Stochastic Models in Probability and Statistics
Vol. -, No. -, 2025, pp. -. Research Article

On the generalized last time buy problem: an application of pseudo-deterministic stopping times to end-of-life management

J. B. G. Frenk[†]*, **Mustafa Hekimoğlu**[‡] **and Ayda Amniattalab**[§]

[†] *Industrial Engineering Department, Faculty of Engineering, Gebze Technical University, Kocaeli, Turkey*
[‡] *Department of Industrial Engineering, Kadir Has University, Istanbul, Turkey*
[§] *Department of Engineering and Natural Sciences, Sabancı University, Gebze,Turkey*

Email(s): frenk@gtu.edu.tr, mustafa.hekimoglu@khas.edu.tr, ayda@sabanciuniv.edu

Abstract. The End-of-life phase is the longest phase of the different life cycles of a product. This phase, characterized by a last time buy order, is known for increasing supply risk and decreasing demand rate. In this paper, starting initially with a repair replacement policy the manufacturer optimizes the order size of the last time buy and selects a time switching to another policy substituting failed components.The defective items stochastic arrival process is given by a non-homogenous Poisson process and the switch time is modeled as a stopping time of this demand process. Since the optimal switching time within the class of stopping times can be difficult to implement, we introduce in this paper the set of so-called pseudo-deterministic stopping times being the minimum of a deterministic stopping time and the time to depletion of the spare part inventory. We show that the optimal pseudo-deterministic stopping time satisfies some nice properties under realistic assumptions on the arrival rate function of the Poissonian defective items arrival process and the substitution cost function of the alternative policy. Using these properties an efficient algorithm is proposed to determine the optimal pseudo-deterministic stopping time. Although in general an optimal stopping time (optimal among the class of all stopping times) does not belong to the class of pseudo-deterministic stopping times, we numerically show and explain in the presence of high penalty costs that the objective value of the optimal pseudo-deterministic stopping time is close to the objective value of the optimal stopping time.

Keywords: End-of-life inventory problem; Martingales; Non-homogeneous Poisson process; Spare parts inventory management; Stopping time.

1 Introduction

Due to rapid technology developments the production time of newly introduced products in the market shortened considerable to sometimes only a few months. By these developments it became more difficult

*Corresponding author

Received: / Revised: / Accepted:
DOI:

 https://smps.um.ac.ir

to deal with after sales services like for example warranty period requirements lasting for several years. At the same time, customers are also seeking support for their equipment even after the expiration date of their warranty, and they value manufacturers based on their aftersales service performance (Ahmad and Butt, 2012). Similarly, some customers are willing to pay extra for longer manufacturer care through extended service contracts (Padmanabhan and Rao, 1993). To respond to this demand and preserve their brand perception in the market, Original Equipment Manufacturers (OEMs) pay increasing attention to aftersale services of products which are at the end of their life cycle (Cohen et al., 2006).

Management of aftersales services for end-of-life products is problematic due to both demand and supply-side problems in spare parts supply chains. Demand-side problems in spare parts supply occur due to the uncertain spare parts demand over time as customers use only the product during a limited amount of time in the market and these products are subject to random failure times. At the supply side, OEMs need to deal with increasing risk of losing suppliers' support (Hekimoğlu et al., 2018; Li et al., 2016) for original spare parts mainly due to technological (Solomon et al., 2000) or economic reasons (Li et al., 2016).

OEMs utilize different strategies to deal with supply-side problems of original spare parts of a given product. The two most common strategies are recognized as last time buy and or development of a substitute component to guarantee the supply of spare parts (Shen and Willems, 2014). When a product at some OEM becomes end-of-life and is out of production, spare part manufacturers will eventually request the OEM to place a last order for original spare parts. For this order, which is referred to as the *Last Time Buy* order, OEMs need to consider the total amount of spare part demand until the planned date of End-of-Support (EoS), at which the OEM no longer provides repair services for that product.

The size of the Last Time Buy order is critical for OEMs as ordering too much original spare parts leads to obsolete spare parts, increasing salvage costs and economic losses. On the other hand, if the size of the Last Time Buy order is insufficient, this results in unsatisfied customer demand, possibly leading to the loss of goodwill and customers' brand loyalty (Padmanabhan and Rao, 1993). To safeguard against these problems an OEM can alternatively next to a repair-replacement policy also utilize a substitution-based policy. This means the company uses as a policy original spare parts in a repair-replacement policy if a defective product, which cannot be repaired, appears until a particular point in time before the End-of-Support date. After that time the OEM switches to an alternative substitution policy not depending on original spare parts. The combination of these two spare parts policies is recognized as *bridge buy* in the literature (Shen and Willems, 2014). We refer to this combined strategy as *Generalized Last Time Buy* since a switching time being equal to the End-of Support date leads to the classic Last Time Buy problem (Teunter and Fortuin, 1999). In the Generalized Last Time Buy problem, substitute products can be obtained by either taking over suppliers' production lines (Shen and Willems, 2014), using 3D printing (Westerweel et al., 2018) or finding an alternative supplier (Shi and Liu , 2020). Therefore, the OEM needs to make two critical decisions for the optimal solution of the Generalized Last Time Buy problem: At time 0, the OEM decides on the size of an Last Time Buy order and then chooses a switch-to-substitute time. The motivation of this study is to investigate how such a more general switching policy affects both the risk of obsolete spare parts and the supply-side risks of obtaining spare parts and how such a general switching policy increases the service level to customers. For a practical example in airline industry related to these risks the reader is referred to (Hekimoğlu et al. (2018)). Since the optimal switching policy is difficult to compute (Frenk et al. (2019b)) we are also motivated in this study in identifying a simple class of switching polices called the class of pseudo-deterministic policies which under certain conditions achieve expected objective costs close to the expected costs of the optimal switching policy among all stopping rules.

1.1 Problem setting

In this paper, we consider a problem setting where customers bring their defective products for repair to the OEM, which accepts them to its repair facility until the time of the End-of Support announcement.

It is assumed in all of the early and later literature (see for example (Behfard et al., 2015; Fortuin, 1980, 1981; Krikke and van Der Laan, 2011; Teunter and Fortuin, 1999)) that in a continuous time setting a non-homogeneous Poissonian defective items arrival process is a good representation of the demand for spare parts. If the arrival of a defective item is before the switch-to-substitute point, then the product's defective component is removed and sent to inspection for repair feasibility. If a repair is possible, the repaired component is installed back to the product and if the component cannot be repaired and there is available stock, the defective component is replaced with an original part to complete the repair service. If there is no original replacement in stock, the repair process uses a substitute part possible from an external source. This business process is depicted in Figure 1.

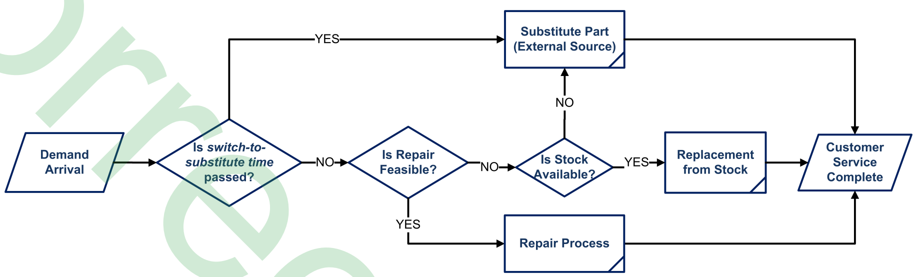

Figure 1: Repair process at the arrival of a defective product

The Generalized Last Time Buy strategy consists of the sequential utilization of a *repair-replacement* and a *repair-with-substitute* policy. This particular structure of the Generalized Last Time Buy strategy, which checks the switch-to-substitute point before checking the repairability of a part, allows OEMs to exploit decreasing substitution cost in time. We may justify this assumption since after the switching time to an alternative policy the repair center is closed and any defective product will be handled according to the new policy. The OEM now needs to make an optimal plan including the optimal size of a Last Time Buy order (at time 0) and the optimal switch-to-substitute time to keep their after sales services running. Note that our *repair-with-substitute* policy is dubbed *alternative* policy by Frenk et al. (2019b). Similar to Frenk et al. (2019a),Frenk et al. (2019b) we follow the same cost setup and our switch-to-substitute decision is formulated as an optimal stopping time problem. As such this paper can be seen as a review, extension and generalization of the results discussed in Frenk et al. (2019a) and Frenk et al. (2019b). To obtain the optimum solution of the Generalized Last Time Buy problem, we first derive the objective function for any switching time represented by an arbitrary stopping time following a simplified approach as done in Frenk et al. (2019a). Next, by substitution, we evaluate the objective function for the new defined subclass of switching times represented by the so-called *pseudo-deterministic stopping times*. These stopping times representing the switching time between different policies are defined as the minimum of a deterministic stopping time as discussed in detail in Frenk et al. (2019a) and the random time of inventory depletion of the Last Time Buy order received at time zero. According to the authors knowledge this is the first time the subclass of pseudo-deterministic switching times is discussed in the Generalized Last Time Buy problem literature and its mathematical properties are derived for both a piece-wise constant substitution cost function and a non-homogeneous Poisson defective items arrival process also having a piecewise constant arrival rate function. The practical and theoretical reason for studying this particular subclass of switching times between policies will be explained in the following paragraphs. First of all, we observe it is more difficult to compute the optimal pseudo-deterministic policy for a non-homogeneous Poisson defective items arrival process having an arbitrary bounded arrival rate function than for an arrival process having a piece-wise constant arrival rate function. In the later case the computation of this optimal pseudo-deterministic policy is very easy. Since any non-homogeneous Poisson process with a bounded arrival rate function can be approximated arbitrarily closely by a non-homogeneous Poisson process with a piece-wise constant arrival rate function it follows that the restriction to this class of arrival processes of defective items is not too restrictive. Our theoretical results and new algorithm for identifying the optimal pseudo-deterministic policy for this class of arrival processes having a piece-wise constant arrival rate function now complement and extend the results discussed in Frenk et al. (2019a), Frenk et al. (2019b) and Javadi (2018).

The motivation for studying this particular class of policies is that our numerical results suggest that the cost of using a switching time represented by the optimal *pseudo-deterministic stopping time* is almost the same as the cost of the optimal switching time represented by the optimal stopping time within the class of arbitrary stopping times. This indicates that the optimal *pseudo-deterministic stopping time* is a good approximation of the optimal stopping time within the class of arbitrary stopping times. The main reason for this empirical observation is the absence of penalty cost within the class of pseudo-deterministic policies. These penalty costs occur when both a defective non-repairable item arrives and at that time no inventory of original spare parts is available and we still apply the repair-replacement policy. If the penalty costs are high at the occurrence of these events any optimal policy tries to avoid these penalty costs by switching in time from the repair-replacement policy to an alternative policy and so such a policy acts like a pseudo-deterministic policy. At the same time, an optimal pseudo-deterministic stopping time is much easier to compute and due to its simplicity easier to implement in practice.

Our problem setting can be motivated by the end-of-life management of electronic products, whose manufacturers aim to provide post-warranty repair service to keep their customers loyal. Due to the rapid pace of technological change in the semi-conductor industry, component manufacturers frequently update their product assortment and shift their capacity to the components of newer models with higher profit potential. Component suppliers usually call for Last Time Buy order, forcing OEMs to decide on the order size of the last order and other supply alternatives.

This paper consists of 5 sections. In the remainder of this section, we briefly review the relevant literature and articulate the contribution of our study. Section 2 describes the mathematical model also discussed in Frenk et al. (2019a) and the different classes of policies and yields simplified proofs of already known results for this model. In Section 3, we derive under some reasonable assumptions properties of the class of pseudo-deterministic stopping times and at the same time give an easy procedure to compute the optimal policy within this class. These results seems to be new in the literature. Section 4 gives by means of easier proofs then the ones used in Frenk et al. (2019a) how to approximate the optimal policy within the class of all bounded stopping times for the generalized last buy decision problem or end of life decision problem. Finally Section 5 includes numerical results, while Section 6 concludes the paper.

1.2 Literature review.

The relevant literature for our study mainly consists of the studies from end-of-life management of durable products.

The last stage of a capital product's life cycle starts with the end of the manufacturing of a product by OEMs. Some time later, component suppliers start announcing their end-of-support dates and ask OEMs to place their last orders. In the literature such an order is called a Last Time Buy order (Bradley and Guerrero, 2008, 2009; Fortuin, 1980, 1981; Frenk et al., 2019a; Teunter and Fortuin, 1999; Teunter and Haneveld, 1998). For determining the size of the Last Time Buy order, OEMs need to make an estimation on the total spare parts demand from their products in use. The first contributions in the literature on determining the size of a Last Time Buy order focused in a discrete time setting solely on using only a repair-replacement policy during the remaining economic life time of the product(Fortuin, 1980; Teunter and Fortuin, 1999). In a recent study, Hur et al. (2018) considers the size of a Last Time Buy order decision in a continuous time setting.

However, in many business cases, companies are interested in complementing decisions about the size of the Last Time Buy order with other policies to replace defective items such as removal of repairable parts from phased out products (Behfard et al., 2015; Frenk et al., 2019a; Van Kooten and Tan, 2009; Krikke and van Der Laan, 2011; Pourakbar et al., 2014), finding an alternative supplier of a substitute

(Frenk et al., 2019b; Pourakbar et al., 2012; Sahyouni et al., 2010) or redesigning the entire production line of components (Shen and Willems, 2014; Shi and Liu , 2020). In Krikke and van Der Laan (2011) a case study and a simple heuristic solution is provided for determining the size of a Last Time Buy order in combination with the availability of repairable spare parts that are removed from phased out products. They address the impact of phaseouts on spare parts demand and the availability of alternative supply sources. The same problem setting is utilized by Pourakbar et al. (2014) for the joint optimization of returned repairable parts from phaseouts and the Last Time Buy Order size. Shi (2019) extended this problem setting with actively collected end-of-life products. They explicitly model the dynamics of a installed base to jointly control the Last Time Buy order size, repairable parts and the size of the installed base. Design refresh options are another important tactic to complement Last Time Buy order decisions. Shi and Liu (2020) considers a problem setting where the Last Time Buy Order size, design refresh and end-of-support decision are jointly optimized in a discrete time setting with respect to the profit maximization criterion. In addition, Inderfurth and Kleber (2013), Inderfurth and Mukherjee (2008) and Bayındır et al. (2007) consider re-manufacturing options to satisfy incoming spare parts demand.

Another relevant research stream for our study is on spare parts management during the warranty period. (van der Heijden and Iskandar, 2013; Huang et al., 2007, 2008; Kim and Park, 2008; Sahyouni et al., 2010). van der Heijden and Iskandar (2013) consider joint optimization of the size of the Last Time Buy order and repair decisions for products with active warranty. They develop cost and service level formulations using a discrete time setting. Huang et al. (2007) consider a discrete-time problem setting where a manufacturer receives demand for new products and warranty claims of old ones. They show the optimality of a state-dependent base stock policy using Markov decision processes. This problem setting is extended with age-dependent warranty claims in the installed base by Huang et al. (2008). Kim and Park (2008) employ continuous time optimal control theory to optimize pricing and production of durable products by considering their spare parts costs during warranty periods. They explicitly model the positive impact of a warranty period on the products' demand. Sahyouni et al. (2010) study the joint optimization of repair and Last Time Buy order quantity for a deterministic repair demand from products under warranty in a continuous time setting. In their model, which is more suited to products having a short life cycle, they focus on the joint optimization of the Last Time Buy order size and end-of-repair point after which failed products are replaced by substitute products.

1.3 Our contribution to the literature.

Our paper contributes to the existing literature of the Generalized Last Time Buy decision problem by analyzing in detail a subclass of switching times from a repair-replacement policy to an alternative policy in the optimal stopping formulation of the Generalized Last Time Buy decision problem as discussed in Frenk et al. (2019a). It is shown that this subclass of switching times satisfies desirable properties under very general assumptions on the non-homogeneous Poisson arrival process of defective items and the substitution cost function of an alternative policy. The optimal policy within this class of these so-called pseudo-deterministic policies can be easily computed and can be used as a heuristic solution replacing the optimal, more complicated, optimal policy within the class of all arbitrary bounded stopping times. It is verified and also explained in the computational section that under certain conditions this can be done without a significant loss in the objective value. At the same time, the optimal policy within the class of pseudo-deterministic policies has a clear interpretation contrary to the optimal policy within the class of all bounded stopping times.

The closest studies to our paper are Frenk et al. (2019a); Javadi (2018); Pourakbar et al. (2012); Shi and Liu (2020). In Frenk et al. (2019a), which extends Pourakbar et al. (2012), the optimal solution of this problem is studied and solved by a dynamic programming algorithm. In Shi and Liu (2020) a similar problem is addressed in a discrete time setting. Under deterministic demand assumptions the same problem is solved in Sahyouni et al. (2010) using mathematical programming techniques. To the best of our knowledge, this paper is the first one addressing the use of pseudo-deterministic policies in the joint optimization of the size of the Last Time Buy order and switching time to an alternative policy in a continuous time setting for a non-homogeneous Poisson defective items arrival process.

2 On the generalized last time buy problem.

In this section we introduce in subsection 2.1 the random arrival process of defective items. In subsection 2.2 we define the costs parameters of the generalized last time buy problem, while in section 2.3 we introduce the objective function for this problem. In subsection 2.4 we formulate the associated minimization problem and derive in subsection 2.5 a lowerbound on the optimal objective value. This is needed to derive an upperbound on the optimal Last Time Buy order size as expressed in Lemma 3 for nonnegative scrapping values and in Lemma 5 for arbitrary scrapping values. Finally in subsection 2.6 we derive some qualitative results for the new class of pseudo-deterministic stopping rules. As such these six subsections reviews, simplifies and extend and complements the model and results in Frenk et al. (2019a) and Frenk et al. (2019b).

2.1 The arrival process of the defective items process.

To formulate our end-of-life inventory or generalized last time buy problem, we assume that defective products arrive according to a non-homogeneous Poisson point process for a repair or replacement. To introduce this arrival process let $(\Omega, \mathscr{H}, \mathbb{P})$ be a probability space hosting the point process $(T_i, R_i)_{i\in\mathbb{N}}$. The random variable T_i, $i \in \mathbb{N}$ denotes the arrival time of the ith customer having a defective product and requesting repair. The counting process of defective products $N \equiv \{N(t) : t \geq 0\}$ is defined by

$$N(t) := \textstyle\sum_{i=1}^{\infty} 1_{\{T_i \leq t\}}, t \geq 0, \tag{1}$$

and assumed to be a non-homogeneous Poisson process with a bounded Borel arrival rate function $\lambda(.)$. The random variables R_i, $i \in \mathbb{N}$, on the other hand, are independent and identically distributed Bernoulli random variables indicating the condition of the defective items. They are defined as

$$R_i = \begin{cases} 1 & \text{if the product can be repaired} \\ 0 & \text{otherwise} \end{cases}$$

with probability $0 \leq q = \mathbb{P}(R_i = 1) \leq 1$. The thinned arrival processes $N_0 := \{N_0(t) : t \geq 0\}$ and $N_1 = \{N_1(t) : t \geq 0\}$ given by

$$N_0(t) := \textstyle\sum_{i=1}^{\infty} (1 - R_i) 1_{\{T_i \leq t\}}, N_1(t) := \textstyle\sum_{i=1}^{\infty} R_i 1_{\{T_i \leq t\}}, \tag{2}$$

count the number of non-repairable and repairable products arriving over time. It is well known that the arrival processes N_0 and N_1 are independent non-homogeneous Poisson processes with arrival rate functions $\lambda_0(.) = (1-q)\lambda(.)$, $\lambda_1(.) = q\lambda(.)$ respectively (cf. Çınlar (2011)) having mean arrival functions $\Lambda_0(.) = (1-q)\Lambda(.)$ and $\Lambda_1(.) = q\Lambda(.)$ with $\Lambda(t) = \int_0^t \lambda(u)du$ for $0 \leq t \leq T$. In the sequel, we let $\mathbb{F} \equiv (\mathscr{F}_t)_{t\geq 0} \subseteq \mathscr{H}$ denote the filtration of the point process $(T_i, R_i)_{i\in\mathbb{N}}$; that is, the flow of information associated with both the arrival times of the products and their condition.

