors, it may also result in losses. The study also examined retail and brand competitiveness. Co-op advertising helped align manufacturing and retailer decisions in supply chains. The manufacturer determined the participation rate and wholesale price, while the retailer responded with effective advertising and pricing. Petrosyan, Yeung, and Pankratova [22] introduced a new characteristic function based on the possibility of stopping interaction by players outside the coalition in each time instant or imposing sanction on players from the coalition.

We mention that our research in this article mostly is related to [27, 14, 29].

Tur and Petrosyan [29] investigated cooperative differential games on networks, building characteristic functions, computing Shapley values, and looking at optimality principles. Our article and Tur and Petrosyan [29] article differ in that our model takes into account the roles of producers and consumers. Tur's model, on the other hand, only takes into account the role of distributors in multi-level marketing, wherein distributors collaborate to increase profit. Additionally, while we have looked into the roles of customers and producers, the majority of the models that we described in the introduction take into account the relationship between the retailer and the producer. As a matter of fact, manufacturers work directly together under this strategy to boost revenue through cooperative advertising. In 2018, Hafezalkotob et al. introduced a mathematical model that calculated the profit from cooperative advertising in social networks. They also suggested cooperative game theory techniques for benefit distribution, showing that coalitions yielded higher profits. One way that our study differs from [14] is that we employ a dynamic game (differential game) as opposed to a static game that is solved using nonlinear programming. In contrast to static games, dynamic games take time into account. In contrast to the Hafezalkotob model, we determine each coalition's earnings using the characteristic function and look at two scenarios depending on advertising efficiency variables to see how they impact revenue of coalition.

3 Problem description and model assumptions

In this section, a cooperative differential game approach on a network is applied to represent the relationship between players. For this purpose, we assume that a graph $G = (\mathcal{V}, E)$ is a network with $\mathcal{V}$ nodes and E edges with prescribed duration of $[0,T]$. Let $\{i_1,\dots,i_n\}$ be the set of companies and $\{k_1,\dots,k_m\}$ be the set of consumers. Each node corresponds to a company i or a costumer j, where $i \in N = \{1,2,\dots,n\}$ and $j \in K = \{1,2,\dots,m\}$, respectively, and also each edge $(i,j) \in E$ represents a connection between company i and costumer j (as illustrated in Figure 1). We introduce the

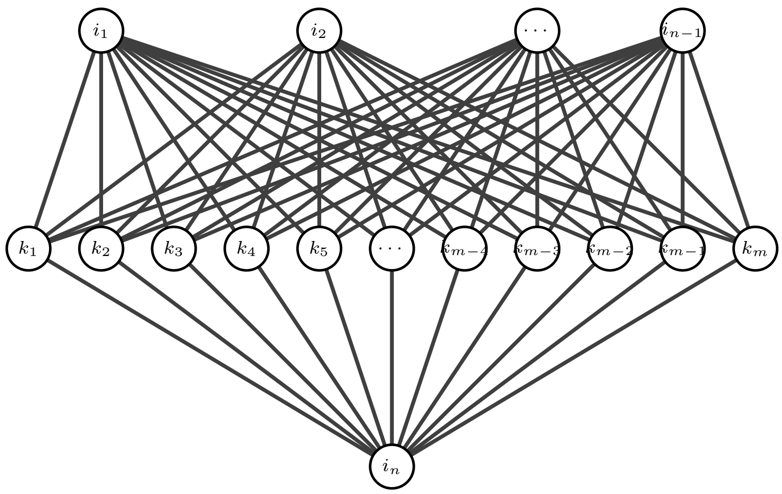

Figure 1: A graph of advertising model with the set $\{i_1,\dots,i_n\}$ for companies and $\{k_1,\dots,k_m\}$ for consumers in a network

assumptions of the model as follows:

Assumption 1. We assume that the Players (companies) in our network are reasonable.

Assumption 2. The advertisement of firms is affected by the sensitivity rate of consumers to companies.

Assumption 3. We assume that the utility of this cooperative game is transferable that means the earnings of a coalition can be calculated.

Assumption 4. If the companies cooperate in network, then they cover the consumers of each other.

Assumption 5. It is supposed that all links are undirected.

The dynamics of company i's sale rate are governed by

$$\dot{x}_i(t) = v_i(t) + \sum_{k=1}^{m} \Big(\rho_{ik}\sqrt{u_{ik}(t)} - \sum_{j\neq i}\delta_{jk}\sqrt{u_{jk}(t)}\Big), \quad i \in \{1,\dots,n\}, \quad (1)$$

where $x_i(t) \subset R^n$ indicates the sales rate of Player i (state variable) at time t and $u_{ik}(t)$ represents the advertisement effort of company i for consumer k (control variable). The amount of basis demand from firm i that is unrelated to the advertising effort is represented by $v_i(t) \geq 0$. The values of parameters $\rho_{ik} \geq 0$ and $\delta_{ik} \geq 0$ signify the sensitivity of consumer k to the advertisement of company i and the sensitivity of consumer k to the advertisement of company j (competitor's advertising), respectively.
They discovered that customer responses to advertising which had an impact on the values of ρ_{ik} and δ_{ik}, which stand for specific parameters connected to consumers characteristics [1]. Because customers are more likely to respond favorably to advertisements from a firm they are familiar with than to those from its rivals, it is often assumed that δ_{ik} is less than ρ_{ik}. Furthermore, the sales rate for company i increases with cooperative advertising efforts and falls in response to competing players' activities.

