Stochastic Models in Probability and Statistics
Vol. 2, No. 2, 2025, pp. 119–132. Research Paper

A note on the signature and dynamic signature of coherent systems

Ahmad Mirjalili† **and Mohammad Khanjari Sadegh**‡*

†*Department of Statistics, Maybod Branch, Islamic Azad University, Maybod, Iran*
‡*Department of Statistics, Faculty of Mathematical Sciences, Ferdowsi University of Mashhad, P. O. Box 1159, Mashhad 91775, Iran*

Email(s): a.mirjalili@maybodiau.ac.ir, mkhanjari@um.ac.ir

Abstract. This paper investigates new properties and applications of the system signature and its dynamic counterpart, which serve as effective tools for analyzing stochastic ordering and aging properties in both coherent and used systems. For coherent systems with exchangeable components, the system signature offers a more powerful comparative framework than the traditional structure function. In the context of used systems with different exchangeable components, we propose an alternative to the dynamic signature that is simpler to implement and often preferable to the standard dynamic signature. This alternative also proves useful in scenarios where the traditional dynamic signature is inapplicable. Additionally, by examining all 28 coherent systems of order $n \leq 4$, we establish a unique property of series systems: under both identical and non-identical independent component lifetimes, series systems are the only ones that are decreasing failure rate (DFR) closed. The results extend several existing findings related to system signatures and their dynamic versions. Illustrative examples are provided to demonstrate the practical relevance of the theoretical results.

Keywords: Coherent systems, Dynamic signature, DFR, IFR, Signature, Stochastic ordering, Used systems.

1 Introduction

One of the most important problems in system reliability analysis is comparison among systems. The most common tools to compare the coherent systems are structure and reliability functions, system signatures, and distortion functions. Assume that the systems are coherent (see, Barlow and Proschan (1975) for details on coherent systems) and suppose that the component lifetimes of the system are independent and identically distributed (IID) or they are exchangeable (EXC) random variables, defined in the next section. This paper is concerned to some more properties and applications of the signatures in system comparisons. Let $T_1,\ldots,T_n$ and $T=\phi(T_1,\ldots,T_n)$ be the component lifetimes and the system lifetime, respectively. ϕ refers to the structure of the system. When T_i's are continuous and IID random variables,

*Corresponding author

Received: 5 March 2025 / Accepted: 29 September 2025
DOI: 10.22067/smps.2025.92495.1043

 https://smps.um.ac.ir

the following well-known and important result is obtained by Samaniego (1985)

$$P(T > t) = \sum_{i=1}^{n} s_i P(T_{i:n} > t), \tag{1}$$

where $T_{i:n}$ is the i-th ordered components lifetime, and $s_i = P(T = T_{i:n})$. The probability vector $\mathbf{s} = (s_1, \ldots, s_n)$ is called the system signature.

Using the system signature, many researchers have studied various aspects of system reliability. In fact, the identity (1) was first proved by Kochar et al. (1999). Navarro et al. (2005) claimed that this equation holds true if T_i's have an absolutely continuous exchangeable joint distribution. Samaniego (2007) published a book on the system signatures and their applications. Also Naqvi et al. (2022) have done a review and bibliometric analysis on system signatures. Navarro et al. (2008) showed that (1) holds true if T_i's be continuous (or discrete) weakly exchangeable random variables and also $P(T_i = T_j) = 0$ for all $i \neq j$ (that is T_i's have no tie). It seems that the strongest result is given by Marichal et al. (2011). They proved that (1) holds true if and only if for $i = 1, \ldots, n$, the following binary random variables are exchangeable

$$X_i(t) = \begin{cases} 1 & T_i > t \\ 0 & T_i \leq t. \end{cases} \tag{2}$$

Navarro et al. (2011) investigated the signatures of coherent systems with heterogeneous and independent components. Rao and Naqvi (2023) studied the stochastic comparisons of coherent systems with heterogeneous and dependent components having proportional reversed failure rates. In the EXC case, a necessary and sufficient condition for comparing two systems was given by Navarro and Rubio (2011). For more details and an extensive study on comparisons of coherent systems see Navarro (2018) and references therein.

Remark 1. *The system signature has the following interesting and important property. It should be noted that in all cases where the Equation (1) holds true (except when the component lifetimes are continuous and IID random variables), the signature is not necessarily defined by* $s_i = P(T = T_{i:n})$*. But in these cases, the value of* s_i *used in (1), is exactly determined by* $s_i = P(T = T_{i:n})$ *when* T_i*'s are continuous and IID random variables. Therefore in all cases, the system signature* $\mathbf{s} = (s_1, \ldots, s_n)$ *remains as a probability vector. For further clarification, consider* $T = min\{T_1, max(T_2, T_3)\}$*, where* T_i*'s are continuous and IID. We have* $\mathbf{s} = (1/3, 2/3, 0)$ *and* $P(T = T_{i:n}) = s_i$*. Now suppose* T_i*'s are discrete and IID with common distribution Binomial(1,1/2). In this case we have*

$$P(T = T_{1:3}) = 1 - P(T_1 = 1, T_2 = 0, T_3 = 1) - P(T_1 = 1, T_2 = 1, T_3 = 0) = 6/8 \neq s_1 = 1/3.$$

But one can easily verify that for

$$P(T = x) = \sum_{1}^{3} s_i P(T_{i:n} = x), \ x = 0, 1.$$

That is the mixture representation (1) holds true.

The concept of the dynamic signature of the system was first defined by Samaniego et al. (2009). It is a common tool in comparisons among coherent used systems. They showed in the case of IID that

$$P(T = T_{k:n} | T > t, N(t) = r) = \frac{s_k}{\sum_{j=r+1}^{n} s_j} = w_k,$$

for $k = r+1, \ldots, n$ and $r = 0, 1, \ldots, n-1$. The probability vector $\mathbf{w}_{(n-r)} = (w_{r+1}, \ldots, w_n)$ is called the dynamic signature of the system at time t. $N(t)$ is the number of failed components of the system up to time t and such a system is said to be a used system. Based on the dynamic signature, they obtained some ordering results for comparing coherent used systems with IID components. Note that if $r = 0$, the dynamic signature reduces to the usual signature as $\sum_1^n s_i = 1$.

Burkschat and Navarro (2013) considered dynamic signatures of coherent systems based on sequential order statistics. Under some partial information about the failure status of the system lifetime, Mahmoudi and Asadi (2011) defined the dynamic signature of coherent systems. Burkschat and Samaniego (2018) examined the concepts of dynamic and conditional increasing failure rates properties in coherent systems. Chahkandi et al. (2015) studied the signature of the repairable systems.