Let T denote the end of service time at which the OEMs' service obligations with respect to the product expires. For the optimal application of the Generalised Last Time Buy policy, the manufacturer makes two decisions: the size of the Last Time Buy order at time 0, $x \in \mathbb{Z}_+ = \{0, 1, 2, \ldots\}$, and the optimal stopping time $\tau \in [0, T]$ of switching from a classical repair-replacement policy to an alternative policy. This is a (possibly random) stopping time belonging to the set of all bounded stopping times with respect to the filtration $\mathbb{F}$. The subset of deterministic stopping times is denoted by $\mathbb{F}_0$. Recall that for the deterministic stopping time $\tau \stackrel{a.s}{=} T$ the Generalised Last Time Buy optimization problem reduces to the Last Time Buy optimization problem in which we only need to determine the size of the Last Time Buy order.

For the decision making problem, we consider the finite horizon, continuous compounding discounted cost criteria with discount rate $\delta \geq 0$. Figure 1 depicts the control process under the Generalized Last Time Buy policy. The total cost of these two decisions for a given (x,τ)-policy consists of cost components related with acquisition, inventory holding, repair and replacement. These cost parameters are explained in the next subsection and are the same as discussed in Frenk et al. (2019a).

2.2 The cost parameters of the generalized last time buy problem.

Acquisition of spare parts take place at time 0 and parts are delivered immediately. The acquisition cost function is denoted by $c:\mathbb{Z}_+ \mapsto \mathbb{R}_+$, where $c(0)=0$. Also the holding cost rate for the delivered parts is $h>0$ per spare part per unit of time.

The used policy up to the end of service time T is now summarized by the following procedure. Starting with x units representing the size of the Last Time Buy order, the OEM utilizes the *repair-replacement* policy until some random stopping time $\tau \leq T$. Under this repair-replacement policy, if a repair is feasible for an arriving defective component, it is always repaired at some repair cost c_{re} plus some service cost c_{se}. If the component is beyond repair and the inventory level of spare parts is non-zero, the defective component in the item is replaced with a spare part from the inventory at service cost c_{se}. If no spare part is available in inventory, the beyond repairable component is replaced with a substitution spare part supplied from an external source like a gray market or a 3D printer (Figure 1). The cost of this substitution is given by the right continuous function $c_a:[0,T]\to\mathbb{R}^+$. Utilizing the substitution policy during the time one should use the repair-replacement policy also leads to an additional penalty cost function $p:[0,\tau]\to\mathbb{R}^+$. This penalty cost function can be motivated by customers' loss-of-goodwill or transportation of the substitute part from another location. Therefore, the total penalty cost of being forced to use the alternative substitution policy at time $0\leq t\leq\tau$ during the time one should use the repair-replacement policy is given by

$$c_{ap}(t) := c_a(t)+p(t). \tag{3}$$

We assume that $c_{ap}(t)\geq c_{se}$ for $0\leq t\leq\tau$ to avoid trivial domination of substitution parts over spare parts available in inventory. It is assumed that the functions c_a and p are both non-increasing and right continuous. Namely, the substitute component gets cheaper over time thanks to technological progress in production processes and the penalty of using a substitute component under the repair-replacement policy is a decreasing function of time.

After time $\tau\leq t\leq T$ until the end of service time T, the OEM first discards at time τ the existing inventory at a scrapping cost of c_{scr} per item and abandons after this time for any defective item arriving at time t the *repair-replacement* policy and replaces it by the substitution policy at the substitution cost $c_a(t)$. All costs of the repair services are indicated next to the associated processes in Figure 1. In our formulation, all cost terms are positive except the scrapping value c_{scr}, which is allowed to be negative. This means there can be a net revenue associated with these scrapped parts. To avoid pathological cases where ordering is profitable because of scrapping, we assume in this study that the function $x\mapsto c(x)-c_{scr}^{-}x$ is increasing with $c_{scr}^{-}:=-\min\{c_{scr},0\}$ and

$$\lim_{x\uparrow\infty} c(x)-c_{scr}^{-}x=\infty. \tag{4}$$

This implies for the special case that the scrapping costs are positive, then the acquisition cost function c is increasing and satisfies $\lim_{\uparrow\infty} c(x)=\infty$. Also for scrapping costs negative this implies that the function $x\to c(x)+c_{scr}x$ is increasing. For the scrapping action to be economically justifiable, we must impose the condition that $h-\delta c_{scr}\geq 0$. If this condition fails to hold, instead of scrapping an item at some time τ, we can keep it indefinitely in inventory at a total cost of $h\int_\tau^\infty e^{-\delta u}du=(h/\delta)e^{-\delta\tau}$ which would be less than $c_{scr}e^{-\delta\tau}$. These three conditions on the cost functions and the parameters listed in the previous relations always hold in this study unless stated otherwise.

2.3 The objective function of the generalized last time buy problem.

In the decision making problem under consideration, a Last Time Buy order of size x and switching time $\tau \leq T$ are determined, which we refer to as a (x,τ)-policy. The first variable x is static, and its value is determined at time zero. The switching decision, on the other hand, can be dynamic, and in the general formulation of the problem τ is a stopping time of the filtration $\mathbb{F}$.

In this section, we derive the expected discounted cost $C(x,\tau)$ of any (x,τ)-policy, $x \in \mathbb{Z}_+$, $\tau \in \mathbb{F}$ and introduce the optimization problem to be solved. To this end, we introduce the random variable $\sigma_x := \inf\{t > 0 : N_0(t) \geq x\}$, denoting the (random) time of inventory depletion for an order size of x. Observe the total expected discounted cost is the sum of the procurement and expected discounted operation costs. The procurement cost is given by $c(x)$. To derive the expected discounted operation costs we utilize the following results about martingales. For any locally bounded Borel measurable function f and any non-homogeneous Poisson process N with locally bounded Borel arrival rate function μ defined on the probability space $(\Omega, \mathscr{H}, \mathbb{P})$ it follows that the stochastic process $X = \{X(t) : t \geq 0\}$, given by $X(t) := \int_0^t f(s)dN(s) - \int_0^t f(s)\mu(s)ds$ for $t \geq 0$, is a right-continuous martingale (cf. Çınlar (2011)). For these so-called Poissonian martingales and any bounded stopping time τ it holds that $\mathbb{E}(X(\tau)) = 0$. This means

$$\mathbb{E}\left(\int_0^\tau f(t)dN(t)\right) = \mathbb{E}\left(\int_0^\tau f(t)\mu(t)dt\right). \tag{5}$$

This is known as Doob's stopping theorem (cf.Çınlar (2011)) and we will make frequently use of this result in the computation of the expected discounted operation costs. The expected discounted operation costs consist of the following components for any (x,τ)-policy with $x \in \mathbb{Z}_+$ and $\tau \in \mathbb{F}$:

- **Inventory holding costs:** We switch to the repair-with-substitute policy at time $\tau \leq T$ and scrap at that time (possibly) leftover inventory of spare parts. Hence, the random discounted inventory holding costs are given by $h\int_0^\tau e^{-\delta u}(x - N_0(u))^+ du$ with $(z)^+ := \max(z, 0)$. This shows that the expected discounted holding costs $C_{inv}(x,\tau)$ of any (x,τ)-policy equal

$$C_{inv}(x,\tau) = h\mathbb{E}\left(\int_0^\tau e^{-\delta u}(x - N_0(u))^+ du\right). \tag{6}$$

- **Service costs:** Service costs arise during the cost of running repair operations for both repairable and non-repairable products. For a repairable product service costs occur during the repair-replacement policy from time 0 to time τ in any (x,τ)-policy. For a non-repairable product, service costs occur in case of positive spare parts inventory at the arrival time of a defective product. Hence, for non-repairable products, service costs only need to be paid from time 0 up to time $\tau \wedge \sigma_x := \min\{\tau, \sigma_x\}$. This shows that the random discounted service costs are given by

$$c_{se}\int_0^\tau e^{-\delta u} dN_1(u) + c_{se}\int_0^{\tau\wedge\sigma_x} e^{-\delta u} dN_0(u).$$

  Applying now relation (5) for properly chosen functions f and using both point processes N_0 and N_1 it follows for the bounded stopping time $\tau \leq T$ that the expected discounted service costs $C_{se}(x,\tau)$ of any (x,τ)-policy equal

$$\begin{aligned} C_{se}(x,\tau) &= c_{se}\mathbb{E}\left(\int_0^\tau e^{-\delta u}\lambda_1(u)du\right) + c_{se}\mathbb{E}\left(\int_0^{\tau\wedge\sigma_x} e^{-\delta u}\lambda_0(u)du\right) \\ &= qc_{se}\mathbb{E}\left(\int_0^\tau e^{-\delta u}\lambda(u)du\right) + (1-q)c_{se}\mathbb{E}\left(\int_0^{\tau\wedge\sigma_x} e^{-\delta u}\lambda(u)du\right). \end{aligned} \tag{7}$$

- **Repair costs:** We incur repair costs during the repair-replacement phase from time 0 to τ. Hence the random discounted repair costs are given by $c_{re}\int_0^\tau e^{-\delta u} dN_1(u)$. Applying the same arguments as for service costs the expected discounted repair costs $C_{re}(x,\tau)$ of any (x,τ)-policy equal to

$$C_{re}(x,\tau) = c_{re}\mathbb{E}\left(\int_0^\tau e^{-\delta u}\lambda_1(u)du\right) = qc_{re}\mathbb{E}\left(\int_0^\tau e^{-\delta u}\lambda(u)du\right). \tag{8}$$

- **Substitution costs:** The random discounted costs of applying the repair-with-substitute policy consist of the substitution cost before and after time τ. The random discounted cost of using the alternative policy is given by

$$\int_\tau^T e^{-\delta u}c_a(u)dN(u) + \int_{\tau\wedge\sigma_x}^\tau e^{-\delta u}c_{ap}(u)dN_0(u).$$

Applying the same arguments as before and using relation (3) the expected discounted alternative policy costs $C_a(x,\tau)$ of any (x,τ)-policy are as follows:

$$\begin{aligned} C_a(x,\tau) &= \mathbb{E}\left(\int_\tau^T e^{-\delta u}c_a(u)\lambda(u)du\right) + (1-q)\mathbb{E}\left(\int_{\tau\wedge\sigma_x}^\tau e^{-\delta u}c_{ap}(u)\lambda(u)du\right) \\ &= \begin{cases} \int_0^T e^{-\delta u}c_a(u)\lambda(u)du + \mathbb{E}(\int_0^\tau e^{-\delta u}[(1-q)c_{ap}(u)-c_a(u)]\lambda(u)du \\ -(1-q)\mathbb{E}\left(\int_0^{\tau\wedge\sigma_x} e^{-\delta u}c_{ap}(u)\lambda(u)du\right) \end{cases} \\ &= \begin{cases} \int_0^T e^{-\delta u}c_a(u)\lambda(u)du + \mathbb{E}(\int_0^\tau e^{-\delta u}[(1-q)p(u)-qc_a(u)]\lambda(u)du \\ -(1-q)\mathbb{E}\left(\int_0^{\tau\wedge\sigma_x} e^{-\delta u}c_{ap}(u)\lambda(u)du\right). \end{cases} \end{aligned} \tag{9}$$

- **Scrapping costs:** The random discounted scrapping costs at time τ are given by $c_{scr}e^{-\delta\tau}(x-N_0(\tau))^+$. This shows that the expected discounted scrapping costs $C_{scr}(x,\tau)$ of any (x,τ) policy equal

$$C_{scr}(x,\tau) = c_{scr}\mathbb{E}(e^{-\delta\tau}(x-N_0(\tau))^+). \tag{10}$$

Adding up the separate operational costs derived in relations (6)-(10), the expected discounted operation cost $C(x,\tau)$ of any (x,τ)-policy is given by

$$C(x,\tau) = \begin{cases} h\mathbb{E}(\int_0^\tau e^{-\delta u}(x-N_0(u))^+du) + c_{scr}\mathbb{E}(e^{-\delta\tau}(x-N_0(\tau))^+) \\ +\mathbb{E}\left(\int_0^\tau e^{-\delta u}[q(c_{re}+c_{se}-c_a(u)) + (1-q)p(u)]\lambda(u)du\right) \\ +(1-q)\mathbb{E}\left(\int_0^{\tau\wedge\sigma_x} e^{-\delta u}[c_{se}-c_{ap}(u)]\lambda(u)du\right) + \int_0^T e^{-\delta u}c_a(u)\lambda(u)du. \end{cases} \tag{11}$$

To rewrite the expression in (11) in a more suitable form, we remind that for any first order stochastic processes $Y = \{Y(t), t \geq 0\}$ and $Z = \{Z(t) : t \geq 0\}$ it holds that

$$Y(t)Z(t) = Y(0)Z(0) + \int_0^t Y(u)dZ(u) + \int_0^t Z(u)dY(u), t > 0$$

This shows for any $0 \leq q \leq 1$ using $N_0(0) = 0$

$$\begin{aligned} e^{-\delta\tau}(x-N_0(\tau))^+ &= e^{-\delta(\tau\wedge\sigma_x)}(x-N_0(\tau\wedge\sigma_x)) \\ &= x - \delta\int_0^{\tau\wedge\sigma_x} e^{-\delta u}(x-N_0(u))du - \int_0^{\tau\wedge\sigma_x} e^{-\delta u}dN_0(u) \end{aligned}$$

and so by Doob's stopping theorem

$$\mathbb{E}(e^{-\delta\tau}(x-N_0(\tau))^+) = x - \delta\mathbb{E}\left(\int_0^{\tau} e^{-\delta u}(x-N_0(u))^+ du\right) - (1-q)\mathbb{E}\left(\int_0^{\tau\wedge\sigma_x} e^{-\delta u}\lambda(u)du\right). \tag{12}$$

Replacing the expectation for the scrapping value in (11) with the expression in (12) and rearranging some terms, we obtain the following more suitable representation of the expected discounted operation costs.

$$C(x,\tau) = \begin{cases} c_{scr}x + (1-q)\mathbb{E}\left(\int_0^{\tau\wedge\sigma_x} e^{-\delta u}\lambda(u)[c_{se}-c_{scr}-c_{ap}(u)]du\right) \\ +\mathbb{E}\left(\int_0^{\tau} e^{-\delta u}\lambda(u)[q(c_{se}+c_{re}-c_a(u))+(1-q)p(u)]du\right) \\ +(h-\delta c_{scr})\mathbb{E}\left(\int_0^{\tau} e^{-\delta u}(x-N_0(u))^+)du\right) + \int_0^T e^{-\delta u}\lambda(u)c_a(u)du. \end{cases} \tag{13}$$

In case we consider the subclass of policies $(x,\tau\wedge\sigma_x), \tau\in\mathbb{F}$ this yields by relation (13)

$$C(x,\tau\wedge\sigma_x) = \begin{cases} c_{scr}x + \mathbb{E}\left(\int_0^{\tau\wedge\sigma_x} e^{-\delta u}\lambda(u)[c_{se}+qc_{re}-(1-q)c_{scr}-c_a(u)]du\right) \\ +(h-\delta c_{scr})\mathbb{E}\left(\int_0^{\tau} e^{-\delta u}(x-N_0(u))^+ du\right) + \int_0^T e^{-\delta u}\lambda(u)c_a(u)du \end{cases} \tag{14}$$

and so

$$C(x,\tau) - C(x,\tau\wedge\sigma_x) = \mathbb{E}\left(\int_{\tau\wedge\sigma_x}^{\tau} e^{-\delta u}[q(c_{se}+c_{re}-c_a(u))+(1-q)p(u)]du\right). \tag{15}$$

In the following lemma, we use relation (15) to show that under certain conditions on the penalty and the substitution cost functions it is always optimal within the class of all stopping times to switch to the substitution policy before or at the moment the inventory level of spare parts is depleted. In Section 2.6 we will analyze this special class of policies with τ a constant.

Lemma 1. *If*

$$\inf_{0\leq t\leq T}\{q(c_{se}+c_{re}-c_a(t))+(1-q)p(t)\}\geq 0, \tag{16}$$

then the class of $(x,\tau\wedge\sigma_x)$-policies with $\tau\in\mathbb{F}$ contains an optimal policy.

Proof. Apply relation (15). □

If no item is repairable ($q=0$!), the (sufficient) condition of Lemma 1 is satisfied. In this case one can find an optimal policy belonging to this special class of policies. In many service systems penalty costs are very high. Hence the condition in Lemma 1 is mostly satisfied in practice especially when q is close to 0. Since the subclass of $(x,\tau\wedge\sigma_x)$-policies avoid the (possibly) high penalty costs occurring in practice, it is therefore worthwhile to study them in detail. We will focus on this set of policies with $\tau\in\mathbb{F}_0$ in subsection 2.6 and section 3. However, it might not always happen that a $(x,\tau\wedge\sigma_x)$-policy is optimal within the class of all bounded stopping time policies. In case of high substitution cost $c_a(.)$ being larger then $c_{se}+c_{re}$ and low penalty costs $p(.)$, and q close to one it might be cheaper (see proof Lemma 1!) to continue with the repair-replacement policy after the inventory level hits zero.

2.4 On the formulation of the generalized last time buy problem.

Applying relation (13) the objective function of using a given (x,τ) policy consists of the summation of the procurement cost $c(x)$ and the operating cost $C(x,\tau)$. The corresponding last time buy optimization problem is then given by

$$\upsilon(P) = \inf_{x\in\mathbb{Z}_+,\tau\in\mathbb{F},0\leq\tau\leq T}\{c(x)+C(x,\tau)\}. \tag{P}$$

Hence we need to determine the parameters of a (x,τ)-policy, if it exists, attaining the infimum in the above optimization problem.

There is one instance of optimization problem (P) which can be solved easily. In case all the defective items are repairable, i.e. $q = 1$, the stochastic counting process N_0 of non-repairable defective components listed in relation (2) becomes the zero process. Since for $q = 1$ all items can be repaired without using spare parts, it is easy to check applying relation (13) it is optimal not to order any spare parts. Also, since the function c_a is decreasing and every time we use for a (repairable) defective item the repair decision against cost $c_{se} + c_{re}$ it is optimal to switch to the substitution policy at the earliest time $t \leq T$ satisfying $c_a(t) \leq c_{se} + c_{re}$. This result is formally stated in the next lemma.

Lemma 2. *If all defective items can be repaired and the three conditions on the parameters introduced in subsection 2.2 hold, then the optimal policy is not to order any spare parts at time* 0 *and the optimal switching time to the repair-with-substitute policy is given by* $\tau = \zeta_1$ *with*

$$\zeta_q = \inf\{0 \leq u \leq T : c_{se} + qc_{re} - c_a(u) \geq 0\}, 0 \leq q \leq 1, \tag{17}$$

with the convention $\inf\{\varnothing\} = T$ *and* $\varnothing$ *denoting the empty set. Also the optimal objective value* $\upsilon(P)$ *equals*

$$\upsilon(P) = \int_0^T e^{-\delta u} \lambda(u) \min\{c_{se} + c_{re}, c_a(u)\} du.$$

Proof. See Appendix. □

Since in Lemma 2 we know for the optimal policy that the time to switch to the alternative policy is already known at time zero this optimal policy is a so-called *static* policy. Hence by Lemma 2 we only need to consider in the remainder of this paper the optimization problem (P) satisfying $0 \leq q < 1$. To solve optimization problem (P) we first compactify the decision space by deriving an upper bound on the optimal order quantity. An easy upper bound valid for $c_{scr} \geq 0$ is given by the following result. Since the function $x \to \inf_{\tau \in \mathbb{F}, 0 \leq \tau \leq T}\{c(x) + C(x, \tau)\}$ does not satisfy in general discrete convexity type properties this restriction to a finite number of possible order sizes is very useful in the computational section. Observe, for the general case of both either positive or negative values of c_{scr} one can show under an additional condition on the function c_{ap} an improved upper bound and this is shown in Lemma 6.