The objective function of company $i \in N$ is considered as follows:

$$\max\ J_i(x_i^0, u_{i1}, \dots, u_{im}) = \int_{t_0}^{T} \Big(px_i(\tau) - \sum_{k=1}^{m} u_{ik}(\tau)\Big)d\tau, \qquad (2)$$

where p denote the price.

The goal of the problem is to maximize company's profit, which is obtained by subtracting advertising expenses from income. Let $\Gamma(x_0, T-t_0)$ be a cooperative differential game on network $(\mathcal{V}, E)$, let the system dynamics (1) and the sets of feasible controls U_i, $i \in N$ be given, and let the players' payoffs be defined by (2). Assume that companies are able to cooperate together in order to maximum possible joint reward

$$\sum_{i\in N} \int_{t_0}^{T} \Big(px_i(\tau) - \sum_{k=1}^{m} u_{ik}(\tau)\Big)d\tau, \qquad (3)$$

subject to dynamics (1).

The optimal cooperative strategies of companies

$$u^*(t) = (u^*_{1,1}(t), \ldots, u^*_{1,m}(t), u^*_{2,1}(t), \ldots, u^*_{2,m}(t), \ldots, u^*_{3,1}(t), \ldots, u^*_{n,m}(t))$$

for $t \in [t_0, T]$ are defined as follows:

$$u^*(t) = \arg \max_{u_{ik}, i\in N, k\in K} \sum_{i\in N} \int_{t_0}^{T} \Big(px_i(\tau) - \sum_k u_{ik}(\tau)\Big) d\tau. \tag{4}$$

The optimal cooperative trajectory is $x^*(t) = \Big(x^*_1(t), x^*_2(t), \ldots, x^*_n\Big)$. This trajectory corresponds to the optimal cooperative strategies vector, $u^*(t)$. We express the maximum joint reward as follows:

$$\begin{aligned} V(N, x_0, T - t_0) &= \sum_{i\in N} \int_{t_0}^{T} \Big(px^*_i(\tau) - \sum_k u^*_{ik}(\tau)\Big) d\tau \\ &= \max_{u_{ik}, i\in N, k\in K} \sum_{i\in N} \int_{t_0}^{T} \Big(px_i(\tau) - \sum_k u_{ik}(\tau)\Big), \end{aligned} \tag{5}$$

subject to dynamics

$$\dot{x}^*_i(t) = v_i(t) + \sum_{k=1}^{m} \Big(\rho_{ik} \sqrt{u^*_{ik}(t)} - \sum_{j\neq i} \delta_{jk} \sqrt{u^*_{jk}(t)}\Big), \quad i \in \{1, \ldots, n\}. \tag{6}$$

It is important to determine the characteristic function of the problem in order to decide how to distribute the greatest total payment of coalition among the participants under an agreeable scheme. Based on the characteristic function of a cooperative game, two disjoint coalitions of companies are at least as excellent when the companies cooperate as when they work apart, and also the empty set has no value. This concept can be stated mathematically as

1. $V(\emptyset) = 0$.
2. If $S, T \subseteq M$ are disjoint coalitions ($S \cap T = \emptyset$), then $V(S) + V(T) \leq V(S \cup T)$.

Let S be a subset of $\mathcal{V}$. A pair $(\mathcal{V}_S, E_S)$ is called a subnet (subgraph) if it only has subset S of the set of vertices (companies) of network $(\mathcal{V}_S, E_S)$, and E_S has all connections from E whose initial and final vertices in network graph are inside subset S. We assume that the value of a coalition S is determined along the cooperative trajectory as follows:

$$V(S, x_0, T - t_0) = \sum_{i\in S} \Big(\int_{t_0}^{T} \Big(px_i^*(\tau) - \sum_{k=1}^{m} u_{ik}^*(\tau)\Big) d\tau \Big), \tag{7}$$

where $x_i(t)$ and $u_{ik}(t)$ are the solutions obtained from (4) and (6), respectively.

Similarly, the cooperative-trajectory characteristic function of the subgame $\Gamma(x^*(t), T-t)$ starting at time $t \in [t_0, T]$ can be evaluated as

$$V(S, x^*(t), T - t) = \sum_{i\in S} \Big(\int_{t}^{T} \Big(px_i^*(\tau) - \sum_{k=1}^{m} u_{ik}^*(\tau)\Big) d\tau \Big). \tag{8}$$

Suppose that the game is played in a cooperative scenario and that players have the opportunity to cooperate in order to achieve maximum total payoff:

$$\max_{u_{ik}, i\in N, k\in K} \sum_{i\in N} \int_{t}^{T} \Big(px_i(\tau) - \sum_{k=1}^{m} u_{ik}(\tau)\Big) d\tau. \tag{9}$$

We apply the Bellman dynamic programming technique to define the cooperative strategies. The Bellman function in a subgame beginning at moment t from state $x(t)$ is denoted by $V(N, x, T - t)$:

$$V(N, x, T - t) = \max_{u_{ik}, i\in N, k\in K} \sum_{i\in N} \int_{t}^{T} \Big(px_i(\tau) - \sum_{k=1}^{m} u_{ik}(\tau)\Big) d\tau. \tag{10}$$