The remainder of this paper is organized as follows. For the sake of completeness, Section 2 provides an extensive review of comparisons among coherent systems based on structure functions, reliability functions, and system signatures. As a minor result, it is shown that, in the case of exchangeable components (EXC), the system signature is a more powerful tool than the structure function. Section 3 focuses on comparisons of coherent used systems. Using the dynamic signature, new stochastic ordering results are derived for comparing coherent used systems with different exchangeable components. A new, simple alternative to the dynamic signature is proposed, which is particularly useful in cases where the standard dynamic signature is not applicable. Due to its ease of use, this alternative should be considered as a first option in such comparisons. Finally, Section 4 examines properties related to the preservation of reliability aging classes in series systems.

Motivated by the result of Samaniego (1985) and by verifying the signatures of all 28 coherent systems of order $n \leq 4$, it is shown that when the lifetimes of the system components are either IID or independent and not identical (INID), the series systems are only decreasing the failure rate (DFR) closed systems. The increasing failure rate (IFR) closedness property of the series systems is also considered. Recall that a random variable X with absolutely continuous distribution function F and density function f has an IFR (DFR) distribution if its failure rate function $h(t) = f(t)/\bar{F}(t)$ is increasing (decreasing) in t, $\bar{F}(t) = 1 - F(t)$. Throughout the paper, when two systems are said to be ordered, it means that their lifetimes are ordered stochastically. Recall that X is less than Y in the usual stochastic order and denote it by $X \leq_{st} Y$ if $\bar{F}_X(t) \leq \bar{F}_Y(t)$ for all $t > 0$.

2 Comparisons of coherent systems

The comparison of two systems can be done at a fixed point in time (static comparison) or during periods of time (dynamic comparison). In order to present the main results of the paper given in the two next sections, and for the sake of completeness, this section mainly deals with a review of the basic results in the literature on comparisons among coherent systems. Some minor contributions are also included. At a fixed point in time, and based on the system structure and reliability functions, we first consider the comparison of two systems. In comparison among the systems with INID components, it is shown that the structure and reliability functions are two equivalent tools. Finally the system signature is used to compare two systems during of times. In the sequel, Lemma 1 shows in IID and EXC cases that the system signature is a more powerful tool than the structure-function.

At a fixed point in time, assume that binary random states are as follows:

$$X_i = \begin{cases} 1 & \text{if } i\text{th component is working} \\ 0 & \text{otherwise,} \end{cases}$$

and

$$\phi(\mathbf{X}) = \phi(X_1, \ldots, X_n) = \begin{cases} 1 & \text{if the system is working} \\ 0 & \text{otherwise.} \end{cases}$$

Their reliabilities are also defined as:

$$p_i = E(X_i) = P(X_i = 1), \; i = 1, 2, \ldots, n$$

$$h(\mathbf{p}) = h(p_1, \ldots, p_n) = E(\phi(\mathbf{X})) = P(\phi(\mathbf{X}) = 1).$$

$\phi(\mathbf{x})$ is called the structure function of the system. In addition, the order of the system is defined as the number of the system components n.

Definition 1. *(Kochar et al., 1999) Let $\phi_1(\mathbf{x})$ and $\phi_2(\mathbf{x})$ be the structure functions of two systems of order n. The second system is said to be better than the first one if*

$$\phi_1(\mathbf{x}) \leq \phi_2(\mathbf{x}) \quad \forall\, \mathbf{x} \in \{0,1\}^n. \tag{3}$$

If $\phi_1(\mathbf{x}) = \phi_2(\mathbf{x})$, $\forall \mathbf{x} \in \{0,1\}^n$ two systems are obviously the same. Therefore (3) should be strictly hold at least for one $\mathbf{x}$. The second system is said to be more reliable than the first one if

$$h_1(\mathbf{p}) = E(\phi_1(\mathbf{X})) \leq h_2(\mathbf{p}) = E(\phi_2(\mathbf{X})) \quad \forall\, \mathbf{p} \in [0,1]^n. \tag{4}$$

Again the inequality should be strict at least for one $\mathbf{p}$, otherwise two systems have the same reliability.

It is claimed in Kochar et al. (1999) that the inequalities (3) and (4) are equivalent when X_i's are independent (INID case). Note that in this case

$$h(\mathbf{p}) = P(\phi(\mathbf{X}) = 1) = \sum_{\mathbf{x}:\phi(\mathbf{x})=1} \prod_{i=1}^{n} p_i^{x_i}(1-p_i)^{1-x_i} = \sum_{\mathbf{x}:\phi(\mathbf{x})=1} \prod_{i\in \mathbf{1_x}} p_i \prod_{i\in \mathbf{0_x}} (1-p_i),$$

where $\mathbf{1_x} = \{1 \leq i \leq n | x_i = 1\}$, $\mathbf{0_x} = \{1 \leq i \leq n | x_i = 0\}$. An argument for their claim is as follows. The inequality (3) simply implies (4). Now suppose (4) holds true. If $\phi_1(\mathbf{x}) = \phi_2(\mathbf{x})$ for all $\mathbf{x}$ obviously $h_1(\mathbf{p}) = h_2(\mathbf{p})$ for all $\mathbf{p}$ which is a contradiction. Now suppose there exists a vector $\mathbf{x}_0$ such that $\phi_1(\mathbf{x}_0) = 1 > \phi_2(\mathbf{x}_0) = 0$. Put $\mathbf{p}_0 = \mathbf{x}_0$ then $p_{0i} = 1$ for $i \in \mathbf{1}_{\mathbf{x}0}$ and $1 - p_{0i} = 1$ for $i \in \mathbf{0}_{\mathbf{x}0}$. Note that $P_{\mathbf{p}_0}(\mathbf{X} = \mathbf{x}) = P(\mathbf{X} = \mathbf{x} | \mathbf{p} = \mathbf{p}_0) = 0$ for $\mathbf{x} \neq \mathbf{x}_0$. It is easy to see that $h_1(\mathbf{p}_0) = 1$ and $h_2(\mathbf{p}_0) = 0$ which is again a contradiction and this shows that (4) implies (3).

The inequalities (3) and (4) are not necessarily equivalent if X_i's are IID that is $p_i = p$, $i = 1, \ldots, n$. See the following example.