Lemma 3. *If* $c_{scr} \geq 0$ *and the three conditions on the parameters introduced in subsection 2.2 hold and* $x_U := \min\left\{x \in \mathbb{Z}_+ : c(x) > \int_0^T e^{-\delta u} \lambda(u) c_a(u) du\right\}$ *then an optimal order quantity* x_* *of the optimization problem (P) exists and it satisfies* $x_* \leq x_U$.

Proof. See Appendix. □

Lemma 3 implies that solving optimization problem (P) is equivalent to solving

$$\upsilon(P) = \min_{x \leq x_U, x \in \mathbb{Z}_+} \inf_{\tau \in \mathbb{F}, 0 \leq \tau \leq T}\{c(x) + C(x, \tau)\}, \tag{18}$$

and this problem can be approximated by a *computable optimal stopping problem*. In Frenk et al. (2019a) an approximate solution to this problem is provided by replacing the set of all stopping times with the set of stopping times attaining only values from a finite subset of [0,T]. To bound the approximation error replacing the set of stopping times by the smaller set of stopping times only attaining values at a finite subset of $[0, T]$ we need a lower bound on $\upsilon(P)$. In the following subsection we provide under some additional assumption on the function $c_{ap}(.)$ an improved upperbound on the optimal order quantity x valid for any scrapping value c_{scr}. This improves the representation in (18).

2.5 An upper bound on the optimal order quantity for arbitrary scrapping values.

In this subsection we derive in Lemma 6 an upper bound on the optimal order quantity of the generalized last buy decision minimization problem for arbitrary scrapping values c_{scr} under general conditions on the function $c_{ap}(.)$. To do so we first derive a lower bound on $\upsilon(P)$ in Lemma 5. To start our analysis introduce the function $L:\mathbb{Z}_{+}\times\mathbb{F}\to\mathbb{R}$ given by

$$L(x,\tau):=\begin{cases} c(x)+h\mathbb{E}\left(\int_0^\tau e^{-\delta u}(x-N_0(u))^{+}du\right)+c_{scr}\mathbb{E}(e^{-\delta\tau}(x-N_0(\tau))^{+}) \\ +\mathbb{E}\left(\int_0^\tau e^{-\delta u}[c_{se}+qc_{re}-c_a(u)]\lambda(u)du\right)+\int_0^T e^{-\delta u}c_a(u)\lambda(u)du. \end{cases} \tag{19}$$

If $c_{ap}(u)=c_{se}$ for every $0\leq u\leq T$ it follows that the costs of the different events given by either an arrival moment of a non-repairable defective item at which one still uses the repair-replacement policy but the inventory level of spare parts is zero or the cost of replacement at a positive spare parts inventory level are the same. If this holds we obtain using $p(u)=c_{ap}(u)-c_a(u)=c_{se}-c_a(u)$ that for every $0\leq u\leq T$

$$q(c_{re}+c_{se}-c_a(u))+(1-q)p(u)=c_{se}+qc_{re}-c_a(u)$$

This implies by relation (11) that for this selection of the cost parameters the value $L(x,\tau)$ is the sum of the procurement and the operational costs for every $0\leq t\leq T$ and so the result in Lemma 4 should not come as a surprise. Also by relation (11) it is easy to check that

$$c(x)+C(x,\tau)-L(x,\tau)=(1-q)\mathbb{E}\left(\int_{\tau\wedge\sigma_x}^{\tau}e^{-\delta u}\lambda(u)[c_{ap}(u)-c_{se}]du\right), \tag{20}$$

and by relation (12)

$$L(x,\tau)=\begin{cases} c(x)+c_{scr}x+(h-\delta c_{scr})\mathbb{E}\left(\int_0^\tau e^{-\delta u}(x-N_0(u))^{+}du\right)+\int_0^T e^{-\delta u}\lambda(u)c_a(u)du \\ -c_{scr}(1-q)\mathbb{E}\left(\int_0^{\tau\wedge\sigma_x}e^{-\delta u}\lambda(u)du\right)+\mathbb{E}\left(\int_0^\tau e^{-\delta u}[c_{se}+qc_{re}-c_a(u)]\lambda(u)du\right). \end{cases} \tag{21}$$

The following result, which immediately follows from relations (20) and (21), is also shown in Frenk et al. (2019a) using a much more complicated proof.

Lemma 4. *If the three conditions on the parameters introduced in subsection 2.2 hold, then for every $\tau\in\mathbb{F}$ the function $x\to L(x,\tau)$ is increasing and $\lim_{x\uparrow\infty}L(x,\tau)=\infty$. Also, if additionally $c_{ap}(u)\geq c_{se}$ for every $0\leq u\leq T$, then $c(x)+C(x,\tau)\geq L(x,\tau)$ for every (x,τ) policy.*

Proof. By relation (21) the first part follows, while the second part is an immediate consequence of relation (20). □

Clearly by the interpretation of c_{ap} being the cost of being forced to apply the substitution policy to a non-repairale defective item at a time one still applies the repair-replacement policy it is natural to assume that this cost is higher then the costs c_{se} representing the cost of applying the repair-replacement policy to that same item under expected conditions. Using Lemma 4 one can derive for every positive or negative scrapping value c_{scr} per item a positive lower bound on $\upsilon(P)$.

Lemma 5. *If the three conditions on the parameters introduced in subsection 2.2 hold and $c_{ap}(u)\geq c_{se}$ for every $0\leq u\leq T$, then the optimal objective value $\upsilon(P)$ of the optimization problem (P) satisfies*

$$\upsilon(P)\geq\int_0^T e^{-\delta u}\min\{c_{se}+qc_{re},c_a(u)\}\lambda(u)du>0. \tag{22}$$

Proof. see Appendix. □

Using Lemma 4 and the natural condition $c_{ap}(u) \geq c_{se}$ for every $0 \leq u \leq T$ it is possible to derive a tighter upper bound $\bar{x}_U \leq x_U$ on the optimal order quantity than the one presented in Lemma 3. Contrary to the (weaker) one in Lemma 3 this upper bound holds for both positive and negative values of the scrapping value c_{scr}.

Lemma 6. *If the three conditions on the parameters introduced in subsection 2.2 hold and $c_{ap}(u) \geq c_{se}$ for every $0 \leq u \leq T$ and*

$$\bar{x}_U = \min\{x \in \mathbb{N} : c(x) + \min\{c_{scr}, 0\}x > \int_0^T e^{-\delta u}\lambda(u)\max\{0, c_a(u) - c_{se} - qc_{re}\}du\}, \tag{23}$$

then an optimal order quantity x_* of the optimization problem (P) exists and it satisfies $x_* \leq \bar{x}_U$.

Proof. See Appendix. □

Observe the upper bound $\bar{x}_U$ is easy to compute by bisection since by our assumptions the function $x \to c(x) + \min\{c_{scr}, 0\}x$ is increasing having limit infinity at infinity. By using the proof of Lemma 6, one can construct an upper bound knowing the value $c(\bar{x}) + C(\bar{x}, \bar{\tau})$ of the objective function for a given $(\bar{x}, \bar{\tau})$-policy. In this case the upper bound $\bar{x}_U(\bar{x}, \bar{\tau})$ is given by

$$\bar{x}_U(\bar{x}, \bar{\tau}) = \min\left\{x \in \mathbb{N} : c(x) + \min\{c_{scr}, 0\}x > c(\bar{x}) + C(\bar{x}, \bar{\tau}) - \int_0^T e^{-\delta u}\lambda(u)\min\{c_{se} + qc_{re}, c_a(u)\}du\right\}. \tag{24}$$

Clearly by Lemma 5 the value

$$c(\bar{x}) + C(\bar{x}, \bar{\tau}) - \int_0^T e^{-\delta u}\lambda(u)\min\{c_{se} + qc_{re}, c_a(u)\}du,$$

is non-negative and the lower the values of this difference leads to tighter bounds.

2.6 On the subclass of deterministic and pseudo-deterministic policies.

Another important class of policies is the class of (x, τ) policies with $\tau \in \mathbb{F}_0$. These so-called *deterministic* policies are studied in detail in Frenk et al. (2019b). Observe these polices are sometimes also called *static* due to the following advantage. The decision maker knows applying those policies already at time 0 the switching time of the repair-replacement policy to the alternative policy and so this policy is easy-to-implement in practice. Not knowing the switching time at time zero, but avoiding possibly high penalty costs instead, leads us to study the class of $(x, \tau \wedge \sigma_x)$ policies with $\tau \in \mathbb{F}$. For arbitrary bounded stopping times $\tau \in \mathbb{F}$ it was shown in Lemma 1 that under some reasonable conditions the class of $(x, \tau \wedge \sigma_x)$ polices indeed contains an optimal one among all (x, τ) policies, $x \in \mathbb{Z}_+$, $\tau \in \mathbb{F}$. We will now study this class of policies in more detail for any $\tau \in \mathbb{F}_0$ and call this class the class of *pseudo-deterministic* policies. These policies are not discussed previously in the literature and are still relatively easy to implement in practice as the decision maker switches from the repair-replacement policy to the alternative policy either at the deterministic time $\tau \in \mathbb{F}_0$ or at the random time σ_x of inventory depletion, whichever occurs first. Since in general penalty cost can be very high, it seems natural to assume that any optimal policy tries to avoid penalty cost or at least tries to minimize the probability of penalty occurrences. Hence it is useful to study this new class of policies. We will compute for different scenarios in Section 5 the probability of occurrence of a penalty and the gap between the cost of the optimal pseudo-deterministic policy and $\upsilon(P)$. In the next section, we will also show that it is extremely simple to compute an optimal pseudo-deterministic policy for the important class of piece-wise constant arrival rate and substitution policy costs functions. For these cost and arrival settings one can compute exactly the optimal policy within the class of pseudco deterministic policies instead of approximating the optimal policy. At the same time these cost and arrival settings approximate by any accuracy the general arrival and cost settings.

For the subset of deterministic and pseudo-deterministic policies, one can simplify considerably the expression for $C(x,\tau)$ in relation (13). By Fubini's theorem it follows for any Borel measurable function $f,\tau \in \mathbb{F}_0$ and $x \in \mathbb{Z}_+$ that

$$\mathbb{E}\left(\int_0^{\tau\wedge\sigma_x} f(u)du\right) = \mathbb{E}\left(\int_0^{\tau} f(u)1_{\{\sigma_x>u\}}du\right) = \int_0^{\tau} f(u)\mathbb{P}(N_0(u)<x)du.$$

This relation and relation (13) imply for $\tau \in \mathbb{F}_0$ that

$$C(x,\tau) = \begin{cases} c_{scr}x + (1-q)\int_0^\tau e^{-\delta u}\lambda(u)[c_{se}-c_{scr}-c_{ap}(u)]\mathbb{P}(N_0(u)<x)du \\ +\int_0^\tau e^{-\delta u}\lambda(u)[q(c_{se}+c_{re}-c_a(u))+(1-q)p(u)]du \\ +(h-\delta c_{scr})\int_0^\tau e^{-\delta u}\mathbb{E}((x-N_0(u))^+)du + \int_0^T e^{-\delta u}\lambda(u)c_a(u)du. \end{cases} \tag{25}$$

By the same argument, we can write relation (14) as

$$C(x,\tau\wedge\sigma_x) = \begin{cases} c_{scr}x + \int_0^\tau e^{-\delta u}\lambda(u)[c_{se}+qc_{re}-(1-q)c_{scr}-c_a(u)]\mathbb{P}(N_0(u)<x)du \\ +(h-\delta c_{scr})\int_0^\tau e^{-\delta u}\mathbb{E}((x-N_0(u))^+)du + \int_0^T e^{-\delta u}\lambda(u)c_a(u)du. \end{cases} \tag{26}$$

To analyze the behavior of both objective functions for a given $\tau \in \mathscr{F}_0$ we need the concept of *discrete convexity*: A function $f:\mathbb{Z}_+ \to \mathbb{R}$ is called *discrete convex* on $\mathbb{Z}_+$ if its first order difference $\Delta_x f(x) := f(x+1)-f(x), x \in \mathbb{Z}_+$ is a non-decreasing function on $\mathbb{Z}_+$. The function f is called *discrete concave* if the function $-f$ is discrete convex. Discrete convexity is applied to our problem through the following result.

Lemma 7. *Let N be a non-homogeneous Poisson process with a locally bounded Borel measurable arrival rate function μ and $f:[0,T]\to\mathbb{R}$ some Borel measurable function. If the function f is non-decreasing and non-positive on $[0,\tau)$ for a given $\tau \in \mathbb{F}_0, 0<\tau\leq T$ then the function*

$$x \mapsto G(x) := \mathbb{E}\left(\int_0^{\tau\wedge\sigma_x} f(v)\mu(u)du\right) \tag{27}$$

is non-increasing and discrete convex on $\mathbb{Z}_+$. If the function f is non-increasing and non-negative on $[0,\tau)$, then this function is non-decreasing and discrete concave on $\mathbb{Z}_+$.

Proof. See Appendix. □

Using Lemma 7 the next result is easy to verify. A similar result was proved in Frenk et al. (2019b) using a much more complicated proof.

Lemma 8. *Let $\tau \in \mathbb{F}_0, 0<\tau\leq T$ be given.*

1. *If the procurement cost function c is discrete convex on $\mathbb{Z}_+$ and $c_{ap}(t) \geq c_{se}-c_{scr}$ for every $t<\tau$, then the function $x \mapsto c(x)+C(x,\tau)$ is discrete convex on $\mathbb{Z}_+$.*

2. *If the procurement cost function c is discrete convex on $\mathbb{Z}_+$ and $c_a(t) \geq c_{se}+qc_{re}-(1-q)c_{scr}$ for every $t<\tau$ then the function $x \mapsto c(x)+C(x,\tau\wedge\sigma_x)$ is discrete convex on $\mathbb{Z}_+$.*

Proof. See Appendix. □

Since we always assume (unless stated otherwise) that $p(u) \geq c_{se}$ and c_{ap} is non-increasing, it follows for $c_{scr} \geq 0$ that $c_{ap}(t) \geq c_{se} - c_{scr}$ for every $0 < t \leq T$. Hence by Lemma 8 the function $x \to C(x,\tau)$ is discrete convex for every $0 < \tau \leq T$ and this result was used in Frenk et al. (2019b) to give an efficient algorithm to compute approximately the optimal deterministic policy. In the next section we will only consider the class of pseudo-deterministic policies and prove that one can identify beforehand a finite set of τ values containing an optimal solution for piece-wise constant arrival rate and (decreasing) piece-wise constant substitution cost functions. For this realistic setting, we will analyze the properties of the following optimization problem

$$\upsilon(Q) = \inf_{x \in \mathbb{Z}_+, \tau \in \mathbb{F}_0, 0 \leq \tau \leq T} \{c(x) + C(x, \tau \wedge \sigma_x)\} \tag{Q}$$

3 On properties of the class of pseudo-deterministic policies.

In this section we show for a Poissonian defective items stochastic arrival process having a piece-wise constant arrival rate function and a piecewise-constant non-increasing substitution cost function c_a that the finite set of breaking points of these functions contain the optimal switching time between the two used policies and so this optimal switching time can be easily computed. Without loss of generality, we may assume that both functions have the same set of breaking points. If this is not the case, we use the union of both sets of breaking points.

To justify the use of a piece-wise constant substitution cost function we first observe one can always find a piece-wise constant function approximating a bounded non-increasing substitution cost function c_a within any given accuracy using the following procedure. Introducing the composite function $c_{a,n}(t) := d_n(c_a(t)), 0 \leq t \leq T, n \in \mathbb{N}$ with the so-called *dyadic* function $d_n : [0,\infty] \to [0,\infty)$ (cf.Çınlar (2011)) given by

$$d_n(r) = \sum_{k=1}^{n2^n} \frac{k-1}{2^n} 1_{[(k-1)2^{-n}, k2^{-n})}(r) + n 1_{[n,\infty)}(r), \tag{28}$$

yield a sequence of piece-wise constant functions taking only finitely many values. Using this representation we obtain that the function $c_{a,n}$ is again a non-increasing right continuous function satisfying $c_{a,n} \leq c_a$ and for every $n \in \mathbb{N}$, $n > c_a(0)$ we have the approximation errors $\| c_{a,n} - c_a \|_\infty \leq 2^{-n}$ with $\| f \|_\infty = \sup_{t \in [0,T]} | f(t) |$. Similarly, one can also approximate any locally bounded, Borel measurable arrival rate function λ by the function $\lambda_n(t) := d_n(\lambda(t))$ for $t \in [0,T]$ and $\| \lambda_n - \lambda \|_\infty \leq 2^{-n}$ for sufficiently large n. By increasing the value of n, we obtain more accurate approximations. Although not proved in this paper it is relatively easy to give an upper bound on the error in the optimal discounted cost replacing the function c_a by its approximation $c_{a,n}$ and the arrival rate function λ by λ_n. Hence up to any given accuracy one can replace the true functions c_a and λ by their piece-wise constant approximations $c_{a,n}$ and λ_n for some properly selected $n \in \mathbb{N}$. Hence this justifies the use of these piece-wise constant functions.

To solve the optimization problem (Q) listed in subsection 2.6 for the set of pseudo-deterministic policies two alternative approaches are possible: The first approach we may use (used in Frenk et al. (2019b) for only the set of static policies) is given by

$$\upsilon(Q) = \inf_{\tau \in \mathbb{F}_0, 0 \leq \tau \leq T} \varphi(\tau),$$

with

$$\varphi(\tau) := \inf_{x \in \mathbb{Z}_+} \{c(x) + C(x, \tau \wedge \sigma_x)\}. \tag{Q(\tau)}$$

Applying this bilevel approach and Lemma 8 we may use the necessary and sufficient first order conditions to determine first the optimal order quantity $x(\tau)$ for each given deterministic switching time $\tau \in \mathbb{F}_0$ and compute for this optimal order quantity the optimal value $\varphi(\tau)$ of optimization problem ($Q(\tau)$). In general the optimal value function $\varphi : [0,T] \to \mathbb{R}$ is not a convex function on $[0,T]$ and so we cannot use a classical one-dimensional optimization algorithm to determine an optimal $\tau \in \mathbb{F}_0$. However, it can be shown that the function φ is Lipschitz continuous with a computable Lipschitz constant and so up to any given accuracy we may construct a finite discretization $\mathscr{D}$ of the set $[0,T]$. Evaluating the finite number of function values $\varphi(\tau), \tau \in \mathscr{D}$ and taking the minimum of those values we can then approximate $\upsilon(Q)$ up to any given accuracy. Although applicable this approximation approach will not be followed in this paper. The second bilevel approach (considered in this paper) yielding an exact procedure to compute the optimal pseudo-deterministic policy for a piecewise-constant arrival rate and piecewise constant substitution cost function is given by

$$\upsilon(Q) = \inf_{x \in \mathbb{Z}_+} \{\psi(x)\}. \tag{Q}$$

with

$$\psi(x) := \inf_{0 \le \tau \le T, \tau \in \mathbb{F}_0} \{c(x) + C(x, \tau \wedge \sigma_x)\}. \tag{$Q(x)$}$$

For the optimization problem (Q) we first identify the following easy instance.