The Hamilton–Jacobi–Bellman (HJB) equation has the following expression [32]:

$$\begin{aligned} -V_t(N, x, T - t) = & \max_{u_{ik}, i\in N, k\in K} \Big\{ \sum_{i\in N} \Big(px_i(\tau) - \sum_{k=1}^{m} u_{ik}(\tau)\Big) + \sum_{i\in N} (v_i(t) \\ & + \sum_{k=1}^{n} (\rho_{ik}\sqrt{u_{ik}} - \sum_{j\neq i} \delta \sqrt{u_{ik}}) V_{x_i} \Big\}, \\ V(T, x(T)) = & \ 0. \end{aligned} \tag{11}$$

Performing the maximization operator in (11) results:

$$\begin{aligned} u_{ik}^* = & \frac{1}{4}\Big(\rho_{ik} V_{x_i} - \sum_{j\neq i} \delta_{ik} V_{x_j}\Big)^2, \\ & i \in N = \{1, \ldots, n\},\ k \in M = \{1, 2, \ldots, m\}, \end{aligned} \tag{12}$$

Substituting u_{ik}^* from (12) into (11) and solving HJB (11), one obtains

$$V(N,x,T-t) = \sum_{i\in N}\Big(A_i(t)x_i + B_i(t)\Big), \tag{13}$$

where $A_i(t)$ and $B_i(t)$ satisfy the following system of ordinary differential equations:

$$\begin{aligned}
\dot{A}_i(t) =&\ -p,\ \ A_i(T)=0,\\
\dot{B}_i(t) =&\ -\sum_{k=1}^{m}\frac{A_i(t)^2(4\delta_{i,k}\rho_{ik}-4\delta_{i,k}^2-\rho_{ik}^2)}{4} - \sum_{k=1}^{m}\frac{A_i^2(t)(2\delta_{i,k}-\rho_{ik})\sqrt{(2\delta_{i,k}-\rho_{ik})^2}}{2}\\
&-A_i(t)v_i(t),\\
B_i(T) =&\ 0.
\end{aligned} \tag{14}$$

The solution of ordinary differential equation (14) is obtained as

$$\begin{aligned}
&A_i(t) = (T-t)p,\\
&B_i(t) = \frac{p(T-t)^2(\sum_{k=1}^{m}(2\delta_{ik}-\rho_{ik})^2p(T-t)+6v_i(t))}{12}.
\end{aligned} \tag{15}$$

The cooperative optimal strategies can be obtained by

$$u_{ik}^* = \frac{1}{4}\Big(\rho_{ik}A_i - \sum_{j\neq i}\delta_{ik}A_j\Big)^2,\ i\in\{1,2,\dots,m\},\ k\in\{1,\dots,m\}. \tag{16}$$

Finally, by substituting (16) in (1) and solving the differential equation, we are able to determine the players' sales rate

$$x_i^* = (\frac{-pt^2}{4}+\frac{Ttp}{2})\Big(\sum_{k=1}^{m}\rho_{ik}(\rho_{ik}-2\delta_{ik}) + \sum_{i\neq j}\sum_{k=1}^{m}2\delta_{jk}^2-\delta_{jk}\rho_{jk}\Big) + tv_i(t) + x_i^0.$$

Now, the value function for the cooperative joint payout of all n companies is obtained as

$$\begin{aligned}
V(N,x_0,T-t_0) = \frac{1}{12}\Big[\Big\{&(\sum_{i=1}^{n}\sum_{k=1}^{m}(2\delta_{ik}-\rho_{ik})^2)(T^2+4t_0T-2t_0^2)p \quad (17)\\
&+6(T+t_0)(\sum_{i=1}^{n}v_i(t)) + 12(\sum_{i=1}^{n}x_0^i)\Big\}p(T-t_0)\Big]. \quad (18)
\end{aligned}$$

Using (7) and substituting $u_{ik}^*(t)$ and $x_i^*(t)$ from (16) and (17), respectively, we obtain $V(S,x_0,T-t_0)$ as follows:

$$V(S,x_0,T-t_0)=\frac{1}{12}\Big[\Big\{(\sum_{i\in S}\sum_{k=1}^{m}(2\delta_{ik}-\rho_{ik})^2)(T^2+4t_0T-2t_0^2)p \\ +6(T+t_0)(\sum_{i\in S}v_i(t))+12(\sum_{i\in S}x_0^i)\Big\}p(T-t_0)\Big]. \quad (19)$$

4 Imputation solution

Exploring coalitions and fair pay distribution in cooperative games is crucial. The characteristic function framework effectively shows coalition possibilities and acceptable payoff distribution schemes among players.