Example 1. (Kochar et al., 1999). Suppose $\phi_1(\mathbf{x}) = \min\{x_1, \max(x_2, x_3)\}$ and $\phi_2(\mathbf{x}) = \min\{x_2, \max(x_1, x_3)\}$ then $\phi_1(1,0,1) = 1 > \phi_2(1,0,1) = 0$ and $\phi_1(0,1,1) = 0 < \phi_2(0,1,1) = 1$ whereas $h_1(p) = 2p^2 - p^3 = h_2(p)$, for all $0 \leq p \leq 1$. That is (4) does not imply (3).

Remark 2. *Note that in INID case, the inequalities (3) and (4) are equivalent even if $\mathbf{p} \in (0,1)^n$. It is because as the system components are independent, the reliability function $h(\mathbf{p})$ is a multilinear, that is, linear in each p_i and therefore is a continuous function.*

Now by using the system signature, the comparison of systems during of times is considered. The first result is given by Kochar et al. (1999) as follows:
Let $T_1 = \phi_1(X_1, \ldots, X_n)$ and $T_2 = \phi_2(X_1, \ldots, X_n)$ denote the lifetimes of two coherent systems where X_i is the lifetime of the ith component and X_i's are IID and continuous and suppose $\mathbf{s}_1$ and $\mathbf{s}_2$ are their signatures respectively. If $\mathbf{s}_1 \leq_{st} \mathbf{s}_2$, then $T_1 \leq_{st} T_2$. If $\mathbf{s}_1 \leq_{hr} \mathbf{s}_2$, then $T_1 \leq_{hr} T_2$ and if $\mathbf{s}_1 \leq_{lr} \mathbf{s}_2$, then $T_1 \leq_{lr} T_2$.
The converse of their results is not true, see Rychlik et al. (2018). (We also refer to Shaked and Shanthikumar (2007) for details on stochastic orders).

Example 2. Consider two coherent systems mentioned in Example 1 where $\mathbf{s}_1 = (1/3, 2/3, 0) = \mathbf{s}_2$ and hence from (1), T_1 and T_2 are identically distributed. Note that these two systems could not be ordered by using of their structure functions.

The comparison of coherent systems with EXC components is studied by Navarro et al. (2005). Recall that the random variables $T_1,\ldots,T_n$ are said to be exchangeable if $P(T_1>t_1,\ldots,T_n>t_n)=P(T_{\pi(1)}>t_1,\ldots,T_{\pi(n)}>t_n)$ for any permutation $\pi=(\pi(1),\ldots,\pi(n))$ of numbers $\{1,\ldots,n\}$.

The relationship between the signature and structure function of a system is obtained by Boland (2001) or Marichal et al. (2011) as follows. In EXC case and for $k=1,\ldots,n$

$$\sum_{i=n-k+1}^{n} s_i = P(\phi(\mathbf{X})=1|\sum X_i=k)=\sum_{\mathbf{x};\sum x_i=k}\phi(\mathbf{x})\frac{1}{\binom{n}{k}}. \tag{5}$$

Lemma 1. *If $\phi_1(\mathbf{x})\leq\phi_2(\mathbf{x})$ for all $\mathbf{x}\in\{0,1\}^n$ then $\mathbf{s}_1\leq_{st}\mathbf{s}_2$.*

Proof. From Equation (5) we see that if $\phi_1(\mathbf{x})\leq\phi_2(\mathbf{x})$ for all $\mathbf{x}\in\{0,1\}^n$ then $\sum_{i=n-k+1}^{n}s_{1,i}\leq\sum_{i=n-k+1}^{n}s_{2,i}$ for $k=1,2,\ldots,n$, that is $\mathbf{s}_1\leq_{st}\mathbf{s}_2$. □

In fact Lemma 1 shows that the system signature is a more stronger tool than the structure function. This section is ended by reviewing the problem of comparison of coherent systems with different sizes.

Definition 2. *(Boland and Samaniego, 2004), or (Samaniego, 2006). Let $\mathbf{r}=(r_1,\ldots,r_k)$ be a probability vector and suppose $T_j=\phi_j(X_1,\ldots,X_n)$, $j=1,\ldots,k$ are the lifetimes of k coherent systems with IID or EXC components. If $P(T=T_j)=r_j$ then T is said to be the lifetime of a mixed-$\mathbf{r}$ system. Note that in general, a mixed system is not coherent, and the class of coherent systems is a subset of the mixed systems. Let $\mathbf{s}_j$ be the signature of the jth coherent system, then $\mathbf{s}=\sum_{j=1}^{k}r_j\mathbf{s}_j$ is defined as the signature of the mixed-$\mathbf{r}$ system.*

For a system of order n, an equivalent system of order $n+1$ is introduced by Samaniego (2006) as follows.

Let T be the lifetime of a coherent or a mixed system of order n with IID components having common distribution F and with signature $\mathbf{s}=(s_1,\ldots,s_n)$. Then $T=^{st}T^*$ where T^* is the lifetime of a mixed system of order $n+1$ with IID components which have the same distribution F and its signature is $\mathbf{s}^*=(s_1^*,\ldots,s_{n+1}^*)$ with

$$s_i^*=\frac{(i-1)s_{i-1}+(n-i+1)s_i}{n+1}, i=1,\ldots,n+1.$$

Assume that $s_0=s_{n+1}=0$. $\mathbf{s}^*$ is of order $n+1$ and is the equivalent of $\mathbf{s}$.

For a system of order k, an equivalent system of order $n(>k)$ can be defined. Note that the above result also holds true for EXC components, also note that the equivalent systems belong in general to the class of mixed systems and all comparison results among the coherent systems also hold true for the mixed systems.

An extended result for coherent systems consisting of EXC components, is given by Navarro et al. (2008) as follows.

Let $T_1=\phi_1(Y_1,\ldots,Y_{n_1})$ and $T_2=\phi_2(Z_1,\ldots,Z_{n_2})$ be the lifetimes of two systems where Y_i's and Z_i's are subsets of the EXC random variables $\{X_1,\ldots,X_n\}$ and suppose $\mathbf{s}_1(n)$ and $\mathbf{s}_2(n)$ are their equivalent signatures of order n, respectively.

(a) If $\mathbf{s}_1(n)\leq_{st}\mathbf{s}_2(n)$, then $T_1\leq_{st}T_2$.

(b) If $\mathbf{s}_1(n)\leq_{hr}\mathbf{s}_2(n)$ and $X_{i:n}\leq_{hr}X_{i+1:n}$, $i=1,\ldots,n-1$, then $T_1\leq_{hr}T_2$.