Lemma 9. *If the substitution cost function c_a of applying the alternative policy is non-increasing and $c_a(0) \le c_{se} + qc_{re} - (1-q)c_{scr}$ it is optimal within the class of pseudo-deterministic policies not to order any spare parts and switch at time 0 immediately to the substitution policy. This yields the optimal objective value $\upsilon(Q) = \int_0^T e^{-\delta u} \lambda(u) c_a(u) du$.*

Proof. See Appendix. □

Due to the low cost of substitution at time 0 and using relation (14) in the proof of Lemma 9 it is easy to see that it is also optimal to switch immediately at time 0 to the alternative policy within the class of all $(x, \tau \wedge \sigma_x)$ policies with τ an arbitrary bounded stopping time. Hence in the remainder of this paper it is assumed that $c_a(0) > c_{se} + qc_{re} - (1-q)c_{scr}$. In practice this condition on the costs parameters is realistic since immediately applying the alternative policy at time 0 is in general much more expensive then applying the repair-replacement policy. We will now analyze the optimization problem ($Q(x)$) for a Poissonian defective items arrival process having a piece-wise constant arrival rate function and decreasing piece-wise constant substitution or alternative policy cost function $c_a(.)$. To introduce this piece-wise constant arrival rate function consider a strictly increasing sequence $(a_i)_{i=1}^{n+1}$ satisfying $0 = a_1 < a_2 < \ldots < a_n < a_{n+1} = T$ and define for $t \ge 0$

$$\lambda(t) = \sum_{i=1}^{n} \lambda_i 1_{[a_i, a_{i+1})}(t) \tag{29}$$

for any arbitrarily selected non-negative sequence $(\lambda_i)_{i=1}^{n}$. At the same time the non-increasing substitution cost function of applying the alternative policy is given by

$$c_a(t) = \sum_{i=1}^{n} c_i 1_{[a_i, a_{i+1})}(t), 0 \le t \le T. \tag{30}$$

with $c_1 \ge c_2 \ge \ldots \ge c_n > 0$ and $c_1 > c_{se} + qc_{re} - (1-q)c_{scr}$. Since by Lemma 9 it is optimal not to order and immediately switch to the substitution policy if $c_1 = c_a(0) \le c_{se} + qc_{re} - (1-q)c_{scr}$, we may assume $c_1 > c_{se} + qc_{re} - (1-q)c_{scr}$. By relation (14) it also follows that the partial derivative of the function $c(x) + C(x, \tau \wedge \sigma_x)$ with respect to $\tau \in \mathbb{F}_0$ for any $a_i < \tau < a_{i+1}$, $i = 1, \ldots, n$ is given by

$$\frac{\partial C}{\partial \tau}(x, \tau \wedge \sigma_x) = e^{-\delta \tau} \left(\lambda_i [c_{se} + qc_{re} - (1-q)c_{scr} - c_i] \mathbb{P}(N_0(\tau) < x) + (h - \delta c_{scr}) \mathbb{E}((x - N_0(\tau))^+) \right). \tag{31}$$

Using relation (31) and introducing the index

$$n^* = \max\{1 \le i \le n : c_i > c_{se} + qc_{re} - (1-q)c_{scr}\} \le n, \tag{32}$$

the next result is easy to show for optimization problem ($Q(x)$).

Lemma 10. *Introducing* $\psi_i(x) = \min_{a_i \le \tau \le a_{i+1}} \{c(x) + C(x, \tau \wedge \sigma_x)\}$, $i = 1, ..., n$ *it follows for every* $x \in \mathbb{Z}_+$ *that* $\psi(x) = \min_{i=1,...,n^*} \psi_i(x)$ *with* $\psi(x)$ *the optimal objective value of optimization problem* ($Q(x)$).

Proof. See Appendix. □

The partial derivative in relation (31) consists of the difference of two positive decreasing functions and so this difference might not be increasing or decreasing. To analyze in detail this partial derivative we introduce the well known Erlang-B formula represented by the function $B : \mathbb{Z}_+ \times \mathbb{R}_+ \to \mathbb{R}$ given by

$$B(x,t) := \frac{\frac{t^x}{x!}}{\sum_{j=0}^{x} \frac{t^j}{j!}}, \ x \in \mathbb{N}, \tag{33}$$

and $B(0,t) = 1$ for every $t > 0$. The value $B(x,t)$ represents the probability that an arriving customer is rejected in a $M/M/x/x$ Markovian loss model with an arrival rate t and departure rate 1 (Kleinrock (1975)). Hence $1 - B(x,t)$ represents the probability that an arriving customer is admitted to the system. By a straightforward computation it is easy to show the following result.

Lemma 11. *If* $N = \{N(t) : t \ge 0\}$ *is a non-homogeneous Poisson process with arrival rate function* θ *and mean arrival function* $\Theta(t) = \int_0^t \theta(u)du, t \ge 0$ *then for any* $x \in \mathbb{N}$ *and* $t > 0$

$$\frac{\mathbb{E}((x - N(t))^+)}{\mathbb{P}(N(t) < x)} = x - \Theta(t)(1 - B(x-1, \Theta(t))). \tag{34}$$

Proof. see Appendix. □

Introducing the function $\kappa_i : \mathbb{Z}_+ \times [0,T] \to \mathbb{R}$ given by

$$\kappa_i(x,\tau) = \lambda_i[c_{se} + qc_{re} - (1-q)c_{scr} - c_i] + (h - \delta c_{scr})x - (h - \delta c_{scr})\Lambda_0(\tau)[1 - B(x-1, \Lambda_0(\tau))], \tag{35}$$

with $\Lambda_0(\tau) = (1-q)\int_0^\tau \lambda(u)du$ and considering relations (31) and (34), we obtain

$$\frac{\partial C}{\partial \tau}(x, \tau \wedge \sigma_x) = e^{-\delta\tau}\mathbb{P}(N_0(\tau) < x)\kappa_i(x,\tau), \tag{36}$$

for $a_i < \tau < a_{i+1}, i = 1, ..., n$. For every $x \in \mathbb{N}$ the following result is well known. We list a short proof of this result in the Appendix.

Lemma 12. *For every* $x \in \mathbb{N}$ *the function* $t \mapsto t(1 - B(x,t))$ *is increasing and satisfies* $lim_{t\uparrow\infty} t(1 - B(x,t)) = x$.

Proof. see Appendix. □

Using Lemma 12 we can slightly improve the result in Lemma 10. By Lemma 12 it follows for every $x \in \mathbb{N}$ and $\tau \ge 0$ that $\Lambda_0(\tau)(1 - B(x-1, \Lambda_0(\tau)) \le x - 1$, and this yields for every $x \in \mathbb{N}$ that the function κ_i listed in relation (35) satisfies

$$\kappa_i(x,\tau) \ge \lambda_i(c_{se} + qc_{re} - (1-q)c_{scr} - c_i) + (h - \delta c_{scr}). \tag{37}$$

Hence it follows for $c_i \le c_{se} + qc_{re} - (1-q)c_{scr} + \lambda_i^{-1}(h - \delta c_{scr})$ that by relation (36)

$$\frac{\partial C}{\partial \tau}(x, \tau \wedge \sigma_x) = e^{-\delta\tau}\mathbb{P}(N_0(\tau) \le x)k_i(x,\tau) \ge 0, \tag{38}$$

and this shows the improved result that the function $\tau \mapsto c(x) + C(x, \tau \wedge \sigma_x)$ is actually increasing for every x on $[a_i, a_{i+1}]$. This shows for

$$c_1 \le \ c_{se} + qc_{re} - (1-q)c_{scr} + \lambda_{\max}^{-1}(h - \delta c_{scr}), \tag{39}$$

with $\lambda_{\max} := \max_{i=1,\ldots,n} \lambda_i$ that again it is optimal not to order in optimization problem (Q). Hence to solve problem (Q) it is sufficient to consider problem instances satisfying

$$c_1 > c_{se} + qc_{re} - (1-q)c_{scr} + \lambda_{\max}^{-1}(h - \delta c_{scr}). \tag{40}$$

For such instances the set $\{1 \le i \le n : c_i > c_{se} + qc_{re} - (1-q)c_{scr} + \lambda_{\max}^{-1}(h - \delta c_{scr})\}$ is nonempty and introducing

$$n_0^* := \max\{1 \le i \le n : c_i > c_{se} + qc_{re} - (1-q)c_{scr} + \lambda_{max}^{-1}(h - \delta c_{scr})\} \le n^*, \tag{41}$$

where n^* is listed in (32). Using similar arguments as in Lemma 10, we obtain

$$\psi(x) = \min_{i=1,\ldots,n_0^*} \psi_i(x). \tag{42}$$

In the next result we show for every $x \in \mathbb{Z}_+$ that the finite set of breaking points $a_1, \ldots, a_{n_0^*+1}$ contains an optimal exit time τ of the optimization problem ($Q(x)$).

Lemma 13. *For every $x \in \mathbb{N}$ an optimal solution of optimization problem ($Q(x)$) is attained at some a_i, $i = 1, \ldots, n_0^* + 1$ with n_0^* listed in (41).*

Proof. See Appendix. □

The result in Lemma 13 implies that for any given $x \in \mathbb{Z}_+$ we only have to evaluate the objective function $c(x) + C(x, \tau \wedge \sigma_x)$ at $\tau = a_i$ for $i = 1, \ldots, n_0^* + 1$. Therefore, our optimization problem (Q) listed in subsection 2.6 reduces to

$$\upsilon(Q) = \min_{i=1,\ldots,n_0^*+1} \min_{x \in \mathbb{Z}_+} \{c(x) + C(x, a_i \wedge \sigma_x)\}.$$

By Lemma 8 the function $x \to C(x, a_i \wedge \sigma_x)$ is discrete convex for $i \le n^* + 1$ since $c_a(u) > c_{se} + qc_{re} - (1-q)c_{scr}$ for every $u < a_{n^*+1}$. For a discrete convex procurement function $c(.)$, an optimal solution $x(a_i)$, $i = 1, \ldots, n_0^* + 1$ of $\inf_{x \in \mathbb{Z}_+}\{c(x) + C(x, a_i \wedge \sigma_x)\}$ is given by

$$x(a_i) = \min\{x \in \mathbb{Z}_+ : c(x+1) - c(x) + \Delta C(x, a_i \wedge \sigma_x) \ge 0\}, \tag{43}$$

where $\Delta C(x, a_i \wedge \sigma_x)$ represents the first order operator of the function $x \mapsto C(x, a_i \wedge \sigma_x)$ given by

$$\Delta C(x, a_i \wedge \sigma_x) := C(x+1, a_i \wedge \sigma_{x+1}) - C(x, a_i \wedge \sigma_x), x \in \mathbb{Z}_+. \tag{44}$$

Observe (44) can be used to compute the objective value $c(x(a_i) + C(x(a_i), a_i \wedge \sigma_x)$ using

$$c(x(a_i)) + C(x(a_i), a_i \wedge \sigma_{x(a_i)}) = \sum_{k=0}^{x(a_i)-1} [c(k+1) - c(k) + \Delta C(k, a_i \wedge \sigma_k)] + C(0, a_i \wedge \sigma_0), \tag{45}$$

for every $i = 2, \ldots, n^* + 1$. In (45), we need to evaluate the first order difference operator for different parameter values and the value of the objective function in case of no ordering. Applying relation (26) to the special instances of piece-wise constant arrival rate and substitution cost functions, we obtain the following simplification for the first order cost difference.

Lemma 14. *If the arrival rate function is given by relation (29) and the substitution cost function by relation (30) then for every $i = 1, \ldots, n+1$*

$$C(0, a_i \wedge \sigma_0) = \int_0^T e^{-\delta u} \lambda(u) c_a(u) du = \sum_{j=1}^n \frac{\lambda_j c_j}{\delta}(e^{-\delta a_j} - e^{-\delta a_{j+1}}), \tag{46}$$

and for every $i = 2, \ldots, n+1$ and $x \in \mathbb{Z}_+$

$$\Delta C(x, a_i \wedge \sigma_x) = \begin{cases} \sum_{j=1}^{i-1} \left(\frac{c_{se} + qc_{re} - c_j}{1-q} - c_{scr}\right) [e^{-\delta a_j} \mathbb{P}(N_0(a_j) \le x) - e^{-\delta a_{j+1}} \mathbb{P}(N_0(a_{j+1}) \le x)] \\ \\ +(h - \delta c_{scr}) \sum_{j=1}^{i-1} \int_{a_j}^{a_{j+1}} e^{-\delta u} \mathbb{P}(N_0(u) \le x) du + c_{scr}. \end{cases} \tag{47}$$

Proof. See Appendix. □

For $a_1 = 0$, relation (26) leads to $\Delta C(0, 0 \wedge \sigma_x) = c_{scr}$. This implies, by relation (43) and $x \to c(x) + \min\{c_{scr}, 0\}x$ being increasing, that an optimal solution is given by $x(a_1) = 0$ with the optimal value

$$\inf_{x \in \mathbb{Z}_+}\{c(x) + C(x, a_1 \wedge \sigma_x)\} = \int_0^T e^{-\delta u} c_a(u)\lambda(u)du = \sum_{j=1}^n \frac{c_j \lambda_j}{\delta}(e^{-\delta a_j} - e^{-\delta a_{j+1}}). \tag{48}$$

To simplify the calculations in Section 5 and determine the optimal pseudo-deterministic policy we will use Lemma 14 together with the following result.

Lemma 15. *If the arrival rate function is given by relation (29) and introducing for every* $1 \le j \le n$ *and* $x \in \mathbb{Z}_+$

$$\upsilon_j(x) := e^{-\delta a_j}\mathbb{P}(N_0(a_j) \le x),$$

then for every $1 \le j \le n$ *and* $x \in \mathbb{Z}_+$

$$\int_{a_j}^{a_{j+1}} e^{-\delta u}\mathbb{P}(N_0(u) \le x)du = \frac{\sum_{m=0}^x \left(\frac{(1-q)\lambda_j}{\delta + (1-q)\lambda_j}\right)^m [\upsilon_j(x-m) - \upsilon_{j+1}(x-m)]}{\delta + (1-q)\lambda_j}. \tag{49}$$

Proof. See Appendix. □

To solve the optimization problem (Q) determining the optimal pseudo-deterministic policy we can now apply the following algorithm.

Solving optimization problem (Q) for piece-wise constant arrival rate and substitution cost function with breaking points a_i, $i = 1, \ldots n$ and c discrete convex

- Compute n_0^* in relation (41).
- Solve for every $i = 1, \ldots, n_0^* + 1$ the discrete convex minimization problem $\inf_{x \in \mathbb{Z}_+}\{c(x) + C(x, a_i \wedge \sigma_x)\}$ by evaluating the first order conditions $x(a_i) = \min\{x \in \mathbb{Z}_+ : c(x+1) - c(x) + \Delta C(x, a_i \wedge \sigma_x) \ge 0\}$ with $\Delta C(x, a_i \wedge \sigma_x)$ given below

$$\Delta C(x, a_i \wedge \sigma_x) = \begin{cases} c_{scr} + \int_0^{a_i} e^{-\delta u}\lambda(u)[c_{se} + qc_{re} - (1-q)c_{scr} - c_a(u)]\mathbb{P}(N_0(u) = x)du \\ +(h - \delta c_{scr})\int_0^{a_i} e^{-\delta u}\mathbb{P}(N_0(u) \le x)du. \end{cases} \tag{50}$$

- Output

$$\upsilon(Q) = \min_{1 \le i \le n^*+1}\{c(x(a_i)) + C(x(a_i), a_i)\},$$

and

$$(x(a_{i^*}), a_{i^*}) = \arg\min_{1 \le i \le n^*+1}\{c(x(a_i)) + C(x(a_i), a_i)\}.$$

In Section 5, we present the results of some computational experiments with piece-wise constant arrival rate and substitution cost functions. We determine both the optimal pseudo-deterministic and the optimal stopping time policies for the most general optimization problem (P). Before presenting the computational results, we shortly discuss in the following section how to (approximately) compute the optimal policy of optimization problem (P).

4 On the optimal policy within the class of all stopping times.

In this section we discuss how to solve optimization problem (P) listed in Section 2.4. Although most of the results except the characterization of the optimal stopping sets S_k in Lemma 18 and 19 already appeared in Frenk et al. (2019a) using a slightly more difficult approach, we list these main results for completeness also in this paper. By relation (18) we know that there exist a computable upperbound x_U satisfying

$$\upsilon(P) = \inf_{x \leq x_U, x \in \mathbb{Z}_+,} \inf_{\tau \in \mathbb{F}, 0 \leq \tau \leq T} \{c(x) + C(x,\tau)\}. \tag{51}$$

To solve for each $x \leq x_U, x \in \mathbb{Z}_+$ the optimization problem $\varphi(x) := \inf_{\tau \in \mathbb{F}, 0 \leq \tau \leq T} \{c(x) + C(x,\tau)\}$ we discretize the set $[0,T]$ and construct a finite set $\Theta = \{t_0, t_1, ..., t_N\} \subseteq [0,T]$ satisfying $0 = t_0 < t_1 < ... < t_N = T$. Its mesh is given by

$$\Delta(\Theta) = \max_{0 \leq j \leq N-1} \mid t_{j+1} - t_j \mid . \tag{52}$$

Depending on the given set Θ we consider the set of stopping times $\tau \in \mathbb{F}$ taking only values in Θ. For each finite set Θ the set of stopping times only attaining value in Θ is denoted by $\mathbb{F}_\Theta$. Consider now for every $x \leq x_U, x \in \mathbb{Z}_+$ the optimization problem $\varphi_\Theta(x) = \inf_{\tau \in \mathbb{F}_\Theta} \{c(x) + C(x,\tau)\}$, and introduce

$$\upsilon(P_\Theta) = \inf_{x \leq x_U, \tau \in \mathbb{F}_\Theta} \varphi_\Theta(x) = \inf_{x \leq x_U, \tau \in \mathbb{F}_\Theta} \{c(x) + C(x,\tau)\}. \tag{P_Θ}$$

Since $\mathbb{F}_\Theta \subseteq \mathbb{F}$ it follows by relation (51) that $0 \leq \upsilon(P_\Theta) - \upsilon(P)$. It is also possibe to construct an upperbound on $\upsilon(P_\Theta) - \upsilon(P)$. If we introduce the functions $f_i : [0,T] \to \mathbb{R}, i = 1,2$ as follows.

$$f_1(t) = (1-q)(c_{se} - c_{scr} - c_{ap}(t)),$$

and

$$f_2(t) = q(c_{se} + c_{re} - c_a(t)) + (1-q)p(t).$$

By using these definitions, one can show the following result. Although this result is shown in Frenk et al. (2019a) we list for completeness an outline of the proof.