Definition 1. ([32]) A vector $\xi(x_0,T-t_0)=(\xi_1(x_0,T-t_0),\xi_2(x_0,T-t_0),\dots,\xi_n(x_0,T-t_0))$ is solution imputation if it satisfies the following conditions:

$$(1)\quad \xi_i(x_0,T-t_0)\geq V(\{i\},x_0,T-t_0),\quad i\in N,$$

$$(2)\quad \sum_{j\in N}\xi_j(x_0,T-t_0)=V(N,x_0,T-t_0).$$

Condition (1) states that each element of the imputation vector must be at least as large as the value of the game for each individual player (i.e., individual rational), and condition (2) states that the total of all the elements of the imputation vector must equal the value of the game for the entire coalition (i.e., group rational). In the game $\Gamma(x_0,T-t_0)$, $L(x_0,T-t_0)$, the set of all imputations is provided by

$$L(x_0,T-t_0)=\Big\{\xi(x_0,T-t_0)=(\xi_1(x_0,T-t_0),\dots,\xi_n(x_0,T-t_0))| \\ \sum_{i\in N}\xi_i(x_0,T-t_0)=V(N,x_0,T-t_0), \\ \xi_i(x_0,T-t_0)\geq V(i,x_0,T-t_0),\ i\in N\Big\}.$$

The problem in computing optimal profitability for all firm coalitions is identifying how to calculate the contributions of the collaborating companies. It will not be easy because each corporation's coalitional participation in network advertising is uncertain. A theoretical approach such as cooperative game theory provides several ways to efficiently evaluate players' cooperative efforts. A few of them are presented in this section.

The core $C(x_0, T-t_0)$ of the game $\Gamma(x_0, T-t_0)$ is the subset of the imputation set $L(x_0, T-t_0)$, such that [29]

$$\begin{aligned} C(x_0, T-t_0) = \; & \Big\{\xi(x_0, T-t_0) = (\xi_1(x_0, T-t_0), \ldots, \xi_n(x_0, T-t_0))| \\ & \sum_{i\in N} \xi_i(x_0, T-t_0) = V(N, x_0, T-t_0), \\ & \sum_{i\in S} \xi_i(x_0, T-t_0) \geq V(S, x_0, T-t_0),\ S \subseteq N\Big\}. \qquad (20) \end{aligned}$$

A new approach to profit allocation in collaboration was developed by [26], it concentrated on the additive, symmetric, efficient, and dummy qualities of participants' contributions. The Shapley value is derived based on the varied shares of players in the coalitions, and it is unique when the core is a set of solution imputation, which is not always unique. If the vector $\Phi(x_0, T-t_0) = \{\Phi_i(x_0, T-t_0), i = 1, 2, \ldots, n\}$ satisfies the following criteria, and the vector is referred to as the Shapley value [32], then

$$\begin{aligned} \Phi_i(x_0, T-t_0) = \; & \sum_{S\subseteq N, i\in S} \frac{(n-s)!(s-1)!}{n!}[V(S, x_0, T-t_0) \\ & -V(S-\{i\}, x_0, T-t_0)], \qquad i = 1, 2, \ldots, n. \qquad (21) \end{aligned}$$

In this model, according to the information in the third section and (21), the Shapley value is obtained as follows:

$$\begin{aligned} \Phi_i(x_0, T-t_0) = \frac{p(T-t)}{12}\Bigg(\Bigg\{ & (2t^2 - 4Tt)\Big(\sum_{k=1}^{m} \delta_{ik}(3\rho_{ik} - 2\delta_{ik}) - \rho_{ik}^2 \\ & + \sum_{i\neq j}\sum_{k=1}^{m} \frac{\delta_{jk}(\rho_{jk} - 2\delta_{jk})}{2})\Big) + T^2(\sum_{k=1}^{m} \rho_{ik}^2 - 4\delta_{ik}^2 \\ & - \sum_{i\neq j}\sum_{k=1}^{m} 2\delta_{jk}(\rho_{jk} - 2\delta_{jk}))\Bigg\}p + 6(T+t)v_i(t) + 12x_0^i\Bigg) (22) \end{aligned}$$

Another cooperative optimality principle is τ-value, which is defined as follows [28]:

A vector $\tau(x_0, T-t_0) = (\tau_1(x_0, T-t_0), \tau_2(x_0, T-t_0), \ldots, \tau_n(x_0, T-t_0))$ is called τ-value which applies to the following equation:

$$\tau_i(x_0, T-t_0) = \alpha M_i(x_0, T-t_0) + (1-\alpha) m_i(x_0, T-t_0),$$

where

$$M_i(x_0, T-t_0) = V(N, x_0, T-t_0) - V(N-\{i\}, x_0, T-t_0), \quad (23)$$
$$m_i(x_0, T-t_0) = \max_{i\in S}\{V(S, x_0, T-t_0) - \sum_{i'\in S-\{i\}} M_{i'}(x_0, T-t_0)\}, \quad (24)$$

and the coefficient $\alpha \in [0,1]$ is determined from the equation

$$\sum_{i\in N}(\alpha M_i(x_0, T-t_0) + (1-\alpha)m_i(x_0, T-t_0)) = V(N, x_0, T-t_0).$$

5 Numerical experiments

Although firms are typically seen as independent agents, they also have the ability to form coalitions and dynamically manage their advertising behavior. In general, the aim is to investigate the effect of cooperation in the proposed generalized model on each of the following cases:

1. The characteristic functions (payoffs of coalitions)
2. Advertising and sale rates of companies
3. Sustainability of cooperation
4. Distribution of profit in case of forming a grand coalition

In this section, we consider an example with $n = 3$ companies and $k = 12$ consumers, as shown in Figure 2. Let $\mathcal{S} = \{S_1 = \{1\}, S_2 = \{2\}, S_3 = \{3\}, S_4 = \{1,2\}, S_5 = \{1,3\}, S_6 = \{2,3\}, S_7 = \{1,2,3\}\}$ be a set of all possible sub-coalitions between companies. We use several procedures of cooperative games such as core (20), Shapley value (22), and τ-value (23) to calculate additional profit of cooperative advertisement in the network (Figure 2).