(c) If $\mathbf{s}_1(n)\leq_{lr}\mathbf{s}_2(n)$ and $X_{i:n}\leq_{lr}X_{i+1:n}$, $i=1,\ldots,n-1$, then $T_1\leq_{lr}T_2$.

For an extensive study on the existence and comparisons of equivalent systems with IID components refer to Lindqvist et al. (2016). It is pointed out there, within the class of mixed systems, one can always find equivalent mixed systems of larger sizes, but not necessary if the size is decreased.

3 Comparisons of used systems

This section begins by considering the comparison of coherent used systems, extends some existing results, and then proposes an alternative for equivalent dynamic signature. Its properties and applications are illustrated. The following result is given in Samaniego et al. (2009).

Let $\mathbf{s}_1$ and $\mathbf{s}_2$ be the signatures of two coherent systems of order n and with IID components having common continuous distribution F and let $T_1 = \phi_1(X_1,\ldots,X_n)$ and $T_2 = \phi_2(X_1,\ldots,X_n)$ be their lifetimes, respectively. Suppose $T_1 > t$, $T_2 > t$, $N_1(t) = r_1$ and $N_2(t) = r_2$. Let $\mathbf{w}_1 = (w_{1,1},\ldots,w_{1,n-r_1})$ and $\mathbf{w}_2 = (w_{2,1},\ldots,w_{2,n-r_2})$ be their dynamic signatures. Also assume that $\mathbf{w}_1^{(n)}$ and $\mathbf{w}_2^{(n)}$ are their equivalent versions of order n. If $\mathbf{w}_1^{(n)} \le_{st} \mathbf{w}_2^{(n)}$ then

$$(T_1 - t|T_1 > t, N_1(t) = r_1) \le_{st} (T_2 - t|T_2 > t, N_2(t) = r_2).$$

As defined in Section 1, $N(t)$ is the number of failed components of the system up to time t.

Samaniego et al. (2009) also provide the following expression:

$$P(T - t > x|T > t, N(t) = r) = \sum_{i=1}^{n-r} w_{r+i}\bar{F}_{i:n-r|t}(x) = \sum_{i=1}^{n} w_i^{(n)}\bar{F}_{i:n|t}(x),$$

where $\bar{F}_{i:n-r|t}(x)$ is the reliability function of the ith order statistic in a random sample of size $n-r$ from $\bar{F}_t(x) = \frac{\bar{F}(x+t)}{\bar{F}(t)}$ for $x > 0$.
Note that $N(t)$ is distributed as Binomial$(n, F(t))$ and $N(t) = r$ is equivalent to $X_{r:n} < t < X_{r+1:n}$. Therefore $P(X_{i:n} - t > x|N(t) = r) = 0$, for $i = 1,\ldots,r$ and for $i = r+1,\ldots,n$, we have

$$\begin{aligned} P(X_{i:n} - t > x|N(t) = r) &= \frac{\binom{n}{r}F^r(t)\sum_{j=0}^{i-r-1}\binom{n-r}{j}[\bar{F}(t) - \bar{F}(t+x)]^j(\bar{F}(t+x))^{n-r-j}}{\binom{n}{r}F^r(t)(\bar{F}(t))^{n-r}} \\ &= \sum_{j=n-i+1}^{n-r}\binom{n-r}{j}(\bar{F}_t(x))^j(1-\bar{F}_t(x))^{n-r-j} \\ &= P(\text{Binomial}(n-r, \bar{F}_t(x)) \ge n-i+1). \end{aligned} \tag{6}$$

Now suppose $T = \phi(X_1,\ldots,X_n)$ is the lifetime of a system with EXC components. For this system, the following equation is given by Sadegh (2016)

$$\begin{aligned} P(T - t > x|T > t, N(t) = r) &= \sum_{k=r+1}^{n} w_k P(X_{k:n} - t > x|N(t) = r) \\ &= \sum_{k=1}^{n} w_k^{(n)} P(X_{k:n} - t > x|N(t) = 0). \end{aligned} \tag{7}$$

Based on the Equation (7), the next theorem extends part (a) of Theorem 2.6 in Samaniego et al. (2009), to the systems with different components.

Theorem 1. *Let $T_1 = \phi_1(X_1,\ldots,X_n)$ and $T_2 = \phi_2(Y_1,\ldots,Y_n)$ be the lifetimes of two coherent systems with EXC components. Suppose $T_1 > t$, $T_2 > t$, $N_1(t) = r_1$ and $N_2(t) = r_2$. Let $\mathbf{w}_1$, $\mathbf{w}_2$, $\mathbf{w}_1^{(n)}$ and $\mathbf{w}_2^{(n)}$ are defined as before. If $\mathbf{w}_1^{(n)} \le_{st} \mathbf{w}_2^{(n)}$ and $(X_{i:n} - t|N_1(t) = 0) \le_{st} (Y_{i:n} - t|N_2(t) = 0)$ for $i = 1,\ldots,n$ then*

$$(T_1 - t|T_1 > t, N_1(t) = r_1) \le_{st} (T_2 - t|T_2 > t, N_2(t) = r_2).$$

Proof. It is known that if $X \leq_{st} Y$ then $Eg(X) \leq Eg(Y)$ for any increasing function g. Let $\mathbf{w}_1^{(n)}$ and $\mathbf{w}_2^{(n)}$ be the probability vectors of two discrete random variables X and Y, respectively. Note that $X \leq_{st} Y$ as $\mathbf{w}_1^{(n)} \leq_{st} \mathbf{w}_2^{(n)}$. Now in view of the second equality of Equation (7) we have

$$\begin{aligned} E[g(X)] &= \sum_1^n w_{1,k}^{(n)} P(X_{k:n} - t > x | N_1(t) = 0) = P(T_1 - t > x | T_1 > t, N_1(t) = r_1) \\ &\leq Eg(Y) = \sum_{k=1}^n w_{2,k}^{(n)} P(X_{k:n} - t > x | N_1(t) = 0) \\ &\leq \sum_{k=1}^n w_{2,k}^{(n)} P(Y_{k:n} - t > x | N_2(t) = 0) \\ &= P(T_2 - t > x | T_2 > t, N_2(t) = r_2). \end{aligned}$$

□

Here is an example that satisfies the conditions of the Theorem 1.

Example 3. Let $T_1 = \phi_1(X_1, X_2, X_3) = \min\{X_1, \max(X_2, X_3)\}$ and suppose $T_2 = \phi_2(Y_1, Y_2, Y_3) = \max Y_i$. We consider two following cases.