Lemma 16. *If the arrival process N of defective items is given by a non-homogeneous Poisson process with arrival rate function $\lambda(.)$ then $\upsilon(P_\Theta) - \upsilon(P) \leq f_0(x_U)\Delta(\Theta)$ and $f_0(x) = h - \delta_{csr}x + \| \lambda f_1 \|_\infty + \| \lambda f_2 \|_\infty$ with $\| g \|_\infty := \sup_{0 \leq t \leq T} \mid g(t) \mid$ denotes the well-known supnorm.*

Proof. see Appendix. □

By Lemma 5 we obtain that the relative error of solving the approximative problem (P_Θ) instead of optimization problem (P) is given by

$$1 \leq \frac{\upsilon(P_\Theta)}{\upsilon(P)} \leq 1 + \frac{f_0(x_U)\Delta(\Theta)}{\int_0^T e^{-\delta u} \lambda(u) \min\{c_{se} + qc_{re}, c_a(u)\} du}. \tag{53}$$

In our computational section we select $\varepsilon > 0$ and use a finite set Θ with mesh $\Delta(\theta)$ satisfying

$$0 < \Delta(\Theta) \leq \frac{\varepsilon \int_0^T e^{-\delta u} \lambda(u) \min\{c_{se} + qc_{re}, c_a(u)\} du}{f_0(x_U)}. \tag{54}$$

If (54) holds, then (53) implies that the relative error is smaller than ε. In Section 5, we will use a piece-wise constant arrival rate and a piece-wise constant substitution cost function with the same set of breaking points. Θ is chosen to be $\Theta = \{j\Delta(\Theta) : j = 0, ..., N\}$ with $N\Delta(\Theta) = T$ and the breaking points of arrival rate substitution cost are included in this set. Δ=$\Delta(\Theta) > 0$ we solve the optimization problem (P_Θ) using shifted stochastic processes.

Definition 1. *For any non-homogeneous Poisson process X with arrival rate function θ, the shifted stochastic process $\{X^{(k)}(t) : t \geq 0\}$ is defined as $X^{(k)}(t) := X(t+k\Delta) - X(k\Delta)$ for $k = 0,...,N-1$.*

It is well known that $X^{(k)}$ is again a non-homogeneous Poisson process with arrival rate function $t \to \theta(t+k\Delta)$. Let us define the shifted stochastic process $N_0^{(k)}, k = 0,...,N-1$ for the arrival process N_0 of non-repairable items with arrival rate function $\lambda_0 = (1-q)\lambda$. For this shifted process the stopping times is defined as $\sigma_x^{(k)} := \inf\{t \geq 0 : N_0^{(k)}(t) \geq x\}$ for every $x \in \mathbb{Z}_+$. At time $k\Delta, k = 0,...,N-1$ we either stop or continue with the repair-replacement policy. If we decide to stop at time $k\Delta$ and the inventory level equals x, then the random discounted costs of switching to the substitution policy (discounted from time $k\Delta$) is given by $c_{scr}x + \int_0^{(N-k)\Delta} e^{-\delta u} c_a(u+k\Delta) dN^{(k)}(u)$, where $N^{(k)}$ is a shifted Poisson process of arrival requests with rate λ. Hence the expected discounted cost of taking action π_0, which is defined as to switch to the repair-with-substitute policy, at time $k\Delta$ with inventory level x equals

$$C_{\pi_0}(x) := c_{scr}x + \int_0^{(N-k)\Delta} e^{-\delta u} c_a(u+k\Delta)\lambda(u+k\Delta)du. \tag{55}$$

Also the random discounted cost of continuing with the repair-replacement policy at time $k\Delta$ equals $b_k(x) + e^{-\delta\Delta} V_{k+1}((x - N_0^{(k)}(\Delta))^+)$ with $b_k(x)$ the random discounted cost of applying the repair-replacement policy during the time interval $[k\Delta, (k+1)\Delta)$. Introducing

$$B_k(x) = \mathbb{E}(b_k(x)), \tag{56}$$

the expected discounted cost of taking action π_1, defined as not to switch to the repair-with-substitute policy, with inventory level x at time $k\Delta$ becomes

$$C_{\pi_1}(x) := B_k(x) + e^{-\delta\Delta}\mathbb{E}(V_{k+1}((x - N_0^{(k)}(\Delta))^+). \tag{57}$$

This shows for $x \leq x_U$ that the Bellman equations for the above stopping problem are given by the functions $V_k : \mathbb{Z}_+ \to \mathbb{R}$, $k = 0,...,N$ given by

$$V_N(x) = c_{scr}x, V_k(x) = \min\{C_{\pi_0}(x), C_{\pi_1}(x)\}. \tag{58}$$

To simplify the recurrent relation for V_k in relation (58), we introduce the function $W_k : \mathbb{Z}_+ \to \mathbb{R}$, $k = 0,...,N$ given by

$$W_k(x) := V_k(x) - \int_0^{(N-k)\Delta} e^{-\delta u} c_a(u+k\Delta)\lambda(u+k\Delta)d. \tag{59}$$

Since it is easy to check that

$$\int_0^{(N-k)\Delta} e^{-\delta u} c_a(u+k\Delta)\lambda(u+k\Delta)du =$$
$$\int_0^{\Delta} e^{-\delta u} c_a(u+k\Delta)\lambda(u+k\Delta)du + e^{-\delta\Delta}\int_0^{(N-(k+1))\Delta} e^{-\delta u} c_a(u+(k+1)\Delta)\lambda(u+(k+1)\Delta)du,$$

we obtain from relations (55)-(59) that we need to solve for $x \leq x_U$ and $k = 0,...,N$ the Bellman equations

$$W_N(x) = c_{scr}x, W_k(x) = \min\{c_{scr}x, \overline{B}_k(x) + e^{-\delta\Delta}\mathbb{E}(W_{k+1}((x - N_0^{(k)}(\Delta))^+)\}, \tag{60}$$

where

$$\overline{B}_k(x) := B_k(x) - \int_0^{\Delta} e^{-\delta u} c_a(u+k\Delta)\lambda(u+k\Delta)du. \tag{61}$$

For $x=0$ and $k=0,...,N-1$ relation (60) reduces to the simplified recurrent relation

$$W_N(0)=0, W_k(0)=\min\{0,\overline{B}_k(0)+e^{-\delta\Delta}W_{k+1}(0)\}. \tag{62}$$

Observe that after hitting the zero inventory level, so $x=0$, it might be cheaper to continue with the repair-replacement policy depending on the sign of $\overline{B}_k(0)$. This holds if the expected cost $q(c_{se}+c_{re})+(1-q)c_{ap}(u)$ of any defective item arriving at u is much lower then the expected costs $c_a(u)$ of substitution at u. Having computed $W_0(x)$ for $x=0,...,x_U$ with the recurrent relations (60) and (61) after N iterations, we set

$$V_0(x)=W_0(x)+\int_0^{N\Delta} e^{-\delta u}c_a(u)\lambda(u)du, \tag{63}$$

for $x=0,...,x_U$ and solve the optimization problem

$$\min_{x\leq x_U}\{c(x)+V_0(x)\}, \tag{64}$$

by enumeration. To compute $\overline{B}_k(x)$ in (60) we proceed as follows. By definition let $C_k(x,\Delta)$ denote the expected discounted operational cost (discounted from time $k\Delta$) from time $k\Delta$ up to time $N\Delta=T$. If we observe inventory level x at time $k\Delta$ and we apply the deterministic (x,Δ) policy at time $k\Delta$, the shifted arrival processes $N_0^{(k)}$ and $N^{(k)}$ are non-homogeneous Poisson processes with arrival rate functions $t\to\lambda_0(k\Delta+t)$ and $t\to\lambda(k\Delta+t)$. Therefore by the definition of $B_k(x)$ in (56), we need to subtract the discounted substitution costs from time $(k+1)\Delta$ up to $N\Delta=T$ from $C_k(x,\Delta)$ and set the scrapping value equal to zero in the expression for $C_k(x,\Delta)$. This means that

$$B_k(x)=C_k(x,\Delta)-\int_{\Delta}^{(N-k)\Delta} e^{-\delta u}\lambda(k\Delta+u)c_a(u+k\Delta)du, \tag{65}$$

and by relation (25) substituting $c_{scr}=0$

$$C_k(x,\Delta)=\begin{cases}(1-q)\int_0^{\Delta}e^{-\delta u}\lambda(k\Delta+u)[c_{se}-c_{ap}(u+k\Delta)]\mathbb{P}(N_0^{(k)}(u)<x)du\\ +\int_0^{\Delta}e^{-\delta u}\lambda(k\Delta+u)[q(c_{se}+c_{re}-c_a(u+k\Delta))+(1-q)p(k\Delta+u)]du\\ +h\int_0^{\Delta}e^{-\delta u}\mathbb{E}((x-N_0^{(k)}(u))^+)+\int_0^{(N-k)\Delta}e^{-\delta u}\lambda(k\Delta+u)c_a(k\Delta+u)du.\end{cases}$$

This implies using relation (65)

$$B_k(x)=\begin{cases}(1-q)\int_0^{\Delta}e^{-\delta u}\lambda(k\Delta+u)[c_{se}-c_{ap}(u+k\Delta)]\mathbb{P}(N_0^{(k)}(u)<x)du\\ +\int_0^{\Delta}e^{-\delta u}\lambda(k\Delta+u)[q(c_{se}+c_{re}-c_a(u+k\Delta))+(1-q)p(k\Delta+u)]du\\ +h\int_0^{\Delta}e^{-\delta u}\mathbb{E}((x-N_0^{(k)}(u)^+)+\int_0^{\Delta}e^{-\delta u}\lambda(k\Delta+u)c_a(k\Delta+u)du.\end{cases}$$

Applying relation (61) we obtain

$$\overline{B}_k(x)=\begin{cases}(1-q)\int_0^{\Delta}e^{-\delta u}\lambda(k\Delta+u)[c_{se}-c_{ap}(u+k\Delta)]\mathbb{P}(N_0^{(k)}(u)<x)du\\ +\int_0^{\Delta}e^{-\delta u}\lambda(k\Delta+u)[q(c_{se}+c_{re}-c_a(u+k\Delta))+(1-q)p(k\Delta+u)]du\\ +h\int_0^{\Delta}e^{-\delta u}\mathbb{E}((x-N_0^{(k)}(u))^+)du.\end{cases} \tag{66}$$

Hence by relation (66) and the definition of the first order difference operator, the next result easily follows.

Lemma 17. *For every $k \in \mathbb{N}$ it follows that*

$$\overline{B}_k(0) = \int_0^\Delta e^{-\delta u} \lambda(k\Delta+u)[q(c_{se}+c_{re}-c_a(k\Delta+u))+(1-q)p(k\Delta+u)]du, \tag{67}$$

and for every $x \in \mathbb{Z}_+$

$$\Delta\overline{B}_k(x) = (1-q)\int_0^\Delta e^{-\delta u} \lambda(k\Delta+u)[c_{se}-c_{ap}(k\Delta+u)]\mathbb{P}(N_0^{(k)}(u)=x)du + h\int_0^\Delta e^{-\delta u}\mathbb{P}(N_0^{(k)}(u) \le x)du. \tag{68}$$

By relation (60) we obtain the following optimal stopping sets S_k.

$$S_k := \{x \in \mathbb{Z}_+ : c_{scr}x \le \overline{B}_k(x) + e^{-\delta\Delta}\mathbb{E}(W_{k+1}((x-N_0^{(k)}(\Delta))^+)\}\}, \tag{69}$$

for $k = 0, \ldots, N-1$. Under certain intuitively clear conditions on the arrival rate and the substitution cost one can show a desirable property for the optimal stopping set S_k. To this end, we first need the next two results.

Lemma 18. *If the arrival process of defective items is a homogeneous Poisson process and the penalty costs are constant over $[0,T]$ then for every $0 \le k \le N-1$ and for every $x \in \mathbb{Z}_+$*

$$W_k(x) \le W_{k+1}(x) \le c_{scr}x.$$

Proof. see Appendix. □

The following result follows from Lemma 18.

Lemma 19. *If the arrival process of defective items is a homogeneous Poisson process and the penalty costs are constant over $[0,T]$ then $S_k \subseteq S_{k+1}$ for every $0 \le k \le N-1$.*

Proof. see Appendix. □

By Lemma 19 we obtain that for every time $0 \le k \le N-1$ there exists no threshold value or there exists some threshold inventory level $x_k^\star$ satisfying we will always stop at time k if and only if our inventory level is above this level. In Section 5, we numerically find that the same structure for our optimal stopping sets in Lemma 19 holds for our piece-wise homogeneous Poisson process. This is probably due to Δ being extremely small, so we have $\mathbb{P}(N_0^{(k)} \le 1) \simeq 1$ at the breaking points which is partly indicated by the proof of Lemma 18. Hence due to (66) the function $x \mapsto \overline{B}_k(x)$ is almost an affine function with approximately the same slopes depending mainly on the inventory cost h and the discount factor δ but with increasing constant terms.

5 Computational results.

In this section we numerically analyze the performance of the optimal stopping problem (P) and determine the optimal pseudo-deterministic policies for different parameter settings. We measure the sensitivity of the performance gap between pseudo-deterministic and deterministic policies to different parameters. Our computational experiments are coded in R Gui and run with a 1.60GHz processor. It took on average 8 minutes to execute the algorithm to compute the algorithm to compute the optimal stopping policy for the dynamic model and 4 seconds to run the algorithm for the optimal pseudo-deterministic policy.

5.1 Experimental setting and base scenario

In our test bed, the planning horizon $[0,T]$ is split into three equal intervals by the sequence of breaking points $(a_i)_{i=1}^4$, where $a_i = (i-1)\frac{T}{3}$. Therefore, our sub-intervals are $[0, \frac{T}{3}]$, $[\frac{T}{3}, \frac{2T}{3}]$ and $[\frac{2T}{3}, T]$, and in each of these sub-intervals the arrival rate and the substitution cost functions are assumed to be equal to λ_i and c_i for $i = 1,2,3$ respectively. The cost of the repair-with-substitute policy over different sub-intervals is now given by $c_j = c_a(0)e^{-\gamma a_j}$ for $j = 1,2,3$, where γ is the decay factor. The arrival rate of the defective products is assumed to be a step function; constant arrival rate of $\lambda_i = l\beta^{i-1}$ over the interval $[a_i, a_{i+1}]$ and jumps at time a_i. For a given β, l is set to a value that makes the expected number of product arrivals over the horizon $[0,T]$ equal to $10T$ (i.e., on average 10 customer requests per unit time). For the general (x,τ)-policy the penalty costs, p, is assumed to be equal to 1290. The base case scenario with these parameters is given in Table 1. Note that our substitution cost and arrival rate functions are also used by Frenk et al. (2019a), to which we benchmark our results.

Table 1: Problem parameters for the base case scenario.
width=0.9

T	c_{scr}	c_{se}	c_{re}	h	q	$\bar{c}$	$c_a(0)$	γ	p	ε	δ	β
66 (months)	30	30	20	3.25	0.5	225	645	0.02	1290	0.001	0.003	0.5

The procurement cost function has the form $c(x) = \bar{c}x$ with rate $\bar{c} > 0$. Also we assume $\bar{c} > c_{scr}^- = -\min\{c_{scr}, 0\}$ which makes $x \mapsto c(x) - c_{scr}^- x$ an increasing non-negative function with limit ∞ satisfying our standard assumption on the cost parameters. This also implies that the first order conditions in (43) reduce to

$$x(a_i) = \min\{x \in \mathbb{Z}_+ : \Delta C(x, a_i \wedge \sigma_x) \geq -\bar{c}\}. \tag{70}$$

To solve the optimization problem (P), we apply the approximation approach proposed in Section 4 and select the finite set Θ having mesh $\Delta > 0$ in such a way that the breaking points are contained in Θ. This means that $M_i\Delta = a_i, i = 1,...,n+1$ and so $M_1 = 0$ and $M_{n+1} = N$. Selecting a relative error of $\varepsilon = 0.001$ in the base case scenario, the mesh Δ of our set Θ is not larger than 0.003 due to (54). Due to the approximation approach, the computed optimal objective values $\upsilon(P)$, in Table 2, are subject to this selected relative error. For the above settings one can also simplify the computation of $\Delta\overline{B}_k(x)$. The next result follows immediately from Lemma 17.

Lemma 20. *If the arrival rate and substitution policy cost functions are given by relations (29), and (30) then for $M_i \leq k \leq M_{i+1} - 1, i = 1,...,n+1$*

$$\overline{B}_k(0) = [q(c_{se} + c_{re} - c_i) + (1-q)p]\lambda_i\delta^{-1}(1 - e^{-\delta\Delta}). \tag{71}$$

Also for every $m \in \mathbb{Z}_+$

$$\Delta\overline{B}_k(m) = [h + \delta(p + c_i - c_{se})]\int_0^\Delta e^{-\delta u}\mathbb{P}(X(u) \leq m)du - (p + c_i - c_{se})[1 - e^{-\delta\Delta}\mathbb{P}(X(\Delta) \leq m)]. \tag{72}$$

and X a homogeneous Poisson process with arrival rate $(1-q)\lambda_i$.

Solving the Bellman equations in relations (60) and (61) we are able to compute the function V_0 in relation (63) and solve optimization problem $\min_{x \leq x_U}\{c(x) + V_0(x)\}$. In Figure 2 the function V_0 is plotted for the values $x = 0, 1, ..., x_U$ under the base case scenario. This plot shows that this function achieves a minimum within this range.

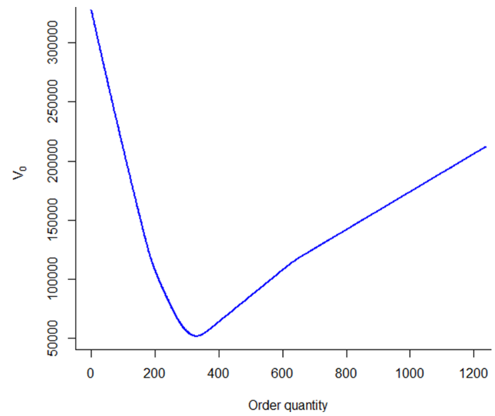

Figure 2: Function $V_0(x)$ plotted over the range of x values for the base case scenario.

5.2 Sensitivity of $\upsilon(P)$ to Problem Parameters.

The optimal solution of (P) consists of the optimal initial order quantity, x, and the optimal stopping time to switch from the repair-replacement policy to the repair-with-substitute policy, τ. Plainly, the set of points (x,τ) represent the times τ at which it is optimal to stop holding inventory x. In order to provide insight into the structure of the optimal policy, we present the optimal (x,τ) policies as blue regions in Figure 3 for different values of holding cost and decay factors. If for a given realization of the arrival process one enters the blue region at a certain time with x items in inventory, it is optimal to switch at that time to the repair-with-substitute policy. Also in the base case scenario it follows for every $1 \le i \le n$ that $q(c_{se}+c_{re}-c_i)+(1-q)p = 670-322.5\theta^{i-1}$. This shows that the conditions of Lemma (1) are satisfied restricting our class of optimal policies for the optimization problem (P). At the optimal stopping time we immediately switch to the substitution policy (from the repair-replacement policy) at the moment the spare parts inventory is zero. This is depicted by the thin blue horizontal lines at the bottom of each plot in Figure 3.

In Figure 3, we observe that as γ increases, the substitution policy cost declines faster and that makes switching to the substitution policy by scrapping inventory more profitable. h has similar but opposite effect on the stopping region. With the increase of h, the repair-replacement policy becomes more expensive. Hence, the size of stopping area is increasing in h. These results are consistent with the results of Frenk et al. (2019a). The main difference between their results and ours stems from the cost of substitution policy, which is larger in our experiments. Therefore, we encounter larger stopping regions than Frenk et al. (2019a) in all scenarios.

In Figure 7, we depict the effect of different parameter values on the optimal cost and order quantity of the problem P. In each plot, the value of a parameter is changed whereas the rest of the parameters are set to the values in Table 1. The optimal cost value is depicted with a dashed line whereas the optimum order size is presented with a straight line on the secondary axis.

Figure 4 indicates that the optimal order size and the optimum cost decrease in γ with different rates. Decrease in the optimum cost values occurs with a constant rate due to decaying cost parameters. The optimum order size converges to 200 for $\gamma \ge 0.05$ due to the fact that order size of less than 200 lead to high penalty costs.

