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Lifetime behavior of a new discrete time shock model

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Abstract. In this paper, a new discrete-time shock model is investigated. Under the proposed model, system failure occurs when the inter-arrival time between successive shocks falls below a random critical threshold, denoted by Δ. This model can be viewed as a randomized-threshold extension of the classical delta shock model. Assuming that Δ follows a uniform distribution, we derive the probability mass function of the system's lifetime and analyze the mean time to failure. Furthermore, the lifetime behavior of the system is examined in a Markovian environment. Illustrative numerical results are presented for both the probability mass function of the system's lifetime and the mean time to failure.

Keywords: Discrete time shock model, δ-shock model, Inter-arrival time, Random threshold.

1 Introduction

Shock models are valuable tools for analyzing systems subjected to random external shocks. Understanding how a system behaves under such shocks is essential; consequently, shock models have attracted considerable attention in applied probability, reliability theory, and engineering. Common types of shock models include cumulative shock models, extreme shock models, and run shock models. In a cumulative shock model, system failure results from the accumulated effect of shocks; in an extreme shock model, failure is caused by a single large shock; and in a run shock model, failure is determined by the occurrence of a specified number of consecutive shocks within a given range. For more on these, see, e.g., Gut (1990), Anderson (1998), Mallor and Omey (2001), and Sumita and Shanthikumar (1985).

There are other shock models that have been developed in recent years. The so-called *δ-shock model* is one of them that has attracted more attention. According to the classical δ-shock model, if the inter-arrival time between two consecutive shocks falls below a critical threshold δ, the system fails. Thus, the system's lifetime is defined as $T = \sum_{i=1}^{N} X_i$, where $X_1, X_2, \ldots$ are inter-arrival times, and N is the number of shocks until the failure of the system. Thus, $\{N = n\}$ iff $\{X_1 > \delta, \ldots, X_{n-1} > \delta, X_n \leq \delta\}$. The δ-shock model was first introduced by Li et al. (1999), after which it was widely studied by many researchers; see, e.g., Li et al. (1999), Xu and Li (2004), Li and Kong (2007), Li and Zhao (2007), Wang and Zhang (2007), and Ma and Li (2010). Many extensions and generalizations have been proposed for the δ-shock model. We briefly review some of them below. Eryilmaz (2012) proposed a generalized δ-shock model, in which the system fails when k consecutive inter-arrival times are less than a threshold δ. Eryilmaz (2013) studied a discrete time version of the δ-shock model, where the shocks occur according to a

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binomial process, i.e. the inter-arrival times between successive shocks follow a geometric distribution. Eryilmaz and Bayramoglu (2014) studied the lifetime behavior of the δ-shock model and the censored δ-shock model under the assumption that shocks arrive according to a renewal process with uniformly distributed inter-arrival times. Tuncel and Eryilmaz (2018) investigated the survival function and the mean lifetime of the system failure under the δ-shock model, considering the proportional hazard rate model. Poursaeed (2019) studied the reliability analysis of an extension of the discrete time version of the δ-shock model by considering two different critical thresholds and a probable failure region. Lorvand et al. (2020) discussed a mixed δ-shock model for multi-state systems by assuming a renewal process of shocks where the system fails in three specific states. Goyal et al. (2022) introduced a general δ-shock model in which the recovery time depends on both the inter-arrival time and the magnitude of the shocks. Finkelstein and Cha (2024) revisited the classical δ-shock model and generalized it to the case of renewal processes of external shocks with arbitrary inter-arrival times and arbitrary distribution of the recovery parameter δ.

In this paper, we study a new discrete time shock model in which the system fails if the inter-arrival time between two successive shocks is less than or equal to a random critical threshold, denoted by Δ. This model can be regarded as a randomized-threshold extension of the classical δ-shock model. The motivation for introducing a random threshold stems from the fact that, in many real-world systems, assuming a fixed threshold is unrealistic. Such situations may arise due to heterogeneity and uncertainty in the strength, tolerance, or operating conditions of systems subjected to shock environments.
Throughout this paper, we study the lifetime of the system under the proposed model. In particular, we derive the probability mass function of the system's lifetime and the mean time to failure of the system. Furthermore, the lifetime behavior of the system in a Markovian environment is investigated.

The remainder of this paper is organized as follows. Section 2 introduces the model and the underlying assumptions. The probabilistic behavior of the system's lifetime under the proposed model is derived in Section 3. A Markovian process approach is presented in Section 4. Finally, Section 5 concludes the paper.