(i) IID case: Suppose X_i's and Y_i's are independent and distributed as $\bar{F}(x) = e^{-\lambda x}$ and $\bar{G}(x) = e^{-\theta x}$, respectively and $\lambda > \theta$. Assume that $N_1(t) = N_2(t) = 1$, $T_1 > t$ and $T_2 > t$. Now these two systems are satisfying the conditions of Theorem 1 as we have:
$\mathbf{s}_1 = (1/3, 2/3, 0)$, $\mathbf{s}_2 = (0, 0, 1)$ and therefore $\mathbf{w}_1 = (1, 0)$ and $\mathbf{w}_2 = (0, 1)$.
Also $\mathbf{w}_1^{(3)} = (2/3, 1/3, 0)$ and $\mathbf{w}_2^{(3)} = (0, 1/3, 2/3)$. Obviously $\mathbf{w}_1^{(3)} \leq_{st} \mathbf{w}_2^{(3)}$ and

$$\bar{F}_t(x) = \bar{F}(x) = e^{-\lambda x} < \bar{G}_t(x) = \bar{G}(x) = e^{-\theta x}.$$

It is known that the Binomial distribution $B(n, p)$ is stochastically increasing in p. Therefore from Equation (6), we have $P(X_{i:n} - t > x | N_1(t) = 0) \leq P(Y_{i:n} - t > x | N_2(t) = 0)$ for $i = 1, 2, 3$. It implies that

$$(T_1 - t | T_1 > t, N_1(t) = 1) \leq_{st} (T_2 - t | T_2 > t, N_2(t) = 1).$$

For more clarity, we compare their survival functions as follows. From structures of these two systems we have

$$\begin{aligned} P(T_1 - t > x | T_1 > t, N_1(t) = 1) &= \frac{2P(X_1 > t + x, X_2 > t + x, X_3 < t)}{2P(X_1 > t, X_2 > t, X_3 < t)} \\ &= \frac{[P(X_1 > t + x)]^2}{[P(X_1 > t)]^2} \\ &= \frac{e^{-2\lambda(t+x)}}{e^{-2\lambda t}} = e^{-2\lambda x}. \end{aligned}$$

Now for the second system we have

$$\begin{aligned} P(T_2 - t > x | T_2 > t, N_2(t) = 1) &= \frac{6P(X_1 > t + x, t < X_2 < t + x, X_3 < t) + 3P(X_1 > t + x, X_2 > t + x, X_3 < t)}{3P(X_1 > t, X_2 > t, X_3 < t)} \\ &= \frac{6e^{-\theta(t+x)}(e^{-\theta t} - e^{-\theta(t+x)}) + 3e^{-2\theta(t+x)}}{3e^{-2\theta t}} \\ &= 2e^{-\theta x} - e^{-2\theta x}. \end{aligned}$$

It is easy to see that

$$e^{-2\lambda x} \leq 2e^{-\theta x} - e^{-2\theta x},$$

as $\lambda > \theta$.

(ii) EXC case: Let $\bar{F}(x_1,x_2,x_3) = e^{-\sum_1^3 x_i - \lambda x_{3:3}}$ and $\bar{G}(y_1,y_2,y_3) = e^{-\sum_1^3 y_i - \theta y_{3:3}}$ where $\lambda > \theta$ and $x_{3:3} = \max(x_1,x_2,x_3)$. Again assume that $N_1(t) = N_2(t) = 1$, $T_1 > t$ and $T_2 > t$. We now verify the conditions of Theorem 1.
If $i = 1$, then

$$\begin{aligned} P(X_{1:3} - t > x | N_1(t) = 0) &= P(X_{1:3} > t + x)/P(X_{1:3} > t) \\ &= \bar{F}(t+x,t+x,t+x)/\bar{F}(t,t,t) \\ &= e^{-3t-3x-\lambda(t+x)}/e^{-3t-\lambda t} = e^{-3x-\lambda x} \\ &\leq e^{-3x-\theta x} \\ &= P(Y_{1:3} - t > x | N_2(t) = 0). \end{aligned}$$

That is $(X_{1:3} - t | N_1(t) = 0) \leq_{st} (Y_{1:3} - t | N_2(t) = 0)$. If $i = 2$, then from exchangeability of X_i's we have

$$\begin{aligned} P(X_{2:3} - t > x | N_1(t) = 0) &= \frac{\bar{F}(t+x,t+x,t+x) + 3P(t < X_1 < t+x, X_2 > t+x, X_3 > t+x)}{\bar{F}(t,t,t)} \\ &= \frac{e^{-3t-3x-\lambda(t+x)} + 3(\bar{F}(t,t+x,t+x) - \bar{F}(t+x,t+x,t+x))}{e^{-3t-\lambda t}} \\ &= \frac{3e^{-3t-2x-\lambda(t+x)} - 2e^{-3t-3x-\lambda(t+x)}}{e^{-3t-\lambda t}} = e^{-\lambda x-2x}(3 - 2e^{-x}) \\ &\leq e^{-\theta x-2x}(3 - 2e^{-x}) \\ &= P(Y_{2:3} - t > x | N_2(t) = 0). \end{aligned}$$

That is $(X_{2:3} - t | N_1(t) = 0) \leq_{st} (Y_{2:3} - t | N_2(t) = 0)$.
Similarly it can be shown that $(X_{3:3} - t | N_1(t) = 0) \leq_{st} (Y_{3:3} - t | N_2(t) = 0)$. Hence all conditions of the Theorem 1 are satisfied and therefore

$$(T_1 - t | T_1 > t, N_1(t) = 1) \leq_{st} (T_2 - t | T_2 > t, N_2(t) = 1).$$

For more clarity, we also obtain the corresponding survival functions to see that the above stochastic ordering between these two systems in fact holds true. For the first system we have

$$\begin{aligned} P(T_1 - t > x | T_1 > t, N_1(t) = 1) &= \frac{2P(X_1 > t+x, X_2 > t+x, X_3 < t)}{2P(X_1 > t, X_2 > t, X_3 < t)} \\ &= \frac{\bar{F}(t+x,t+x,0) - \bar{F}(t+x,t+x,t)}{\bar{F}(t,t,0) - \bar{F}(t,t,t)} \\ &= \frac{e^{-2(t+x)-\lambda(t+x)} - e^{-3t-2x-\lambda(t+x)}}{e^{-2t-\lambda t} - e^{-3t-\lambda t}} = e^{-\lambda x-2x}. \end{aligned}$$