Figure 5 also depicts the effect of the probability q on the optimum order size and the optimum cost value. As q increases, less spares will be needed and thus the optimum order is smaller. In addition, we apply the repair-replacement policy for most of the time horizon which in turn incurs lower expected total cost. For the extreme case $q=1$, which represents all items can be repaired, the optimal order quantity is 0 and the total expected cost of the optimal policy is 30.757 thousand. On the other hand, for $q=0$,

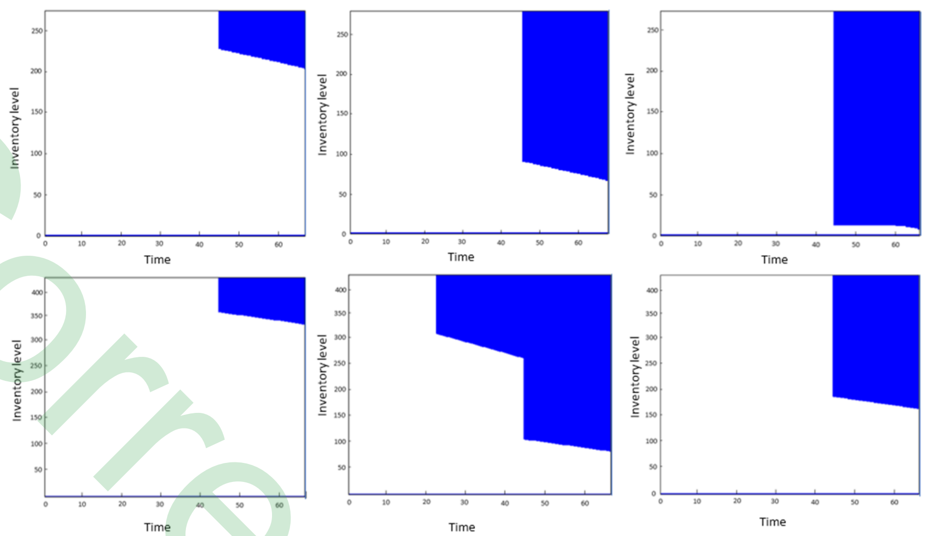

Figure 3: Stopping regions for different γ and h values. First row plots are for $\gamma \in \{0.03, 0.05, 0.07\}$. Second row plots are for $h \in \{3.25, 6.5, 13\}$.

we order initially a large quantity to cover the demand and therefore, we incur a large optimum expected cost. This shows that the effect of the repair probability is rather significant on the system behavior and optimum results. Figure 6 shows the relation between the optimum results and the inventory holding cost rate. Expectedly, increasing values of h lead to lower order size and higher total cost in the optimum solutions.

5.3 Performance of pseudo-deterministic policy for different parameters.

To investigate the effect of different problem parameters on $\upsilon(P)$ and $\upsilon(Q)$, we conduct numerical experiments by altering parameter values of the base scenario in one-at-a-time fashion. For each parameter setting, we report the optimal values of $\upsilon(P)$ and $\upsilon(Q)$ together with their relative percent difference given below:

$$\upsilon(P) \text{ vs } \upsilon(Q) = 100 \times \frac{\upsilon(Q) - \upsilon(P)}{\upsilon(P)}. \tag{73}$$

In addition, we report the optimal initial order quantities for both optimization problems and switching times for $\upsilon(Q)$. The results of our numerical experiments are reported in Table 2.

In all considered scenarios the optimal policies of $\upsilon(P)$ and $\upsilon(Q)$ are almost the same and this explains the small relative deviation. The relative error between the optimal objective $\upsilon(P)$ and $\upsilon(Q)$ is on average 0.15 percent. Such a low difference justifies using the optimal pseudo-deterministic policy as an approximation to the general optimization problem. At the same time, the small difference between the two policy shows that one cannot prove that the optimal pseudo-deterministic policy is optimal within the larger class of stopping times and that within the class of pseudo-deterministic policies the order quantity is the most important decision variable to control the cost. For many parameter settings, the optimal order quantity of the problem (P) satisfies

$$x_{opt} \leq \min\{x \in \mathbb{Z}_+ : (x,\tau) : W_\tau(x) = c_{scr}x\}, \tag{74}$$

showing that the optimal policy never switches to the repair-with-substitute policy before the end of the horizon with a positive inventory. The switch to substitution takes place before the end of the horizon

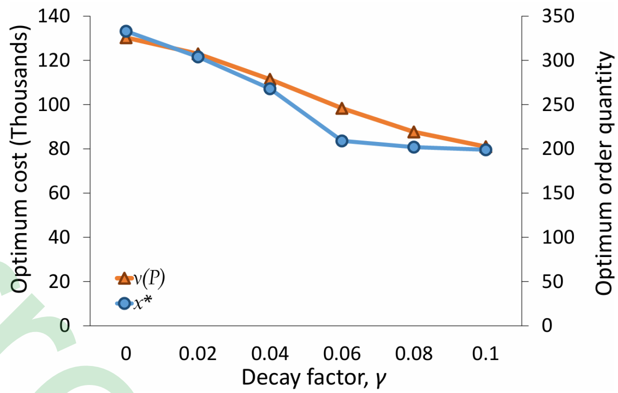

Figure 4: Effect of decay rate (γ)

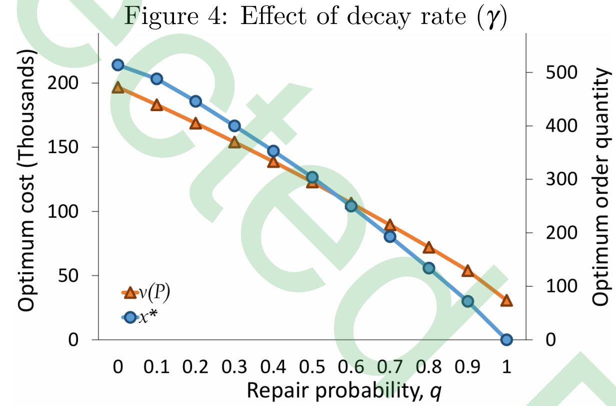

Figure 5: Effect of non-repairable probability (q)

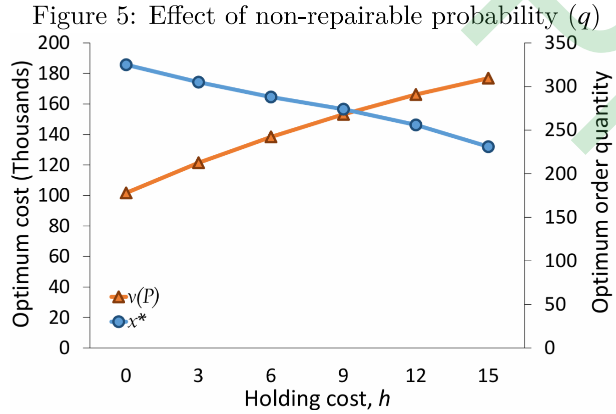

Figure 6: Effect of holding cost (h)

Figure 7: Sensitivity of the optimum solution of the problem $\boldsymbol{P}$ to different parameters.

with 0 inventory. This means that the optimal dynamic policy actually coincides with the optimal pseudo-deterministic policy, explaining the extremely small relative deviation, which is mainly caused by our approximation to $\upsilon(P)$. For the pseudo-deterministic policy, we calculate the probability of $\{\tau_0 \leq \sigma_x\}$, which represents the event of switching to the repair-with-substitute policy with positive inventory. Table 2 presents the calculated probabilities, which are very close to zero in almost all cases. For the cases failing to satisfy (74), the highest probabilities of switching with positive inventory are related to high substitution costs either directly by means of $c_a(0)$ or indirectly through γ and q. The probability is almost 0 in case of low h, high initial $c_a(0)$ or high q. All of these findings are intuitively clear. In addition, we computed the probability of switching to substitution with a positive spare parts inventory in the optimum solution of (P). This probability, which we didn't report in Table 2 due to space limitation, is again found to be extremely small. Hence in both optimal policies (recall that we will only stop at the end of the horizon with a positive inventory level under the pseudo-deterministic policy) switching to the repair-with-substitute policy with a positive spare parts inventory occurs rarely when the system starts with the optimal order quantity.

Generally we observe that time to switch to substitution coincides with the breaking point at which substitution becomes cheaper and inventory is depleted in the optimal policies. In Table 2 we observe that the optimal order quantities of both policies are the same and the objective values of the two problems are very close to each other. This is partly due to the fact that the majority of the cases in Table 2 satisfy the condition in Lemma 1. In majority of the cases, the deterministic switching time of the optimal pseudo-deterministic policy is equal to 66, i.e. $\tau_{opt} = 66$ except two cases where the substitution is relatively inexpensive at time 0. In those cases $\tau_{opt} = 44$. $\tau_0 = 66$ represents the utilization of the repair-replacement policy until σ_x, after which substitution is utilized.

In Table 2, we notice that the decay factor, γ, of the substitution cost function has a negative effect on order sizes and costs of both optimal policies. Decreasing values of γ make substitution expensive leading to a higher total expected cost. For the same reason, we order a higher initial inventory so that we can avoid switching to the repair-with-substitute policy due to reaching earlier inventory level 0. The initial substitution cost $c_a(0)$ has a similar effect on the optimal cost and the optimal initial inventory. In our analyses we find that the optimum cost and policy parameters are insensitive to c_{scr} value. This is mainly because switch time mostly corresponds to σ_x in almost all cases. Similarly, p values do not have major effect on the objective value as long as (16) in Lemma 1 is satisfied (cases that do not satisfy (16) are evaluated in Section 5.5).

Furthermore, we find that q has a large negative effect on the expected total costs for both policies. In case of high q, a lot of the returned products need only a repair service, which leads to smaller order size and lower cost. Parameters h and β have similar effect on the optimal order quantity and the objective value. Expectedly, a higher h increases the average cost of the repair-replacement policy and decreases the optimum order size. For higher values of β most of the arrivals happen in later periods (recall that the total number of expected arrival is $10T$). This decreases the optimal order quantity as substitution gets cheaper over time.

5.4 End-of-life management for different product types.

We also compute the optimal dynamic policy and the optimal pseudo-deterministic policy for different lengths of the warranty period by considering three different product types. Each of these cases is presented in Table 3 and the different length T refers to a certain type of product. First, we compare the performance of the two policies on the second product type in the table (row with $T = 66$). The box plot in Figure 8 indicates a very small and always positive difference between the optimal policy in problem (P) and the optimal pseudo-deterministic policy. Also one can observe that both policies order the same amount of spare parts at the beginning of the planning horizon.

The first product in the table represents a high-demand electronic device with $T = 24$ months of warranty. The product type can be exemplified with cell phones. Since, the product has a short life cycle,

Table 2: Sensitivity analysis of the problem parameters for different policies.

		$\upsilon(P)$		$\upsilon(Q)$			$\mathbb{P}(\tau_0 < \sigma_x)$	$\upsilon(P)$ vs $\upsilon(Q)$
		x	cost	x	τ_0	cost		(%)
γ	0.05	227	105,455.8	227	66	105,552.9	1.19×10^{-9}	0.09
	0.02	304	122,965.6	304	66	122,974.6	7.88×10^{-2}	0.09
	0.01	323	127,140.5	323	66	127,259.6	3.64×10^{-1}	0.09
	0.005	329	128,755.6	329	66	128,878.2	4.93×10^{-1}	0.10
$c_a(0)$	250	191	100,289.7	191	44	100,382.0	4.16×10^{-9}	0.09
	322.5	216	109,735.0	216	44	109,732.0	2.01×10^{-5}	0.09
	1290	329	128,991.9	329	66	129,111.9	4.93×10^{-1}	0.09
	2580	342	132,955.64	342	66	133,083.2	7.56×10^{-1}	0.10
p	322.5	304	122,845.2	304	66	122,974.6	7.88×10^{-2}	0.11
	645	304	122,850.3	304	66	122,974.6	7.88×10^{-2}	0.10
	2580	304	122,854.2	304	66	122,974.6	7.88×10^{-2}	0.10
	5160	304	122,853.5	304	66	122,974.6	7.88×10^{-2}	0.10
h	0.8125	320	107,237.1	320	66	107,350.5	3.03×10^{-1}	0.11
	1.625	315	112,679.2	315	66	112,787.6	2.14×10^{-1}	0.10
	6.5	285	140,809.4	285	66	140,953.0	6.24×10^{-3}	0.10
	13	248	169,876.7	248	66	170,039.8	1.41×10^{-6}	0.10
c_{scr}	−30	304	122,829.7	304	66	122,945.8	7.88×10^{-2}	0.09
	10	304	122,841.2	304	66	122,965.0	7.88×10^{-2}	0.10
	60	304	122,864.2	304	66	122,989.1	7.88×10^{-2}	0.10
	90	303	122,879.7	303	66	123,001.8	7.09×10^{-2}	0.10
q	0.4	353	138,842.7	353	66	138,849.5	1.51×10^{-2}	0.10
	0.6	250	106,489.7	250	66	106,500.5	2.04×10^{-1}	0.09
	0.8	134	72,127.8	134	66	72,145.2	5.92×10^{-1}	0.13
	1	0	30,772.5	1	66	31,232	1	1.54
β	0.25	319	119,943.2	319	66	119,958.8	2.84×10^{-1}	0.10
	1	270	130,079.9	270	66	130,078.7	3.73×10^{-4}	0.09
	1.5	244	134,660.0	244	66	134,652.9	4.20×10^{-7}	0.09
	2	227	137,495.7	227	66	137,485.0	1.19×10^{-9}	0.09

the average number of service requests per unit of time $\Lambda(T)/T$, the substitution cost and its decay factor are expected to be high. The second row in Table 3 represents products with medium economic lifetime (5-10 years) with relatively cheaper substitution and repair costs. As it has longer lifetime, its repair demand is slower than the electronic products. This product type can be exemplified with household appliances such as dishwasher. The last row refers to a product with a long warranty period and very low value of $\Lambda(T)/T$. An example of such a product can be an expensive medical machine/equipment. The average monthly service demand as well as the decay of the substitution cost are very low for these types of products. The box plot of the operational costs is given in Figure 9. The box plot shows that short and long-life products have closer optimum cost whereas the medium-life product has a much lower optimum cost. The proximity of the optimum costs of short and long-life products, despite the difference between the cost rates, mainly stems from the difference between their demand rates. Lower optimum cost of medium-life products, on the other hand, is due to cheaper substitution, service and repair costs.

In order to investigate the relative cost difference between (P) and (Q), we calculate the statistics in

Table 3: Three types of products with different characteristics.

Type	T	$\Lambda(T)/T$	parameter ranges
Short life, medium cost (Cell phone)	24	50	$c_{scr} \in [-10,80]$, $c_{se} \in [20,70]$, $c_{re} \in [10,70]$, $h \in [0.5,1.5]$, $q \in [0.2,0.5]$, $c \in [250,750]$, $c_a(0) \in [1250,3000]$, $\gamma \in [0.025,0.05]$, $p \in [1000,3000]$, $\delta \in [0.001,0.01]$, $\beta \in [0.7,1]$
Medium life, low cost (Dishwasher)	66	10	$c_{scr} \in [-10,40]$, $c_{se} \in [10,50]$, $c_{re} \in [10,40]$, $h \in [1,5]$, $q \in [0.3,0.7]$, $\bar{c} \in [100,400]$, $c_a(0) \in [750,2000]$, $\gamma \in [0.01,0.03]$, $p \in [500,3000]$, $\delta \in [0.001,0.01]$, $\beta \in [0.5,0.7]$
Long life, high cost (MRI Scanner)	120	1	$c_{scr} \in [-1000,2000]$, $c_{se} \in [200,700]$, $c_{re} \in [1000,3000]$, $h \in [10,30]$, $q \in [0.5,0.9]$, $c \in [7500,20000]$, $c_a(0) \in [15000,30000]$, $\gamma \in [0.005,0.015]$, $p \in [10000,30000]$, $\delta \in [0.001,0.01]$, $\beta \in [0.5,0.7]$

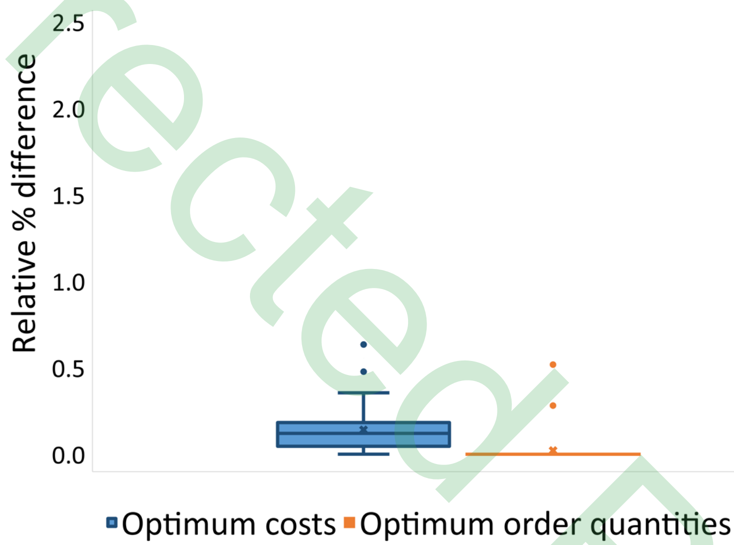

Figure 8: Difference between the dynamic and the pseudo-deterministic policies for a medium-life product ($T = 66$).

(73) for each generated parameter sets for the three product types. Percent cost difference between the two problems are depicted in Figure 10. Results of our calculations indicate that the pseudo-deterministic policy provides a very good approximation to the optimal policy for short-life products. However, as product lifetimes increase (while spare parts demand decreases), the cost difference increases up to 300%. This indicates that the pseudo-deterministic policy has a questionable performance for long-life capital products due to the importance of penalty costs, which is ignored by the pseudo-deterministic policy.

5.5 Worst case performance of pseudo-deterministic policy for short-life products.

In Sections 5.2 and 5.3, we consider parameter settings that mostly satisfy the condition (16) in Lemma 1 and similar to short-life medium-cost products in Table 3. Specifically, 24 out of 28 cases in Table 2 satisfy (16) in Lemma 1, which proves that the pseudo-deterministic policy is optimal for the GLTB problem. In order to investigate the worst case performance of the pseudo-deterministic policy, we conduct numerical experiments with the parameter values, given in Table 4, which violate the condition (16). To this end,

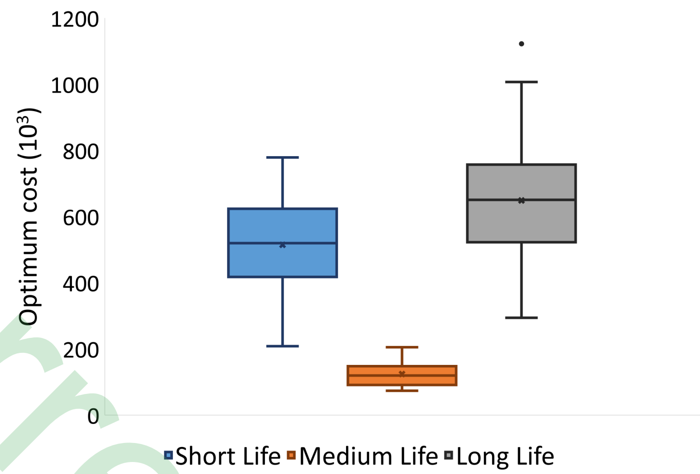

Figure 9: Operational costs for the three different product types in Table 3.