2 Description of the model

Assume that a system is subject to a sequence of external random shocks. Let X_i be the inter-arrival time between the ith and $(i+1)$th shocks for $i = 1, 2, \ldots$, and let also inter-arrival times $X_1, X_2, \ldots$ take nonnegative integer values and are independent and identically distributed by an arbitrary discrete probability distribution with the probability mass function (pmf) $\Pr(X = x)$, where X is a generic random variable of X_i's. The performance of the system is such that for a random critical threshold Δ, which is independent of X_i's, if $X_n \leq \Delta$, the system fails. Thus, the lifetime of the system is defined as follows:

$$T_\Delta = \sum_{i=1}^{N} X_i,$$

where

$$\{N = n\} \Leftrightarrow \{X_1 > \Delta, X_2 > \Delta, \ldots, X_{n-1} > \Delta, X_n \leq \Delta\}. \tag{1}$$

By considering an appropriate probability distribution for Δ, the relationship of the new model to the classical δ-shock model can be expressed as in the following remark:

Remark 1. *If Δ has a degenerate distribution at point δ, i.e., $\Pr(\Delta = \delta) = 1$, the model reduces to the classical δ-shock model*

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The proposed model can be applied to a variety of real-world systems, an example of which is described below. Many electrical components used in electronic devices may overheat and fail when exposed to electric shocks. Examples of such components include capacitors, transistors, and CPUs, which can overheat due to electrical disturbances such as high voltage and low current.

Consider a specific component operating in an environment with instantaneous temperature changes that heats up with each shock arriving over discrete time periods. If the next shock reaches the component shortly after the previous one, it may not have sufficient time to cool down and may therefore fail due to overheating. If the heat generated by each shock is the same and the ambient temperature remains constant, a fixed critical threshold for the inter-arrival time between two consecutive shocks could be determined. However, because the ambient temperature varies over time, the cooling time of the component is also affected. As a result, assuming a fixed critical threshold becomes unrealistic, and the critical threshold is more appropriately modeled as a random variable. Therefore, in such scenarios, using the proposed model can provide a more realistic assessment of system survival.

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3 Probability behavior of the system's lifetime

3.1 Properties of the stopping time N

In the following theorem, we derive the pmf of random variable N under the model described in Section 2. We obtain the pmf of N in the following theorem, assuming that the random critical threshold Δ is uniformly distributed.

Theorem 1. *Consider the model described in Section 2. If Δ has support in $\{0,1,2,\dots,a\}$ with the pmf $\Pr(\Delta=\delta)=\frac{1}{a+1}$ for $\delta\in\{0,1,2,\dots,a\}$, then the pmf of N is*

$$\Pr(N=n)=\frac{1}{a+1}\sum_{\delta=0}^{a}(1-\Pr(X\le\delta))^{n-1}\Pr(X\le\delta).$$

Proof. Since X_i's are independent of Δ, by using the definition of N (see (1)) we have

$$\Pr(N=n)=\Pr(X_1>\Delta,X_2>\Delta,\dots,X_{n-1}>\Delta,X_n\le\Delta).$$

Besides, since $\Pr(\Delta=\delta)=\frac{1}{a+1}$ for $\delta=0,1,2,\dots,a$, then

$$\begin{aligned}\Pr(N=n)&=\sum_{\delta=0}^{a}\Pr(X_1>\Delta,X_2>\Delta,\dots,X_{n-1}>\Delta,X_n\le\Delta,\Delta=\delta)\\&=\sum_{\delta=0}^{a}\Pr(X_1>\delta,X_2>\delta,\dots,X_{n-1}>\delta,X_n\le\delta,\Delta=\delta)\\&=\sum_{\delta=0}^{a}\Pr(X_1>\delta,X_2>\delta,\dots,X_{n-1}>\delta,X_n\le\delta)\Pr(\Delta=\delta)\\&=\sum_{\delta=0}^{a}(\Pr(X>\delta))^{n-1}\Pr(X\le\delta)\left(\frac{1}{a+1}\right)\\&=\frac{1}{a+1}\sum_{\delta=0}^{a}(1-\Pr(X\le\delta))^{n-1}\Pr(X\le\delta).\end{aligned}$$

This completes the proof. □

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Corollary 1. *Let the inter-arrival time X have a geometric distribution with mean $\frac{1}{p}$ for $0 < p < 1$. Then, the pmf of N is*

$$\Pr(N = n) = \frac{1}{a+1} \sum_{\delta=0}^{a} \left(q^{\delta+1}\right)^{n-1} (1 - q^{\delta+1}),$$

where $q = 1 - p$.