Now for the second system we have

$$\begin{aligned}
P(T_2-t>x|T_2>t,N_2(t)=1) &= \frac{6P(X_1>t+x,t<X_2<t+x,X_3<t)+3P(X_1>t+x,X_2>t+x,X_3<t)}{3P(X_1>t,X_2>t,X_3<t)} \\
&= \frac{6[\bar{F}(t+x,t,0)-\bar{F}(t+x,t,t)-\bar{F}(t+x,t+x,0)+\bar{F}(t+x,t+x,t)]}{3[\bar{F}(t,t,0)-\bar{F}(t,t,t)]} \\
&+ \frac{3[\bar{F}(t+x,t+x,0)-\bar{F}(t+x,t+x,t)]}{3[\bar{F}(t,t,0)-\bar{F}(t,t,t)]} \\
&= \frac{6[\bar{F}(t+x,t,0)-\bar{F}(t+x,t,t)]-3[\bar{F}(t+x,t+x,0)-\bar{F}(t+x,t+x,t)]}{3[\bar{F}(t,t,0)-\bar{F}(t,t,t)]}.
\end{aligned}$$

After algebraic simplifications, it can be shown that

$$P(T_2-t>x|T_2>t,N_2(t)=1)=2e^{-x(1+\theta)}-e^{-x(2+\theta)}.$$

As $\lambda>\theta$, it is easy to verify that

$$e^{-\lambda x-2x}\leq 2e^{-x(1+\theta)}-e^{-x(2+\theta)}.$$

Therefore again we see that the stochastic ordering between these two used systems with EXC components holds true.

3.1 A proposed dynamic signature

Let $T=\phi(X_1,\ldots,X_n)$, $T>t$ and $N(t)=r$. Also suppose $\mathbf{w}_{(n-r)}=(w_{r+1},\ldots,w_n)$ is the vector of dynamic signature defined in Equation (2) and $\mathbf{w}^{(n)}$ is its corresponding usual equivalent dynamic signature which is of order n. Here a simple modified dynamic signature $\mathbf{w}^*$ for $\mathbf{w}^{(n)}$ is proposed that have some useful properties. For $i=1,\ldots,n$ its coordinates are given below

$$w_i^*=\begin{cases}0 & i\leq r\\ w_i & i>r.\end{cases} \tag{8}$$

In EXC case and in view of (7), it is obvious that

$$P(T-t>x|T>t,N(t)=r)=\sum_{k=1}^{n}w_k^*P(X_{k:n}-t>x|N(t)=r). \tag{9}$$

Theorem 2. *The results of Theorem 1 hold true if* $\mathbf{w}_1^{(n)}$ *and* $\mathbf{w}_2^{(n)}$*, are replaced by* $\mathbf{w}_1^*$ *and* $\mathbf{w}_2^*$*, respectively and also* $(X_{i:n}-t|N_1(t)=r_1)\leq_{st}(Y_{i:n}-t|N_2(t)=r_2)$ *for* $i=min(r_1,r_2)+1,\ldots,n$.

Proof. From Equation (9) and in view of the first equality of Equation (7) we have

$$
\begin{aligned}
Eg(X) = \sum_{1}^{n} w^*_{1,k} P(X_{k:n} - t > x | N_1(t) = r_1) &= P(T_1 - t > x | T_1 > t, N_1(t) = r_1) \\
&\leq Eg(Y) = \sum_{k=1}^{n} w^*_{2,k} P(X_{k:n} - t > x | N_1(t) = r_1) \\
&= \sum_{k=r_1+1}^{n} w^*_{2,k} P(X_{k:n} - t > x | N_1(t) = r_1) \\
&= \sum_{k=\min(r_1,r_2)+1}^{n} w^*_{2,k} P(X_{k:n} - t > x | N_1(t) = r_1) \\
&\leq \sum_{k=\min(r_1,r_2)+1}^{n} w^*_{2,k} P(Y_{k:n} - t > x | N_2(t) = r_2) \\
&= \sum_{k=r_2+1}^{n} w^*_{2,k} P(Y_{k:n} - t > x | N_2(t) = r_2) = P(T_2 - t > x | T_2 > t, N_2(t) = r_2).
\end{aligned}
$$

We note that $P(X_{i:n} - t > x | N(t) = r) = 0$, for $i = 1, \ldots, r$. This completes the proof of the theorem. □

Remark 3. *In comparisons of used systems, Theorem 1 is an extension of part (a) of Theorem 2.6 in Samaniego et al. (2009). On the other hand the conditions of Theorem 2 are less and easier than those of Theorem 1. In Theorem 2, we do not need to compute the usual equivalent dynamic signatures* $\mathbf{w}_1^{(n)}$ *and* $\mathbf{w}_2^{(n)}$*, which are usually obtained after lengthy computations on dynamic signatures. Also the number of stochastic comparisons of conditional random variables in Theorem 2 is* $n - min(r_1, r_2)$ *which is less than that of Theorem 3.1, that is* n*. One may verify the possibility of happening the following situation:* $\mathbf{w}_1^{(n)} \leq_{st} \mathbf{w}_2^{(n)}$ *but* $\mathbf{w}_1^*$ *and* $\mathbf{w}_2^*$ *are not ordered.*
Obviously, in this case the proposed dynamic signature $\mathbf{w}^*$ *is not applicable and one should use* $\mathbf{w}^{(n)}$*. But the converse of this case may also happen (see the following example when the system components lifetimes are IID).*

Therefore as the above discussed, among $\mathbf{w}^{(n)}$ and $\mathbf{w}^*$ one should clearly first take $\mathbf{w}^*$ into consideration, particularly when $\mathbf{w}^{(n)}$ is not usable. See the following example.

Example 4. Let $\mathbf{s}_1 = (0, 0, 0, 4/5, 1/5)$ and $\mathbf{s}_2 = (0, 0, 3/5, 1/5, 1/5)$ be the signatures of two systems consisting of different IID components and suppose $N_1(t) = 2$ and $N_2(t) = 3$. From Equation (2), it implies that $\mathbf{w}_1 = (0, 4/5, 1/5)$ and $\mathbf{w}_2 = (1/2, 1/2)$. Also it is easy to see that the equivalent vectors of $\mathbf{w}_1$ and $\mathbf{w}_2$ are $\mathbf{w}_1^{(5)} = 0.01 \times (0, 24, 34, 30, 12)$ and $\mathbf{w}_2^{(5)} = (1/5, 1/5, 1/5, 1/5, 1/5)$, respectively. Note that in view of the usual stochastic order, $\mathbf{w}_1^{(5)}$ and $\mathbf{w}_2^{(5)}$ are not ordered. Therefore, regardless the system components be common or not, for comparing of two mentioned used systems, part (a) of Theorem 2.6 in Samaniego et al. (2009), and Theorem 1, both are not usable. Whereas,

$$\mathbf{w}_1^* = (0, 0, 0, 4/5, 1/5) \leq_{st} \mathbf{w}_2^* = (0, 0, 0, 1/2, 1/2).$$

Hence, under the assumptions of Theorem 2, we have $(T_1 - t | T_1 > t, N_1(t) = 2) \leq_{st} (T_2 - t | T_2 > t, N_2(t) = 3)$.