For numerical simulation, we fix the parameters in the model as shown in Tables 1 and 2 for scenarios 1 and 2, respectively. A sensitivity analysis is provided to check the impact of changing key parameter values $\rho_{ik} \geq 0$ and $\delta_{ik} \geq 0$ on sustainability of cooperation between companies.

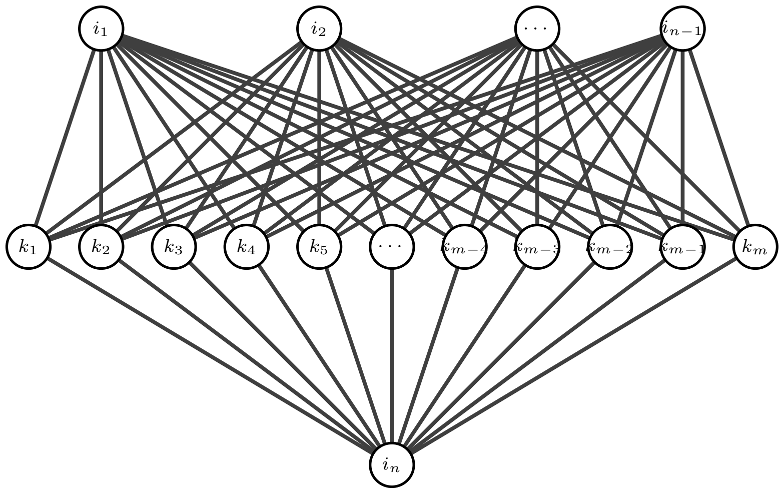

Figure 2: example of cooprative network with 3 advertising company

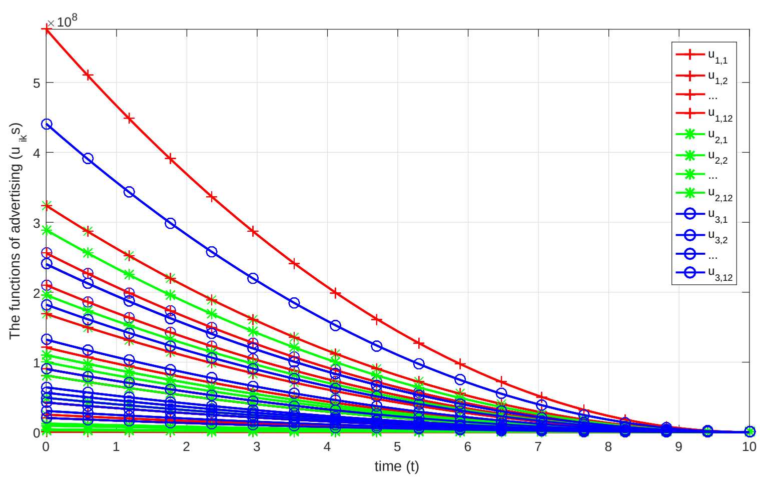

Figure 3: Promotional functions of each player over time

Table 1: Data of numerical example in scenario 1.

Players	k	ρ	δ	v	p
Company 1	1	5	2	5	100
	2	15	3		
	3	30	4		
	4	50	1		
	5	40	2		
	6	20	5		
	7	20	1		
	8	30	2		
	9	40	4		
	10	25	3		
	11	35	3		
	12	15	1		
Company 2	1	5	1	5	100
	2	10	2		
	3	20	3		
	4	30	1		
	5	25	2		
	6	15	4		
	7	10	3		
	8	20	1		
	9	40	3		
	10	25	2		
	11	30	5		
	12	20	3		
Company 3	1	20	3	5	100
	2	15	2		
	3	25	1		
	4	50	4		
	5	35	2		
	6	15	1		
	7	20	2		
	8	25	3		
	9	35	4		
	10	25	5		
	11	35	2		
	12	15	3		

Table 2: Data of numerical example in scenario 2.

Players	k	ρ	δ	v	p
Company 1	1	1	0.5	5	100
	2	5	3		
	3	7	0.5		
	4	2	1.1		
	5	1	0.5		
	6	6	3		
	7	2	1		
	8	1	0.4		
	9	1	0.9		
	10	1.5	1		
	11	6	5.9		
	12	5	3		
Company 2	1	5	0.25	5	100
	2	4	0.75		
	3	6	0.8		
	4	2	1		
	5	1	0.5		
	6	5	0.25		
	7	1	0.1		
	8	10	1		
	9	3	0.1		
	10	1	0.4		
	11	2.1	1.8		
	12	4	0.2		
Company 3	1	4	0.9	5	100
	2	6	0.75		
	3	2	0.75		
	4	3	0.3		
	5	2	1		
	6	4	2		
	7	6	2		
	8	2	1		
	9	6	0.9		
	10	2	1		
	11	2	1.5		
	12	5	3		

5.1 Scenario 1

In this scenario, we solve the problem using the data from Table 1 to generate Figures 3–8 and determine the quantity of coalition revenue and allocation to participants. In Figure 3, we show the advertising function (advertising efforts) over time, for all customers, in red, green, and blue for company 1, 2, and 3, respectively. For every feasible combination, we solve the mathematical model (7)–(10). The observed differences in advertising expenditures can be attributed to varying marketing strategies, differences in budget al-