Proof. We have $\Pr(X = x) = p(1-p)^{x-1}$ for $x = 1, 2, \ldots$ and $0 < p < 1$. Using this in Theorem 1, we have

$$\Pr(N = n) = \frac{1}{a+1} \sum_{\delta=0}^{a} \left(q^{\delta+1}\right)^{n-1} (1 - q^{\delta+1}), \qquad q = 1 - p.$$

Now, we shall show that $\sum_{n=1}^{\infty} \Pr(N = n) = 1$. We have

$$\begin{aligned} \sum_{n=1}^{\infty} \Pr(N = n) &= \sum_{n=1}^{\infty} \left(\frac{1}{a+1} \sum_{\delta=0}^{a} \left(q^{\delta+1}\right)^{n-1} (1 - q^{\delta+1}) \right) \\ &= \frac{1}{a+1} \sum_{\delta=0}^{a} \left(\left(1 - q^{\delta+1}\right) \sum_{n=1}^{\infty} \left(q^{\delta+1}\right)^{n-1} \right) \\ &= \frac{1}{a+1} \sum_{\delta=0}^{a} (1 - q^{\delta+1}) \frac{1}{1 - q^{\delta+1}} \\ &= \frac{1}{a+1} \sum_{\delta=0}^{a} 1 \\ &= 1. \end{aligned}$$

This completes the proof. □

In the next theorem, we obtain the mean of the random variable N.

Theorem 2. *Consider the model described in Section 2. If Δ has support in $\{0, 1, 2, \ldots, a\}$ with the pmf $\Pr(\Delta = \delta) = \frac{1}{a+1}$ for $\delta \in \{0, 1, 2, \ldots, a\}$, then the mean of N is*

$$E(N) = \frac{1}{a+1} \sum_{\delta=0}^{a} \frac{1}{\Pr(X \le \delta)}.$$

Proof. We have

$$\begin{aligned} E(N) &= \sum_{n=1}^{\infty} \frac{1}{a+1} \sum_{\delta=0}^{a} n \Pr(X \le \delta) \left(1 - \Pr(X \le \delta)\right)^{n-1} \\ &= \frac{1}{a+1} \sum_{\delta=0}^{a} \sum_{n=1}^{\infty} n \Pr(X \le \delta) \left(1 - \Pr(X \le \delta)\right)^{n-1} \\ &= \frac{1}{a+1} \sum_{\delta=0}^{a} \Pr(X \le \delta) \sum_{n=1}^{\infty} n \left(1 - \Pr(X \le \delta)\right)^{n-1} \\ &= \frac{1}{a+1} \sum_{\delta=0}^{a} \Pr(X \le \delta) \frac{1}{(\Pr(X \le \delta))^2} \\ &= \frac{1}{a+1} \sum_{\delta=0}^{a} \frac{1}{\Pr(X \le \delta)}. \end{aligned}$$

The proof is complete. □

3.2 The pmf of the system's lifetime

Under the model described in Section 2, the pmf of the system's lifetime is given below.

Theorem 3. *Consider the model described in Section 2, and let the random critical threshold* Δ *take values from* $\{0,1,2,\ldots,a\}$ *with the pmf* $\Pr(\Delta=\delta)=\frac{1}{a+1}$. *If the inter-arrival times follow the geometric distribution with mean* $\frac{1}{p}$, *the pmf of* T_Δ *is*

$$\begin{aligned}\Pr(T_\Delta=t) &= \sum_{\delta=0}^{a}\Pr(T_\Delta=t|\Delta=\delta)\Pr(\Delta=\delta)\\ &= \sum_{\delta=0}^{a}\sum_{n=2}^{\lfloor\frac{t+\delta+1}{\delta+2}\rfloor}\Pr(\Delta=\delta)\left[\binom{t-(n-2)\delta-(n-1)}{n-1}-\binom{t-(n+1)\delta-(n-2)}{n-1}\right]p^n(1-p)^{t-n}.\end{aligned}$$

Proof. The following pattern shows the lifetime of the system. In that way, 1 means that a shock has occurred, and 0 means that no shock has occurred.

$$\underbrace{\underbrace{0\ldots0}_{y_1\geq 0}1\underbrace{0\ldots0}_{y_2>\Delta}1\underbrace{0\ldots0}_{y_3>\Delta}1\ldots1\underbrace{0\ldots0}_{y_{n-1}>\Delta}1\underbrace{0\ldots0}_{y_n\leq\Delta}1}_{t}.$$

Indeed, the number of 0s until a 1 occurs is considered inter-arrival time. The total possible states of the above pattern are equal to the number of integer solutions of the following equation:

$$\underbrace{y_1+y_2+\cdots+y_{n-1}+y_n}_{y_1\geq 0, y_2>\Delta,\ldots,y_{n-1}>\Delta, y_n\leq\Delta}=t-n,$$

where n is the number of 1's. Then we have

$$\underbrace{y_1+y_2+\cdots+y_{n-1}}_{y_1\geq 0, y_2>\Delta+1,\ldots,y_{n-1}>\Delta+1, y_n\leq\Delta}=t-n.$$