4 Some unique aging properties of the series systems

The preservation of reliability aging classes under the formation of coherent systems is an important topic in reliability studies. This section considers the preservation of IFR and DFR classes in series systems and

particularly shows that among the coherent systems with INID or IID components, the series systems are only DFR closed systems. This result is mainly obtained based on the system signatures. For a detailed study on the preservation of reliability aging classes under the formation of coherent systems refer to Navarro et al. (2013).
Let $T = \phi(X_1,\ldots,X_n)$ be the lifetime of a coherent system in which the component lifetimes X_i's are IID and have a common absolutely continuous distribution F.

Definition 3. *(Samaniego (1985)) A system is said to be IFR (DFR) closed if T has an IFR (DFR) distribution whenever F is an IFR (DFR) distribution. A class of systems is IFR (DFR) closed if each system in the class is IFR (DFR) closed.*

Samaniego (1985) showed that a coherent system with lifetime $T = \phi(X_1,\ldots,X_n)$ and with IID components is IFR closed if and only if

$$k(x) = \frac{\sum_{i=0}^{n-1}(n-i)s_{i+1}\binom{n}{i}x^i}{\sum_{i=0}^{n-1}\left(\sum_{j=i+1}^{n}s_j\right)\binom{n}{i}x^i}, \; x \in (0,\infty)$$

is increasing in x where $\mathbf{s} = (s_1,\ldots,s_n)$ is the system signature.

It is well known that the class of k-out-of-n systems (i.e., systems that work when at least k of their n components work) is IFR closed (see for example Barlow and Proschan (1975)).
Note that all coherent systems with IID components are not necessary IFR closed. For example, in the system with lifetime $T = \max\{X_1,\min(X_2,X_3)\}$, $k(x)$ is not a monotone function.

Remark 4. *It is easy to see that the system is DFR closed if and only if $k(x)$ is decreasing in x. Also note that if a system be IFR and DFR closed both, then $k(x)$ should be a constant function. That is, $s_i = 0$ for $i = 2,\ldots,n$ and therefore $s_1 = 1$. It means that, only the series systems are both IFR and DFR closed. This result is also proved in Navarro et al. (2013). Note that only in a series system $\mathbf{s} = (1,0,\ldots,0)$.*

Remark 5. *Using the signatures of all 28 coherent systems of order $n \leq 4$ and verifying the monotonicity of $k(x)$, we observed that in each case it is not decreasing except for series systems in which obviously we have $k(x) = n$. Based on this observation, we conjectured that the series systems are only DFR closed systems. This unique property of the series systems is proved in the following lemma, for both IID and INID cases. First see an example given in Barlow and Proschan (1975) which is needed in the sequel.*

Example 5. Let $T = \max(X_1,X_2)$ where X_1 and X_2 are independent and X_i is exponentially distributed with parameter λ_i, $i = 1,2$. Then the failure rate function of T is not monotone if $\lambda_1 \neq \lambda_2$, but T has an IFR distribution if $\lambda_1 = \lambda_2$.

Lemma 2. *A coherent system with INID components is a DFR closed system if and only if it be series.*

Proof. It is well-known that if $T = \min(X_1,\ldots,X_n)$ and X_i's are independent with $h_{X_i}(t)$ as the failure rate function of X_i then $h_T(t) = \sum_1^n h_{X_i}(t)$ and therefore a series system is obviously both IFR and DFR closed system. Now to prove the Lemma, we use induction on the values of n. For $n = 2$ and from Example 5, note that T can not be $\max(X_1,X_2)$, as its failure rate function is not monotone and therefore $T = \min(X_1,X_2)$. Now assume that the result of the Lemma holds true for the systems of order $n-1$ that is if $\phi(X_1,\ldots,X_{n-1})$ is DFR when $X_1,\ldots,X_{n-1}$ are DFR then $\phi(X_1,\ldots,X_{n-1}) = \min(X_1,\ldots,X_{n-1})$ or equivalently its structure function is $\phi(x_1,\ldots,x_{n-1}) = \prod_1^{n-1} x_i$.
Now suppose $\phi(X_1,\ldots,X_n)$ is DFR when $X_1,\ldots,X_n$ are DFR. One can consider the system of order n as a system of order 2 in which the subsystem consisting of $x_1,\ldots x_{n-1}$ is its first component and x_n is the second. We note that the main system of order n is DFR closed, therefore both subsystems $\phi(x_1,\ldots,x_{n-1},1)$ and $\phi(x_1,\ldots,x_{n-1},0)$, which are of order $n-1$ should be DFR closed (otherwise one can easily give the examples of some structures $\phi(X_1,\ldots,X_n)$, such that when the substructures $\phi(X_1,\ldots,X_{n-1},1)$ and

$\phi(X_1,\ldots,X_{n-1},0)$ are not DFR closed then the main structure $\phi(X_1,\ldots,X_n)$ is also not DFR closed). Now by pivoting on component n and using induction assumption, we conclude that, both subsystems $\phi(x_1,\ldots,x_{n-1},1)$ and $\phi(x_1,\ldots,x_{n-1},0)$, which are of order $n-1$, have the series structures. On the other hand, in view of the case where $n=2$, these two subsystems and component n are also in series. That is $\phi(x_1,\ldots,x_{n-1},1)=\prod_1^{n-1} x_i \times 1$ and $\phi(x_1,\ldots,x_{n-1},0)=\prod_1^{n-1} x_i \times 0$. Therefore we have

$$\phi(x_1,\ldots,x_n)=x_n\phi(x_1,\ldots,x_{n-1},1)+(1-x_n)\phi(x_1,\ldots,x_{n-1},0)=x_n\times\prod_1^{n-1}x_i+0=\prod_1^{n}x_i.$$

This completes the proof of the lemma. □

Note that Lemma 2 holds true in IID case. That is the series systems with IID components are only DFR closed systems. Therefore Lemma 2 extends a previous result in Lindqvist and Samaniego (2019) for IID case.