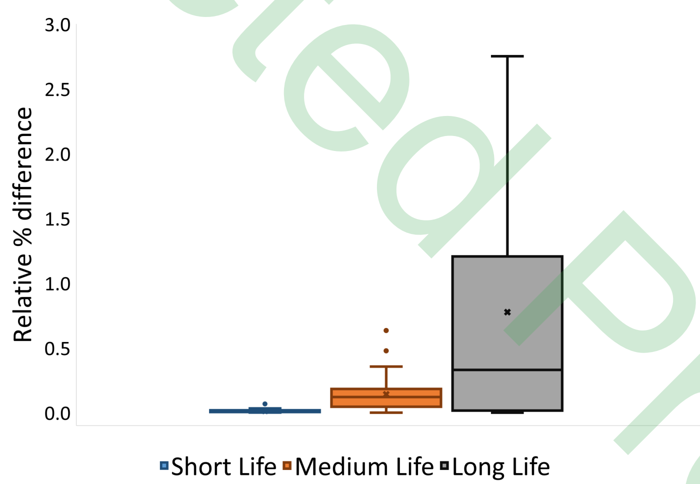

Figure 10: Percentage of relative cost efficiency for the three product types.

we set q to the values close to 1 and choose $c_a(0)$ such that (16) fails at least one part of the planning horizon. Note that all unspecified parameters are equal to the values given in Table 1.

Our results in Table 4 indicate that pseudo-deterministic policy performs near-optimally. The difference between the optimal policy and the pseudo-deterministic policy appears when we $q = 0.999$. Even in those cases the cost difference does not exceed 0.02 %. Recall that the optimal solution of the GLTB problem is presented in Lemma 2 for $q = 1$. Therefore, the parameters of the pseudo-deterministic solution can easily be modified for those cases using our analytical results (Lemma 2).

Table 4: Worst case performance of pseudo-deterministic policy.

q	$c_a(0)$	$x^*(P)$	$x^*(Q)$	$\tau^*(Q)$	$\upsilon(Q)$ vs. $\upsilon(P)$
0.999	35	0	0	0	0.00
	70	0	1	22	0.01
	105	0	1	44	0.01
	140	0	1	66	0.02
0.95	70	16	16	22	0.00
	120	27	27	44	0.00
	140	29	29	66	0.00
	210	34	34	66	0.00
	280	36	36	66	0.00
0.9	120	44	44	66	0.00
	240	62	62	66	0.00
	360	67	67	66	0.00
	480	70	70	66	0.00

6 Conclusion

This paper is a continuation of Teunter and Fortuin (1999), Pourakbar et al. (2012), Frenk et al. (2019b) and Frenk et al. (2019a). In Teunter and Fortuin (1999) it is assumed that during the so-called end-of-life phase of a product one always uses the so-called repair-replacement policy and never switches to an alternative policy. In many practical problems, other end-of-life management strategies are utilized to avoid (possibly) high holding costs of Last Time Buy orders. Finding a substitute part is recognized to be a good alternative policy that can be executed. As the cost of such a policy becomes cheaper over time Pourakbar et al. (2012) proposed the idea of switching to a alternative policy instead of holding inventory for the rest of the planning horizon. Accordingly, they extended the class of policies and proposed a heuristic procedure to determine a optimal policy among this larger class. Subsequently for the important subclass of deterministic policies Frenk et al. (2019b) gives an exact procedure to determine the optimal deterministic policy. Also, in Frenk et al. (2019a) an algorithm was proposed to identify the optimal so-called dynamic policy by studying in more detail the problem originally considered in Pourakbar et al. (2012). In the current paper we consider the same problem and introduce among the set of all dynamic policies the subclass of pseudo-deterministic policies. Under the assumption that the substitution policy cost function and the arrival intensity function are piece-wise constant functions we show it is easy to identify the optimal pseudo-deterministic policy. In our computational section we then compare the objective value of the optimal pseudo-deterministic policy with the optimal objective value of the more general optimal stopping problem and show numerically for different scenarios that these objective values are close. This can be explained since the class of deterministic polices avoid possibly high penalty costs and in this case an optimal policy avoids these penalty costs. This empirical evidence and its intuitive explanation suggests that the simpler set of pseudo-deterministic policies can serve as an approximation of the optimal policy within the more general optimal stopping problem.This justifies a detailed study of these class of policies.

Acknowledgement

The authors like to thank the anonymous referees for their constructive comments which improved the presentation of the paper.

References

Ahmad, S., and Butt, M. M. (2012). Can after sale service generate brand equity? *Marketing Intelligence & Planning*, **30**, 307–323. (Cited on page 2.)

Bayındır, Z. P., Erkip, N., and Güllü, R. (2007). Assessing the benefits of remanufacturing option under one-way substitution and capacity constraint. *Computers and Operations Research*, **34**, 487–514. (Cited on page 6.)

Behfard, S., van der Heijden, M. C., Al Hanbali, A., and Zijm, W. H. (2015). Last time buy and repair decisions for spare parts. *European Journal of Operational Research*, **244**, 498–510. (Cited on pages 3 and 5.)

Bradley, J. R., and Guerrero, H. H. (2008). Product design for life-cycle mismatch. *Production and Operations Management*, **17**, 497–512. (Cited on page 5.)

Bradley, J. R., and Guerrero, H. H. (2009). Lifetime buy decisions with multiple obsolete parts. *Production and Operations Management*, **18**, 114–126. (Cited on page 5.)

Çınlar, E. (2011). *Probability and Stochastic.* Vol 261, Springer Science & Business Media. (Cited on pages 7, 9, and 18.)

Cohen, M. A., Agrawal, N., and Agrawal, V. (2006). Winning in the aftermarket. *Harvard Business Review*, **84**, 129. (Cited on page 2.)

Dietrich S. and Daryl J. D. (1983). *Comparison Methods for Queues and Other Stochastic Models*. Wiley, New York. (Cited on page 44.)

Fortuin, L. (1980). The all-time requirement of spare parts for service after sales-theoretical analysis and practical results. *International Journal of Operations & Production Management*, **1**, 59–70. (Cited on pages 3 and 5.)

Fortuin, L. (1981). Reduction of all-time requirements for spare parts. *International Journal of Operations & Production Management*, **2**, 29–37. (Cited on pages 3 and 5.)

Frenk, J. B. G., Javadi, S., Pourakbar, M., and Sezer, S. O. (2019). An exact static solution approach for the service parts end-of-life inventory problem. *European Journal of Operational Research*, **272**, 496–504. (Cited on pages 3, 4, 5, 6, 7, 8, 13, 14, 23, 24, 28, 29, 36, and 37.)

Frenk, J. B. G., Javadi, S., Pourakbar, M., and Sezer, S. O. (2019b). An exact static solution approach for the service parts end-of-life inventory problem. *European Journal of Operational Research*, **272**, 496–504. (Cited on pages 3, 4, 5, 7, 16, 17, 18, 36, and 37.)

van der Heijden, M., and Iskandar, B. P. (2013). Last time buy decisions for products sold under warranty. *European Journal of Operational Research*, **224**, 302–312. (Cited on page 6.)

Hekimoğlu, M., van der Laan, E., and Dekker, R. (2018). Markov-modulated analysis of a spare parts system with random lead times and disruption risks. *European Journal of Operational Research*, **269**, 909–922. (Cited on pages 2 and 3.)

Huang, W., Kulkarni, V., and Swaminathan, J. M. (2007). Coordinated inventory planning for new and old products under warranty. *Probability in the Engineering and Informational Sciences*, **21**, 261–287. (Cited on page 6.)

Huang, W., Kulkarni, V., and Swaminathan, J. M. (2008). Managing the inventory of an item with a replacement warranty. *Management Science*, **54**, 1441–1452. (Cited on page 6.)

Hur, M., Keskin, B. B., and Schmidt, C. P. (2018). End-of-life inventory control of aircraft spare parts under performance based logistics. *International Journal of Production Economics*, **204**, 186–203. (Cited on page 5.)

Inderfurth, K., and Kleber, R. (2013). An advanced heuristic for multiple-option spare parts procurement after end-of-production. *Production and Operations Management*, **22**, 54–70. (Cited on page 6.)

Inderfurth, K., and Mukherjee, K. (2008). Decision support for spare parts acquisition in post product life cycle. *Central European Journal of Operations Research*, **16**, 17–42. (Cited on page 6.)

Javadi S. (2018). On the end-of-life inventory problem (Doctoral dissertation). PhD thesis, Sabanci University, Istanbul, Turkey. (Cited on pages 4 and 6.)

Kim, B., and Park, S. (2008). Optimal pricing, EOL (end of life) warranty, and spare parts manufacturing strategy amid product transition. *European Journal of Operational Research*, **188**, 723–745. (Cited on page 6.)

Van Kooten, J. P., and Tan, T. (2009). The final order problem for repairable spare parts under condemnation. *Journal of the Operational Research Society*, **60**, 1449–1461. (Cited on page 5.)

Kleinrock, L. (1975). *Queueing Sytems*. Vol 1. Wiley, New York. (Cited on page 20.)

Krikke, H., and van Der Laan, E. (2011). Last Time Buy and control policies with phase-out returns: a case study in plant control systems. *International Journal of Production Research*, **49**, 5183–5206. (Cited on pages 3 and 5.)

Li, X., Dekker, R., Heij, C., and Hekimoğlu, M. (2016). Assessing end-of-supply risk of spare parts using the proportional hazard model. *Decision Sciences*, **47**, 373–394. (Cited on page 2.)

Padmanabhan, V., and Rao, R. C. (1993). Warranty policy and extended service contracts: Theory and an application to automobiles. *Marketing Science*, **12**, 230–247. (Cited on page 2.)

Pourakbar, M., van derLaan, E., and Dekker, R. (2014). End-of-life inventory problem with phaseout returns. *Production and Operations Management*, **23**, 1561–1576. (Cited on page 5.)

Pourakbar, M., Frenk, J. and Dekker, R, (2012). End-of-life inventory decisions for consumer electronics service parts. *Production and Operations Management*, **21**, 889–906. (Cited on pages 5, 6, 36, and 37.)

Ross, S. M. (1997). *Introduction to Probability Models*. Academic Press, New York. (Cited on pages 44 and 47.)

Sahyouni, K., Savaskan, R. C., and Daskin, M. S. (2010). The effect of lifetime buys on warranty repair operations. *Journal of the Operational Research Society*, **61**, 790–803. (Cited on pages 5 and 6.)

Shen, Y., and Willems, S. P. (2014). Modeling sourcing strategies to mitigate part obsolescence. *European Journal of Operational Research*, **23**, 522–533. (Cited on pages 2 and 5.)

Shi, Z. (2019). Optimal remanufacturing and acquisition decisions in warranty service considering part obsolescence. *Computers and Industrial Engineering*, **135**, 766–779. (Cited on page 5.)

Shi, Z., and Liu, S. (2020). Optimal inventory control and design refresh selection in managing part obsolescence. *European Journal of Operational Research*, **287**, 133–144. (Cited on pages 2, 5, and 6.)

Solomon, R., Sandborn, P. A., and Pecht, M. G. (2000). Electronic part life cycle concepts and obsolescence forecasting. *IEEE Transactions on Components and Packaging Technologies*, **23**, 707–717. (Cited on page 2.)

Teunter, R. H., and Fortuin, L. (1999). End-of-life service. *International Journal of Production Economics*, **59**, 487–497. (Cited on pages 2, 3, 5, and 36.)

Teunter, R. H., and Haneveld, W. K. K. (1998). The final orders problem. *European Journal of Operational Research*, **107**, 35–44. (Cited on page 5.)

Westerweel, B., Basten, R. J., and van Houtum, G. J. (2018). Traditional or additive manufacturing? Assessing component design options through lifecycle cost analysis. *European Journal of Operational Research*, **270**, 570–585. (Cited on page 2.)

A Appendix: Proof of the results.

In this Appendix we list the proofs of the results shown in this paper.

Proof of Lemma 2. Since $q=1$ the arrival process $N_0=\{N_0(t):t\geq 0\}$ of non repairable items is the zero process and so by relation (13) we obtain introducing

$$k(T):=\int_0^T e^{-\delta u}\lambda(u)c_a(u)du \tag{75}$$

that

$$C(x,\tau)=c_{scr}x+\mathbb{E}\left(\int_0^\tau e^{-\delta u}[c_{se}+c_{re}-c_a(u)]\lambda(u)du\right)+x(h-\delta c_{scr})\mathbb{E}\left(\int_0^\tau e^{-\delta u}du\right)+k(T). \tag{76}$$

This shows for every $\tau\in\mathbb{F}$ that $\inf_{x\in\mathbb{Z}_+}\{c(x)+C(x,\tau)\}=\gamma(\tau)+\mathbb{E}\left(\int_0^\tau e^{-\delta u}[c_{se}+c_{re}-c_a(u)]\lambda(u)du\right)+k(T)$ with the value $\gamma(\tau)$ given by

$$\gamma(\tau):=\inf_{x\in\mathbb{Z}_+}\left\{c(x)+c_{scr}x+x(h-\delta c_{scr})\mathbb{E}\left(\int_0^\tau e^{-\delta u}du\right)\right\}.$$

Since $h-\delta c_{scr}\geq 0$ we obtain by our remark after relation (4) that $\gamma(\tau)=0$ and so

$$\upsilon(P)=\inf_{\tau\in\mathbb{F}}\left\{\mathbb{E}\left(\int_0^\tau e^{-\delta u}[c_{se}+c_{re}-c_a(u)]\lambda(u)du\right)\right\}+k(T).$$

Since the integrand function in the above optimization problem does not depend on the stopping time τ it is easy to see that an optimal switching time is given by some deterministic stopping time $\tau\in\mathbb{F}_0$ and so

$$\inf_{\tau\in\mathbb{F}}\left\{\mathbb{E}\left(\int_0^\tau e^{-\delta u}[c_{se}+c_{re}-c_a(u)]\lambda(u)du\right)\right\}=\inf_{\tau\in\mathbb{F}_0}\left\{\int_0^\tau e^{-\delta u}[c_{se}+c_{re}-c_a(u)]\lambda(u)du\right\}.$$

We know that the function c_a is non-increasing and by standard first order condition arguments the optimal solution of the above optimization problem is given by ς_1 and no ordering. This has the objective value $\int_0^T e^{-\delta u}\lambda(u)\min\{c_{se}+c_{re},c_a(u)\}du$ and we have shown the result of Lemma 2.

□

Proof of Lemma 3. Since $c_{csr}\geq 0$ we obtain that all operational costs components mentioned in the beginning of subsection 2.3 are non-negative and so we obtain $C(x,\tau)\geq 0$ for every $x\geq 0$ and $\tau\in\mathbb{F}$. Since $c_{scr}\geq 0$ and hence c is increasing and the vector $(0,0)$ is a feasible solution this implies that for every $x\geq x_U+1$

$$c(x)+C(x,\tau)\geq c(x)>\int_0^T e^{-\delta u}\lambda(u)c_a(u)du=C(0,0)\geq\upsilon(P) \tag{77}$$

showing the desired result. □

Proof of Lemma 5. Since by Lemma 4 we know that the objective function is bounded below by the function $L(x,\tau)$ and $x\to L(x,\tau)$ is increasing for every $\tau\in\mathbb{F}$, we obtain

$$\upsilon(P)=\inf_{x\in\mathbb{Z}_+,\tau\in\mathbb{F},0\leq\tau\leq T}\{c(x)+C(x,\tau)\}\geq\inf_{\tau\in\mathbb{F},0\leq\tau\leq T}\{L(0,\tau)\}. \tag{78}$$

Similar as in the proof of Lemma 2 it follows using relation (21) that

$$\begin{aligned}\inf_{\tau\in\mathbb{F},0\leq\tau\leq T}\{L(0,\tau)\}&=\inf_{\tau\in\mathbb{F}_0}\left\{\int_0^\tau e^{-\delta u}\lambda(u)[c_{se}+qc_{re}-c_a(u)]du\right\}\\&=\int_0^{\zeta_q}e^{-\delta u}\lambda(u)[c_{se}+qc_{re}-c_a(u)]du\end{aligned}$$

with ζ_q given in relation (17) and we have verified the result. □

Proof of Lemma 6. It follows by relations (19) for $c_{scr} \geq 0$ and (21) for $c_{csr} \leq 0$ that

$$L(x,\tau) \geq q(x) + \mathbb{E}\left(\int_0^\tau e^{-\delta u}\lambda(u)[c_{se} + qc_{re} - c_a(u)]du\right) + \int_0^T e^{-\delta u}\lambda(u)c_a(u)du$$

with $q(x) := c(x) + \min\{c_{scr}, 0\}x$. Since $c_{ap}(u) \geq c_{se}$ for every $0 \leq u \leq T$ we may apply Lemma 4 and so for every $x \in \mathbb{Z}_+$ we obtain by the previous inequality

$$\inf_{\tau\in\mathbb{F},0\leq\tau\leq T}\{c(x) + C(x,\tau)\} \geq q(x) + \inf_{\tau\in\mathbb{F},0\leq\tau\leq T}\left\{\mathbb{E}\left(\int_0^\tau e^{-\delta u}\lambda(u)[c_{se} + qc_{re} - c_a(u)]du\right)\right\} + k(T)$$

with $k(T)$ listed in relation (75). This implies using the same arguments as in Lemma 5 that

$$\inf_{\tau\in\mathbb{F},0\leq\tau\leq T}\{c(x) + C(x,\tau)\} \geq q(x) + \int_0^T e^{-\delta u}\lambda(u)\min\{c_{se} + qc_{re}, c_a(u)\}du. \tag{79}$$

Hence it follows for every $x \geq \bar{x}_U$ that

$$\inf_{\tau\in\mathbb{F},0\leq\tau\leq T}\{c(x) + C(x,\tau)\} > \int_0^T e^{-\delta u}\lambda(u)c_a(u)du = C(0,0) \geq \upsilon(P)$$

and the result is verified. □

Proof of Lemma 7. It is sufficient to give the proof of the first result only. The second claim follows replacing f by $-f$. Since f is non-positive it is obvious that the function G is non-increasing. To show that the function G is discrete convex, we note by Doob's stopping theorem for any $0 < \tau \leq T$

$$\mathbb{E}\left(\int_0^{\tau\wedge\sigma_x} f(u)\mu(u)du\right) = \mathbb{E}\left(\int_0^{\tau\wedge\sigma_x} f(u)dN(u)\right) = \mathbb{E}\left(\textstyle\sum_{k=1}^x f(\sigma_k)1_{\{\sigma_k\leq\tau\}}\right).$$

with σ_k the hitting time at level k of the nonhomogeneous Poisson process N. Since σ_k has a continuous cdf this yields

$$\mathbb{E}\left(\int_0^{\tau\wedge\sigma_x} f(u)\mu(u)du\right) = \mathbb{E}\left(\textstyle\sum_{k=1}^x f(\sigma_k)1_{\{\sigma_k<\tau\}}\right)$$

and hence for every $x \in \mathbb{Z}_+$ it follows

$$\Delta_x G(x) := G(x+1) - G(x) = \mathbb{E}\left(f(\sigma_{x+1})1_{\{\sigma_{x+1}<\tau\}}\right). \tag{80}$$

Using $\sigma_{x+1} \leq \sigma_{x+2}$ and hence $1_{\{\sigma_{x+1}<\tau\}} \geq 1_{\{\sigma_{x+2}<\tau\}}$ and f non-decreasing and non-positive on $[0,\tau)$ we obtain

$$f(\sigma_{x+1})1_{\{\sigma_{x+1}<\tau\}} \leq f(\sigma_{x+2})1_{\{\sigma_{x+2}<\tau\}}.$$