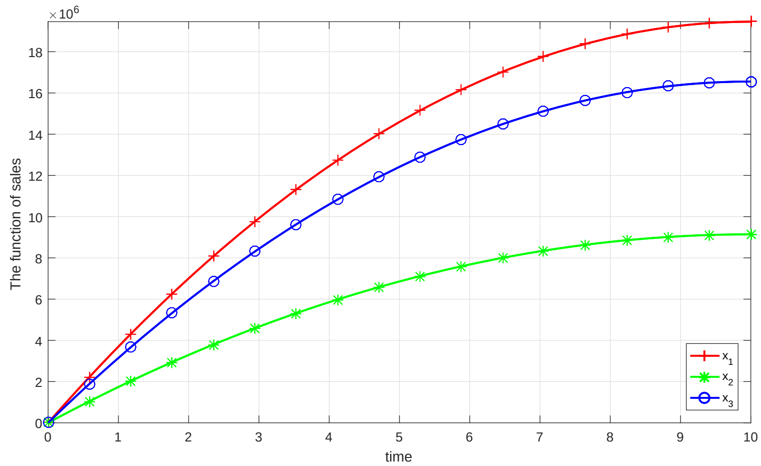

Figure 4: The functions of sales for each player over time

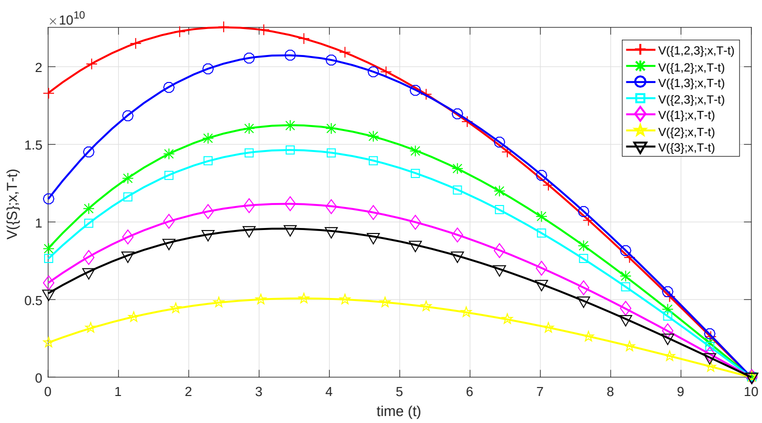

Figure 5: Earnings of all coalitions over time

locations, or distinct target market characteristics across firms. This figure illustrates the advertising expenditure trends for each company. For example, Company 1 initially invests heavily in advertising but experiences a subsequent decline, whereas Company 2 maintains a more stable pattern. These differences likely reflect divergent advertising strategies, with firms adopting distinct approaches for customer retention versus new customer acquisition. Table 3 contains a list of all conceivable coalitions' results. Table

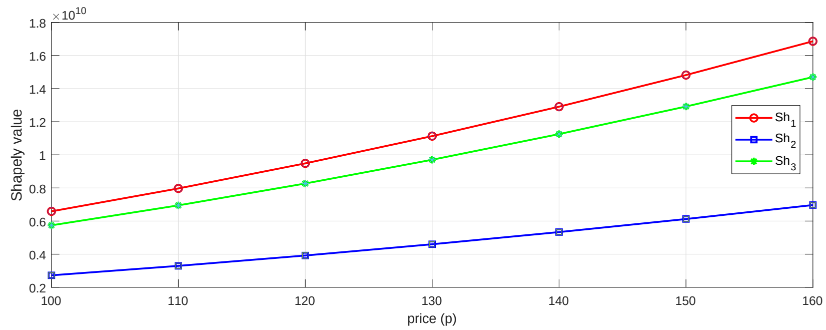

Figure 6: Sensitivity of Shapley value

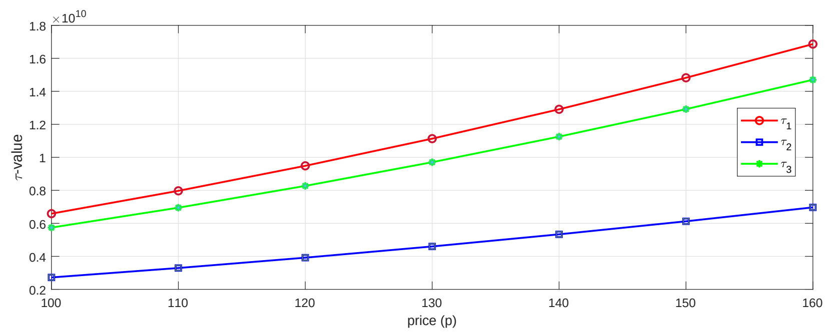

Figure 7: Sensitivity of τ-value

3 shows that the larger the coalition, the higher its income will be. For Player 1, joining coalition S_5 is preferable to joining coalition S_4 because $V(S_5, x_0, T - t_0) \geq V(S_4, x_0, T - t_0)$, similarly, Player 2 and Player 3 are benefited by joining coalitions S_4 and S_5, respectively. In addition, this table shows that the game has a additive property for for all S_1, S_2 we have $(V(\{S_1, S_2\}, x_0, T - t_0) = V(\{S_1\}, x_0, T - t_0) + V(\{S_2\}, x_0, T - t_0))$.