Remark 6. *Based on Lemma 2, one can similarly show that the series systems with INID components are only IFR closed systems. Note that unlike the unique property of DFR closedness of the series systems in both IID and INID cases, the class of IFR closed systems with IID components is not contained only by the series systems, as mentioned before it also includes the k-out-of-n systems.*

The result of the Lemma 2 and the mentioned property in Remark 6, extend the Proposition 2.1 in Navarro et al. (2013) from IID to INID case. Proposition 2.5 in Navarro et al. (2013), gives a sufficient condition for a coherent system with INID components to be IFR(DFR) closed. Therefore in view of Remark 6, Lemma 2 presents a more stronger result than the Proposition 2.5 in Navarro et al. (2013). Also for the systems with dependent components, some conditions are obtained in Navarro et al. (2013). They showed that for different coherent systems, some aging classes are preserved.

5 Concluding remarks

This paper presents new properties and applications of system signatures and dynamic signatures, which are essential tools for analyzing the stochastic behavior and aging characteristics of coherent and used systems. Following an extensive review in Section 2, we explored comparisons among coherent systems and examined the relationships between the structure function, reliability function, and system signature. The main contributions are presented in Sections 3 and 4. In Section 3, we investigate comparisons of the residual lifetimes of coherent used systems. By introducing the concept of an equivalent dynamic signature, we derive extended results for systems composed of different exchangeable components. Additionally, we propose an alternative to the equivalent dynamic signature, which exhibits several useful properties for comparing used systems.
Section 4 focuses on unique aging properties of series systems. Specifically, we study the relationship between system signatures and the concepts of IFR and DFR closed systems. It is shown that, under INID and IID assumptions, series systems are only DFR closed, thereby extending existing results in the literature. Illustrative examples are provided throughout the paper to clarify and support the theoretical findings.

Acknowledgement

The authors would like to thanks referees for giving valuable and constructive comments, which led to a significantly improvement in the revised version of the paper.

References

Barlow, R.E. and Proschan F. (1975). *Statistical Theory of Reliability and Life Testing*, Holt, Rinehart and Winston.

Boland, P. (2001). Signatures of indirect majority systems. *Journal of Applied Probability*, **38**, 597–603.

Boland, P. J., and Samaniego, F. J. (2004). The signature of a coherent system and its applications in reliability. In *Mathematical Reliability*: An expository perspective (pp. 3-30). Springer, Boston.

Burkschat, M. and Navarro, J. (2013). Dynamic signatures of coherent systems based on sequential order statistics. *Journal of Applied Probability*, **50**, 272–287.

Burkschat, M. and Samaniego, F. J. (2018). Dynamic IFR concepts for coherent systems. *Statistics & Probability Letters*, **142**, 1–7.

Chahkandi, M., Ruggeri, F., and Suárez-Llorens, A. (2015). A generalized signature of repairable coherent systems. *IEEE Transactions on Reliability*, **65**, 434–445.

Kochar, S., Mukerjee, H., and Samaniego, F. J. (1999). The "signature" of a coherent system and its application to comparisons among systems. *Naval Research Logistics*, **46**, 507–523.

Lindqvist, B. H., and Samaniego, F. J. (2019). Some new results on the preservation of the NBUE and NWUE aging classes under the formation of coherent systems. *Naval Research Logistics*, **66**, 430–438.

Lindqvist, B. H., Samaniego, F. J., and Huseby, A. B. (2016). On the equivalence of systems of different sizes, with applications to system comparisons. *Advances in Applied Probability*, **48**, 332–348.

Navarro, J. (2018). Stochastic comparisons of coherent systems. *Metrika*, **81**, 465–482.

Navarro, J., Balakrishnan, N., and Samaniego, F. J. (2008). Mixture representations of residual lifetimes of used systems. *Journal of applied Probability*, **45**, 1097–1112.

Navarro, J., del Águila, Y., Sordo, M. A., and Suárez-Llorens, A. (2013). Preservation of reliability classes under the formation of coherent systems. *Applied Stochastic Models in Business and Industry*, **30**, 444–454.

Navarro, J., Ruiz, J. M., and Sandoval, C. J. (2005). A note on comparisons among coherent systems with dependent components using signatures. *Statistics and Probability Letters*, **72**, 179–185.

Navarro, J., Samaniego, F. J., and Balakrishnan, N. (2011). Signature-based representations for the reliability of systems with heterogeneous components. *Journal of Applied Probability*, **48**, 856–867.

Naqvi, S., Chan, P. S., and Mishra, D. B. (2022). System signatures: A review and bibliometric analysis. *Communications in Statistics-Theory and Methods*, **51**, 1993–2008.

Navarro, J. and Rubio, R. (2011). A note on necessary and sufficient conditions for ordering properties of coherent systems with exchangeable components. *Naval Research Logistics*, **58**, 478–489.

Mahmoudi, M. and Asadi, M. (2011). The dynamic signature of coherent systems. *IEEE Transactions on Reliability*, **60**, 817–822.

Marichal, J. L., Mathonet, P., and Waldhauser, T. (2011). On signature-based expressions of system reliability. *Journal of Multivariate Analysis*, **102**, 1410–1416.

Samaniego, F. J. (2007). *System Signatures and their Applications in Engineering Reliability*. Springer, Boston.

Rao, T. V., and Naqvi, S. (2023). Comparisons of coherent systems with two types of heterogeneous components having proportional reversed hazard rates. *Applied Stochastic Models in Business and Industry*, **40**, 483–511.

Rychlik, T., Navarro, J. and Rubio, R. (2018). Effective procedure of verifying stochastic ordering of system lifetimes. *Journal of Applied Probability*, **55**, 1261–1271.

Sadegh, M. K. (2006). Dynamic signature of a coherent system with exchangeable components. *In: Proceeding of the 13th Iranian statistical conference*, Kerman.

Samaniego, F. J. (1985) On closure of the IFR class under formation of coherent systems. *IEEE Transactions on Reliability*, **34**, 69–72.

Samaniego, F. J. (2006). On the comparison of engineered systems of different sizes. In *Proceedings of the 12th Annual Army Conference on Applied Statistics*, Aberdeen Proving Ground, Army Research Laboratory.

Samaniego, F. J., Balakrishnan, N. and Navarro, J. (2009). Dynamic signatures and their use in comparing the reliability of new and used systems. *Naval Research Logistics*, **56**, 577–591.

Shaked, M. and Shanthikumar, J. G. (2007). *Stochastic Orders*, Springer, New York.