This shows applying relation (80) that for every $x \in \mathbb{Z}_+$

$$\Delta G(x) = \mathbb{E}(f(\sigma_{x+1})1_{\{\sigma_{x+1}<\tau\}}) \leq \mathbb{E}(f(\sigma_{x+2})1_{\{\sigma_{x+2}<\tau\}}) = \Delta G(x+1),$$

and we have verified the discrete convexity property. □

Proof of Lemma 8. We will only verify the first part of this result. The proof of the second part is similar. Since $c_{ap}(u) \geq c_{se} - c_{scr}$ for every $u < \tau$ and c_{ap} is non-increasing we obtain that the function $u \mapsto e^{-\delta u}(c_{se} - c_{scr} - c_{ap}(u))$ is non-positive and non-decreasing on $[0,\tau)$. Hence, by Lemma 7 the function

$$x \mapsto \mathbb{E}\left(\int_0^{\tau\wedge\sigma_x} e^{-\delta u}\lambda_0(u)(c_{se} - c_{scr} - c_{ap}(u))du\right)$$

is discrete convex. Since the random function $x \mapsto ((x - N_0(u))^+)$ is also discrete convex and $h - \delta c_{scr} \geq 0$ it follows from relation (13) that the function $x \mapsto C(x,\tau)$ is discrete convex again. Finally, the discrete convexity of c completes the proof. □

Proof of Lemma 9. For any $x \in \mathbb{Z}_+$ using c_a decreasing it follows immediately due to $c_a(0) \leq c_{se} + qc_{re} - (1-q)c_{scr}$ and relation (26) that optimization problem ($Q(x)$) has optimal solution $\tau = 0$. This shows using $x \to c(x) + c_{scr}x$ is increasing that it is optimal in optimization problem (Q) not to order and immediately start with the alternative policy with objective value $\int_0^T e^{-\delta u}\lambda(u)c_a(u)du$. □

Proof of Lemma 10. For the selected arrival rate and alternative policy cost function in (30) and (29) it is clear for every $x \in \mathbb{Z}_+$ that the function $\tau \to c(x) + C(x \wedge \sigma_x)$ with $C(x \wedge \sigma_x)$ listed in relation (26) is continuous on $(0,T)$ satisfying both $c(x) + C(x, 0 \wedge \sigma_x) = \lim_{\tau \downarrow 0} c(x) + C(x, \tau \wedge \sigma_x)$ and $c(x) + C(x, T \wedge \sigma_x) = \lim_{\tau \uparrow T} c(x) + C(x, \tau \wedge \sigma_x)$. This shows $\psi(x) = \min_{i=1,\ldots,n} \psi_i(x)$. By relation (26) it follows for every $x \in \mathbb{Z}_+$ that the derivative of the function $c(x) + C(x, \tau \wedge \sigma_x)$ with respect to τ for any $a_i < \tau < a_{i+1}, i = 1, \ldots, n$ is given by

$$\frac{\partial C}{\partial \tau}(x, \tau \wedge \sigma_x) = e^{-\delta\tau}\left(\lambda_i[c_{se} + qc_{re} - (1-q)c_{scr} - c_i]\mathbb{P}(N_0(\tau) < x) + (h - \delta c_{scr})\mathbb{E}((x - N_0(\tau))^+)\right).$$

Clearly for $c_i \leq c_{se} + qc_{re} - (1-q)c_{scr}$ it follows that $\frac{\partial C}{\partial \tau}(x, \tau \wedge \sigma_x) \geq 0$ and so we may immediately conclude using $c_1 \geq c_2 \geq \ldots \geq c_n$ that the continuous function $\tau \mapsto c(x) + C(x, \tau \wedge \sigma_x)$ is increasing on $[a_{n^*+1}, T]$ for every x. This shows $\psi(x) = \min_{i=1,\ldots,n^*} \psi_i(x)$ and we have shown the result. □

Proof of Lemma 11. For every $x \in \mathbb{N}$ we obtain by writing out the expectation

$$\mathbb{E}((x - N(t))^+) = x\mathbb{P}(N(t) < x) - \sum_{n=1}^{x-1} n\mathbb{P}(N(t) = n)$$

that

$$\frac{\mathbb{E}((x - N(t))^+)}{\mathbb{P}(N(t) < x)} = x - \frac{\sum_{n=1}^{x-1} n\mathbb{P}(N(t) = n)}{\mathbb{P}(N(t) < x)}. \tag{81}$$

Since the random variable $N(t)$ has a Poisson cdf with parameter $\theta(t)$ (cf.Ross (1997)) it follows

$$\frac{\sum_{n=1}^{x-1} n\mathbb{P}(N(t) = n)}{\mathbb{P}(N(t) < x)} = \frac{\sum_{n=1}^{x-1} \frac{\Theta(t)^n}{(n-1)!}}{\sum_{n=0}^{x-1} \frac{\Theta(t)^n}{n!}} = \Theta(t)(1 - B(x-1, \Theta(t)))$$

and substituting this into relation (81) the result follows. □

Proof of Lemma 12. By the interpretation of $B(x,t)$ and expected service time equals 1 it follows by the formula of Little (cf.**?**) that $t(1 - B(x,t))$ denotes the long run average number of busy serves in the system in a pure Markovian loss system with arrival rate $\lambda = t > 0$ and departure rate $\mu = 1$. Since each server serves exactly one customer and we consider a pure loss system the long run average number of busy servers equals the long run average number of customers in the system and so

$$t(1 - B(x,t)) = \mathbb{E}(\mathbf{Z}^{(t)}(\infty)) \tag{82}$$

with

$$\mathbf{Z}^{(t)}(\infty) := \lim_{s \uparrow \infty} \frac{1}{s}\int_0^s \mathbf{Z}^{(t)}(v)dv$$

denoting the longrun average number of customers in the Markovian loss system in the equilibrium situation having arrival rate t and departure rate 1. Since the Markovian loss system is a birth-death process we obtain by Proposition 4.2.10 of Stoyan and Daley (1983) that

$$\mathbf{Z}^{(t)}(\infty) \geq_d \mathbf{Z}^{(s)}(\infty)$$

for $t > s$ with $\geq_d$ denoting first order dominance and this shows that $\mathbb{E}(\mathbf{Z}^{(t)}(\infty)) \geq \mathbb{E}(\mathbf{Z}^{(s)}(\infty))$. Hence by (82) it follows that the function $t(1 - B(x,t)$ is increasing in t. Since $t(1 - B(x,t))$ represents the number of busy servers and so for the arrival rate going to infinity on average all x servers will be busy it follows that $\lim_{t \uparrow \infty} t(1 - B(t,x)) = x$. □

Proof of Lemma 13. Applying relation (42) it is sufficient to verify for any $i \leq n_0^*$ that

$$\psi_i(x) = \min\{c(x) + C(x, a_i \wedge \sigma_x), c(x) + C(x, a_{i+1} \wedge \sigma_x)\}.$$

By Lemma 12 and using $h - \delta c_{scr} \geq 0$ and Λ_0 increasing it follows that the function κ_i listed in (35) is decreasing. This shows for $\kappa_i(x, a_i) \leq 0$ that by (36) the partial derivative $\frac{\partial C}{\partial \tau}(x, \tau \wedge \sigma_x)$ is non-positive on (a_i, a_{i+1}) and hence the minimum is attained at a_{i+1}. Similar for $\kappa_i(x, a_i) > 0$ we distinguish the mutually exclusive cases $\kappa_i(x, a_{i+1}) > 0$ and $\kappa_i(x, a_{i+1}) \leq 0$. For the first subcase the partial derivative $\frac{\partial C}{\partial \tau}(x, \tau \wedge \sigma_x)$ is non-negative on (a_i, a_{i+1}) and so the minimum is attained at a_i. For the second subcase we conclude using again a standard calculus argument as before that the minimum is either attained at a_i or a_{i+1} showing the desired result. □

Proof of Lemma 14. For $x = 0$ and $i = 1, ..., n+1$ it follows by relation (26) that $C(0, a_i \wedge \sigma_0) = \int_0^T e^{-\delta u} \lambda(u) c_a(u) du$ and we obtain the result in relation (46) using (29) and (30). To verify relation (47) we first observe for every $x \in \mathbb{Z}_+$ that

$$\mathbb{E}((x+1-N_0(u))^+) - \mathbb{E}(x - N_0(u))^+) = \mathbb{P}(N_0(u) \leq x)$$

Applying again relation (26) this implies for $i = 2, ..., n+1$ and $x \in \mathbb{Z}_+$

$$\Delta C(x, a_i \wedge \sigma_x) = \begin{cases} c_{scr} + \int_0^{a_i} e^{-\delta u} \lambda(u)[c_{se} + qc_{re} - (1-q)c_{scr} - c_a(u)] \mathbb{P}(N_0(u) = x) du \\ +(h - \delta c_{scr}) \int_0^{a_i} e^{-\delta u} \mathbb{P}(N_0(u) \leq x) du. \end{cases} \tag{83}$$

Substituting now into relation (83) the particular choice of the arrival rate and the alternative policy cost function given in (29) and (30) yields

$$\Delta C(x, a_i \wedge \sigma_x) = \begin{cases} c_{scr} + \sum_{j=1}^{i-1} (c_{se} + qc_{re} - (1-q)c_{scr} - c_j) \lambda_j \int_{a_j}^{a_{j+1}} e^{-\delta u} \mathbb{P}(N_0(u) = x) du \\ +(h - \delta c_{scr}) \sum_{j=1}^{i-1} \int_{a_j}^{a_{j+1}} e^{-\delta u} \mathbb{P}(N_0(u) \leq x) du. \end{cases} \tag{84}$$

To simplify the above expression we observe, since the non-homogenous Poisson process N_0 has the constant arrival rate $(1-q)\lambda_j$ on the interval (a_j, a_{j+1}), that by Lemma 21 using $\rho(u) = e^{-\delta u}$ and applying relation (84) that for $i = 2, ..., n+1$

$$\Delta C(x, a_i \wedge \sigma_x) = \begin{cases} c_{scr} + \sum_{j=1}^{i-1} \left(\frac{c_{se} + qc_{re} - (1-q)c_{scr} - c_j}{1-q}\right) [e^{-\delta a_j} \mathbb{P}(N_0(a_j) \leq x) - e^{-\delta a_{j+1}} \mathbb{P}(N_0(a_{j+1}) \leq x)] \\ + \sum_{j=1}^{i-1} [h + \delta(\frac{c_j - c_{se} - qc_{re}}{1-q})] \int_{a_j}^{a_{j+1}} e^{-\delta u} \mathbb{P}(N_0(u) \leq x) du. \end{cases} \tag{85}$$

and this shows the desired result. □

Proof of Lemma 15. To give a proof of Lemma 15 we first show by induction that the sequence $\alpha_{kj}, k \in \mathbb{Z}_+$ given by

$$\alpha_{kj} := \int_{a_j}^{a_{j+1}} e^{-\delta u} \mathbb{P}(N_0(u) = k) du \tag{86}$$

satisfies

$$\alpha_{kj} = \frac{1}{\delta + (1-q)\lambda_j} \sum_{m=0}^{k} \left(\frac{(1-q)\lambda_j}{\delta + (1-q)\lambda_j} \right)^m \theta_j(k-m) \tag{87}$$

with

$$\theta_j(k) := e^{-\delta a_j} \mathbb{P}(N_0(a_j) = k) - e^{-\delta a_{j+1}} \mathbb{P}(N_0(a_{j+1}) = k), k \in \mathbb{Z}_+$$

Introducing the functions $f:(0,T]\to\mathbb{R}$ and $g_k:(0,T]\to\mathbb{R}$, $k\in\mathbb{Z}_+$ given by $f(u):=e^{-(\delta u+(1-q)\Lambda(u))}$ and $g_k(u):=\frac{((1-q)\Lambda(u))^k}{k!}$ it follows for every $k\in\mathbb{Z}_+$ and $a_j<u<a_{j+1}$ that

$$(fg_k)(u)=e^{-\delta u}\mathbb{P}(N_0(u)=k),(fg'_{k+1})(u)=(1-q)\lambda_j e^{-\delta u}\mathbb{P}(N_0(u)=k) \tag{88}$$

Since

$$f'(u)=-(\delta+(1-q)\lambda_j)f(u) \tag{89}$$

for every $a_j<u<a_{j+1}$ we obtain

$$\int_{a_j}^{a_{j+1}} f(u)du=\frac{f(a_j)-f(a_{j+1})}{\delta+(1-q)\lambda_j}=\frac{\theta_j(0)}{\delta+(1-q)\lambda_j}$$

and this verifies relation (87) for $k=0$. To show relation (87) for $k+1$ we assume that relation (87) holds for k. By partial integration and relations (88) and (89) we conclude

$$-(\delta+(1-q)\lambda_j)\alpha_{k+1j}=-(\delta+(1-q)\lambda_j)\int_{a_j}^{a_{j+1}} f(u)g_{k+1}(u)du=-\theta_j(k+1)-(1-q)\lambda_j\alpha_{kj}$$

This shows that

$$\begin{aligned}\alpha_{k+1j} &= \tfrac{(1-q)\lambda_j}{\delta+(1-q)\lambda_j}\alpha_{kj}+\tfrac{\theta_j(k+1)}{\delta+(1-q)\lambda_j}\\ &= \tfrac{1}{\delta+(1-q)\lambda_j}\textstyle\sum_{m=0}^{k}\left(\tfrac{(1-q)\lambda_j}{\delta+(1-q)\lambda_j}\right)^{m+1}\theta_j(k-m)+\tfrac{\theta_j(k+1)}{\delta+(1-q)\lambda_j}\\ &= \tfrac{1}{\delta+(1-q)\lambda_j}\textstyle\sum_{m=0}^{k+1}\left(\tfrac{(1-q)\lambda_j}{\delta+(1-q)\lambda_j}\right)^{m}\theta_j(k+1-m)\end{aligned}$$

and we have verified relation (87). Using this result we obtain

$$\begin{aligned}\textstyle\int_{a_j}^{a_{j+1}} e^{-\delta u}\mathbb{P}(N_0(u)\le x)du &= \textstyle\sum_{k=0}^{x}\alpha_{kj}\\ &= \tfrac{1}{\delta+(1-q)\lambda_j}\textstyle\sum_{k=0}^{x}\sum_{m=0}^{k}\left(\tfrac{(1-q)\lambda_j}{\delta+(1-q)\lambda_j}\right)^{m}\theta_j(k-m)\\ &= \tfrac{1}{\delta+(1-q)\lambda_j}\textstyle\sum_{m=0}^{x}\sum_{k=m}^{x}\left(\tfrac{(1-q)\lambda_j}{\delta+(1-q)\lambda_j}\right)^{m}\theta_j(k-m)\\ &= \tfrac{1}{\delta+(1-q)\lambda_j}\textstyle\sum_{m=0}^{x}\left(\tfrac{(1-q)\lambda_j}{\delta+(1-q)\lambda_j}\right)^{m}\sum_{k=0}^{x-m}\theta_j(k)\end{aligned}$$

and this completes the proof. □

Proof of Lemma 16. Introduce for every $\tau\in\mathbb{F}$ the stopping time $\tau_\Theta=d_\Theta(\tau)$ with

$$d_\Theta(r)=\textstyle\sum_{k=0}^{N-2}t_{k+1}1_{[t_k,t_{k+1})}(r)+T1_{[T_{N-1},T]}(r).$$

By its definition we obtain $\tau_\Theta\ge\tau,\tau_\Theta\in\mathbb{F}_\Theta$ and $\mathbb{E}(\tau_\Theta-\tau)\le\Delta(\Theta)$. Applying relation (13) it follows by some standard upper bound arguments applied to each separate term in this relation that for every $x\in\mathbb{Z}_+$

$$\mid C(x,\tau_\Theta)-C(x,\tau)\mid\le f_0(x)\Delta(\Theta)$$

This shows

$$\upsilon(P_\Theta)-\upsilon(P)\le\sup_{x\le x_U}f_0(x)\Delta(\Theta)=f_0(x_U)\Delta(\Theta)$$

and we have verified the result. □

Proof of Lemma 18. By the definition of W_N and W_{N-1} in relation (59) and (60) it is obvious that $W_{N-1}(x) \leq c_{scr}x = W_N(x)$ and so the result is verified for $k = N-1$. Assume now that $W_{k+1}(x) \leq W_{k+2}(x)$ for every $x \leq x_U$ and $k \leq N-2$. Since the arrival process N_0 is a homogeneous Poisson process it follows that $N_0^{(k)} \stackrel{d}{=} N_0^{(k+1)}$ and this shows using $W_{k+1}(x) \leq W_{k+2}(x)$ that

$$\mathbb{E}(W_{k+1}((x - N_0^{(k)}(\Delta))^+) \leq \mathbb{E}(W_{k+2}((x - N_0^{(k+1)}(\Delta))^+).$$

If it can be shown that $\overline{B}_k(x) \leq \overline{B}_{k+1}(x)$ we obtain by relation (60) that

$$\begin{aligned} W_k(x) &= \min\{c_{scr}x, \overline{B}_k(x) + e^{-\delta\Delta}\mathbb{E}(W_{k+1}((x - N_0^{(k)}(\Delta))^+) \\ &\leq \min\{c_{scr}x, \overline{B}_{k+1}(x) + e^{-\delta\Delta}\mathbb{E}(W_{k+1}((x - N_0^{(k+2)}(\Delta))^+)\} \\ &= W_{k+1}(x) \end{aligned}$$

and this shows the result. Hence it is sufficient to verify that $\overline{B}_k(x) \leq \overline{B}_{k+1}(x)$ and using relation (66) and c_a a decreasing function this is easy to verify for penalty costs being constant on $[0,T]$. □

Proof of Lemma 19. It follows for $x \in S_k$ that $W_k(x) = c_{scr}x$. Applying Lemma 18 we obtain

$$c_{scr}x = W_k(x) \leq W_{k+1}(x) = \min\left\{c_{scr}x, \overline{B}_{k+1}(x) + e^{-\delta\Delta}\mathbb{E}(W_{k+2}((x - N_0^{(k+1)}(\Delta))^+)\right\} \leq c_{scr}x.$$

This shows $W_{k+1}(x) = c_{scr}$ and we have verified the desired result. □

Lemma 21. *Let N be a non-homogeneous Poisson process with a continuous arrival intensity function μ and ρ some differentiable function. Then it follows for every $x \in \mathbb{Z}_+$ and $1 \leq j \leq n$ that*

$$\int_{a_j}^{a_{j+1}} \rho(u)\mu(u)\mathbb{P}(N(u) = x)du = \int_{a_j}^{a_{j+1}} \rho'(u)\mathbb{P}(N(u) \leq x)du + \rho(a_j)\mathbb{P}(N(a_j) \leq x) - \rho(a_{j+1})\mathbb{P}(N(a_{j+1}) \leq x).$$

Proof of Lemma 21. It is well known for every $x \in \mathbb{Z}_+$ (Ross (1997)) that the function $\chi(u) := \mathbb{P}(N(u) \leq x)$ is differentiable and satisfies $\chi'(u) = \mu(u)\mathbb{P}(N(u) = x)$. This shows that

$$\begin{aligned} \rho(a_{j+1})\mathbb{P}(N(a_{j+1}) \leq x) - \rho(a_j)\mathbb{P}(N(a_{j+1}) \leq x) &= \int_{a_j}^{a_{j+1}} (\rho\chi)'(u)du \\ &= \int_{a_j}^{a_{j+1}} \rho'(u)\chi(u)du + \int_{a_j}^{a_{j+1}} \rho(u)\chi'(u)du \\ &= \int_{a_j}^{a_{j+1}} \rho'^{(u)}\chi(u)du + \int_{a_j}^{a_{j+1}} \rho(u)\chi'(u)du \\ &= \int_{a_j}^{a_{j+1}} \rho'(u)\mathbb{P}(N(u) \leq x)du \\ &\quad + \int_{a_j}^{a_{j+1}} \rho(u)\mu(u)\mathbb{P}(N(u) = x)du \end{aligned}$$

and we obtain the desired result. □