For fair distribution, several cooperative game theory techniques have been developed. Researchers may refer to [3] and [5] for further details. The imputations derived from different cooperative game methods are shown in Table 4, which includes the Shapley value and the τ-value. From Table 4, it can be seen that the Shapley value and the τ-value lead to nearly identical allocations. Furthermore, because the game has an additive property, the Shapley value and τ-value represent the amount of cash earned by each

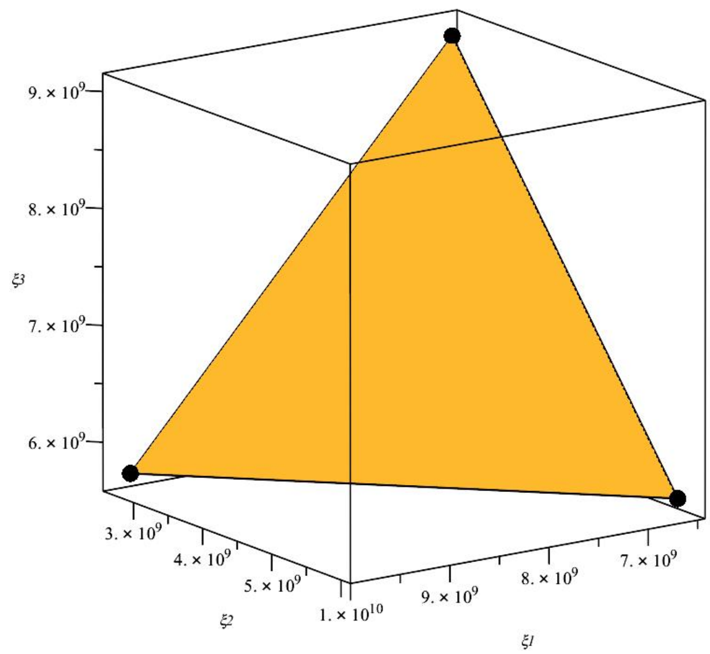

Figure 8: Core for companies in the numerical example.

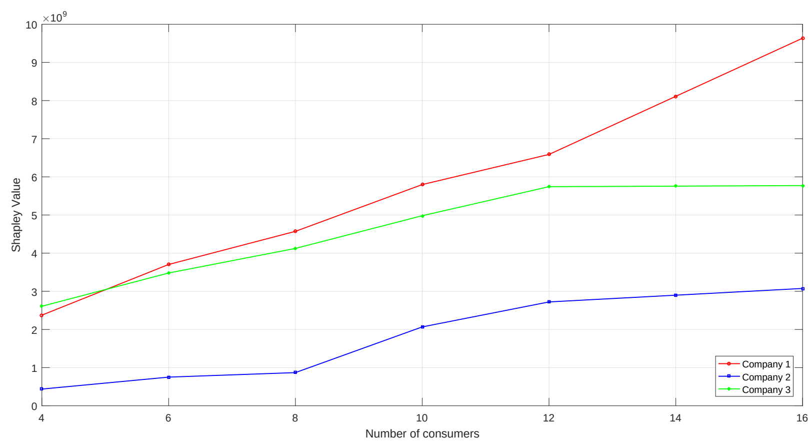

Figure 9: Sensitivity analysis of the Shapley value of each company with respect to the number of consumers (m)

player individually. Based on the advertising efforts of each company, the sales rate of each player is obtained, as shown in Figure 4. This figure reveals that Company 1 has achieved significantly higher sales volumes, indicating that its advertising campaigns have generated more enduring customer acquisition effects. This suggests that Company 1 not only attracts customers temporarily but also fosters long-term customer loyalty. Collectively, Company 1's extensive advertising efforts, specialized sales strategies, and product quality likely constitute the primary success factors.

Company 3 demonstrates the second highest sales performance after Company 1. While this company initially exhibited favorable growth trends, its expansion rate remained slower than Company 1's. This performance gap may stem from either less effective advertising strategies or reduced effectiveness in new customer acquisition compared to Company 1. Nevertheless, Company 3 has gradually established more stable growth patterns and has frequently outperformed Company 2 across multiple time periods.

Conversely, Company 2 has consistently recorded the lowest sales figures among the three competitors. This under performance could be attributed to deficiencies in advertising strategies, product quality issues, or inadequate new customer acquisition capabilities. It appears that Company 2 has failed to establish a competitive position comparable to Companies 1 and 3, with its initial advertising impacts showing diminishing returns over time. Figure 5 shows the earnings of all possible coalitions over time that illustrates the profit generated by each coalition. It is evident that the grand coalition—comprising all companies—achieves the highest total profit. This outcome reflects the principle of superadditivity, where the collective profit from cooperation exceeds the sum of individual profits gained through independent operations. From a strategic standpoint, the result underscores the potential advantages of forming alliances and coalitions in marketing and advertising contexts. Figures 6 and 7, respectively, illustrate the effect of a company's price sensitivity on Shapley values and τ-value obtained from a grand coalition. In these figures, the distribution of profits among coalition members is shown based on the Shapley value and τ-value. These methods help to understand the contribution of each company to the overall profit. According to this approach, companies that play a larger role in the success of the coalition will receive a greater share of the profit. For instance, Company 1, due to its marketing efforts and provision of higher-quality products, receives a larger share of the profit. The results show that the higher the price, the higher the amount of revenue allocated to each player. Figure 9 illustrates how the Shapley value allocated to each company in the grand coalition changes as the number of consumers increases. As expected, increasing the number of consumers from 4 to 16 raises the value assigned to each company, since a larger customer base leads to higher overall revenue, which is then distributed among the members.

For most values of m, Company 1 receives the highest share. This is likely due to stronger advertising efforts, better market positioning, or greater influence within the coalition.

A notable point in the figure is that the Shapley value of Company 3 remains relatively stable at higher consumer levels (e.g., $m = 12$, 14, and 16). This indicates that its additional contribution to the coalition diminishes as the market grows—possibly due to limitations such as restricted service capacity, limited advertising strategies, or reduced competitive differentiation.

Interestingly, when $m = 4$, Company 3 holds the highest Shapley value, suggesting it may have an advantage in smaller markets. This could result from effective early-stage marketing or closer engagement with a smaller customer base.

As the consumer count rises, however, Company 1's share grows significantly. This trend reflects the competitive advantage of companies with broader reach, higher advertising budgets, or greater scalability in larger markets. The range of core is determined by (20). The equation and inequalities (25) are drawn, and the space between them is taken into consideration as Figure 8, in order to calculate the range of the core using Table 3. Note that if the core is empty, it indicates the instability of the game, and companies may have an incentive to deviate from the coalitions.

$$\xi_1(x_0, T - t_0) + \xi_2(x_0, T - t_0) + \xi_3(x_0, T - t_0) = 15052091670,$$
$$\xi_1(x_0, T - t_0) \geq 6587625000,$$
$$\xi_2(x_0, T - t_0) \geq 2721791667,$$
$$\xi_3(x_0, T - t_0) \geq 5742675000,$$
$$\xi_1(x_0, T - t_0) + \xi_2(x_0, T - t_0) \geq 9309416667,$$

$$\xi_1(x_0, T-t_0) + \xi_3(x_0, T-t_0) \geq 12330300000,$$
$$\xi_2(x_0, T-t_0) + \xi_3(x_0, T-t_0) \geq 8464466667. \quad (25)$$

Table 3: Characteristic function for coalitional advertisement.

	$S_1 = \{1\}$	$S_2 = \{2\}$	$S_3 = \{3\}$	$S_4 = \{1,2\}$	$S_5 = \{1,3\}$	$S_6 = \{2,3\}$	$S_7 = \{1,2,3\}$
$V(S)$	6587625000	2721791667	5742675000	9309416667	12330300000	8464466667	15052091670

Table 4: Imputation vectors of different cooperative game methods.

Company	Shapley value	τ-value
{1}	6587625000	6587625000
{2}	2721791667	2721791667
{3}	5742675000	5742675000

5.2 Scenario 2

According to the data in Table 2 and scenario 2, the results for the characteristic functions have been reported in Table 5. The results show that for this set of parameters, the profit of coalition S_3 is negative, which means that if Player 3 plays alone, he not only does not make a profit, but also incurs a loss. In other words, Player 3 must enter the coalition game, otherwise he faces the risk of bankruptcy. However, as can be seen, the value of the S_6 coalition is lower than when Player 2 plays alone. Therefore, it is not rational for Player 2 to join this coalition because the condition of individual rationality does not exist, similarly for Player 1 in coalition S_5. The S_4 coalition is a reasonable two-person coalition.

The payoff of the coalition S_4 is greater than the grand coalition S_7. Hence, forming a grand coalition is not logical. In such cases, the grand coalition is not stable and the goal of the companies is reduced to finding an optimal sub-coalition. The results show that parameters ρ_{ik} and δ_{ik} are key in determining the amount of coalition profits, and changing these parameters

Table 5: Characteristic functions for scenario 2.

	$S_1 = \{1\}$	$S_2 = \{2\}$	$S_3 = \{3\}$	$S_4 = \{1,2\}$	$S_5 = \{1,3\}$	$S_6 = \{2,3\}$	$S_7 = \{1,2,3\}$
$V(S)$	24141666	80058333	-5516666	104200000	18625000	74541667	98683333

have a significant influence on the outcomes of cooperation in the proposed model.

6 Conclusion

Most dynamic cooperative advertising research focuses on the interaction between retailers and manufacturers in the framework of cooperative adversarial games. In this paper, we have generalized the relationship between firms and customers in the form of a cooperative differential game in a network. The characteristic functions and benefits of network collaboration for the proposed model were studied. The core, Shapley value, and τ-value are obtained for a numerical example. We solved the proposed model for two different sets of parameters to investigate sustainability of cooperation between companies using advertising effectiveness parameters.

In the first scenario, we found that the larger the coalition, the higher its earnings, and we also discovered that the game has the additive property. In the second scenario, it was discovered that a larger coalition does not always yield greater income, and it is possible that players will earn more if they play alone. The influence of price on the model revealed that as the price grows, so does the profit allotted to the firms. According to Table 2, the assigned earnings in the coalition were larger than those in the non-collaborative state, ensuring long-term collaboration. Several topics might be recommended for further investigation. First, the model's collaboration process is explored in the infinite horizon time mode. The second step is to assess the type of communication in the network and its influence on earnings. Third, applying a stochastic dynamic system and solving it for this model would be both interesting and challenging.

Acknowledgements

Authors are grateful to there anonymous referees and editor for their constructive comments.

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