This is equivalent to the equation

$$\underbrace{y_1+y_2-(\Delta+1)+\cdots+y_{n-1}-(\Delta+1)+y_n}_{y_1\geq 0, y_2\geq\Delta+1,\ldots,y_{n-1}\geq\Delta+1, y_n\leq\Delta}=t-n-(n-2)(\Delta+1),$$

that is,

$$\underbrace{y_1+y_2'+\cdots+y_{n-1}'+y_n}_{y_1\geq 0, y_2'\geq 0,\ldots,y_{n-1}'\geq 0, y_n\leq\Delta}=t-n-(n-2)\Delta-(n-2).$$

The number of integer solutions of the above equation is equal to the difference between the number of integer solutions of the equations

$$\underbrace{y_1+y_2'+\cdots+y_{n-1}'+y_n}_{y_1\geq 0, y_2'\geq 0,\ldots,y_{n-1}'\geq 0, y_n\geq 0}=t-n-(n-2)\Delta-(n-2), \tag{2}$$

and

$$\underbrace{y_1+y_2'+\cdots+y_{n-1}'+y_n}_{y_1\geq 0, y_2'\geq 0,\ldots,y_{n-1}'\geq 0, y_n\geq\Delta+1}=t-n-(n-2)\Delta-(n-2). \tag{3}$$

The number of integer solutions of (2) is

$$\binom{t-(n-2)\Delta-(n-1)}{n-1},$$

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and the number of integer solutions of (3) is

$$\binom{t-(n+1)\Delta-(n-2)}{n-1}.$$

This completes the proof. □

In the next theorem, we obtain an explicit formula for the mean lifetime of the system, which defines the system's mean time to failure (MTTF).

Theorem 4. *Consider the model described in Section 2, and let the random critical threshold* Δ *take values from* $\{0,1,2,\ldots,a\}$ *with the pmf* $\Pr(\Delta=\delta)=\frac{1}{a+1}$*. The MTTF of the system is*

$$E(T_\Delta)=\frac{E(X)}{a+1}\sum_{\delta=0}^{a}\frac{1}{\Pr(X\leq\delta)}.$$

Proof. The random variable N is a stopping time for $X_1,X_2,\ldots$, i.e., the event $\{N=n\}$ is independent of inter-arrival times $X_{n+1},X_{n+2},\ldots$ for all integers $n\geq 1$. Hence, by using the well-known Wald's identity, the MTTF of the system can be computed as follows:

$$E(T_\Delta)=E\left(\sum_{i=1}^{N}X_i\right)=E(N)E(X_1).$$

By using Theorem 2 for $E(N)$, the desired result is immediately obtained. □

3.3 Numerical results

In this section, we give some numerical results on the pmf and MTTF of the system's lifetime. To this end, suppose a system is subject to random shocks under the model described in Section 2, and let the inter-arrival times between successive shocks have the geometric distribution with mean $\frac{1}{p}$ for $p>0$. Tables 1 and 2 show the pmf and MTTF of the system's lifetime T_Δ for the cases that the random critical threshold Δ follows the uniform distribution on the set $\{0,1,2,3\}$ and $\{0,1,2,3,4,5\}$, respectively. In both tables, incremental values of 0.2, 0.5, and 0.6 are considered for the parameter p, which is equivalent to a decreasing trend for the mean of inter-arrival times between successive shocks.

Table 1: The pmf and MTTF of T_Δ for when $\Delta\sim Unif\{0,1,2,3\}$.

p	$E(T_\Delta)$	t	$\Pr(T_\Delta=t)$
0.2	2.7801	3	0.0798
		4	0.0591
		5	0.0434
		7	0.0269
0.5	0.3678	3	0.0201
		4	0.0101
		5	0.0065
		7	0.0021
0.6	0.3589	3	0.0021
		4	0.0010
		5	0.0002
		7	0.0001

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Table 2: The pmf and MTTF of T_Δ for when $\Delta \sim Unif\{0,1,2,3,4,5\}$.

p	$E(T_\Delta)$	t	$\Pr(T_\Delta = t)$
0.2	2.1571	3	0.0591
		4	0.0487
		5	0.0391
		7	0.0222
0.5	0.7650	3	0.0551
		4	0.0301
		5	0.0011
		7	0.0005
0.6	0.3671	3	0.0014
		4	0.0009
		5	0.0001
		7	0.0000

From Tables 1 and 2, it can be seen that the values of MTTF and pmf decrease with increasing parameter p. This seems reasonable because increasing p leads to a decrease in the mean of inter-arrival times between shocks and, consequently, the probability of occurrence of inter-arrival times in the sets $\{0,1,2,3\}$ and $\{0,1,2,3,4,5\}$ increases, which is equivalent to a decrease in the probability of the system's lifetime T_Δ.

3.4 Estimation problem

This section deals with the estimation of the parameter of the critical threshold distribution. In the model under discussion, we have $\Delta \sim Unif\{0,1,2,\ldots,a\}$ and the inter-arrival times between successive shocks are distributed geometrically with mean $\frac{1}{p}$ for $p > 0$. Assuming that p (and then $q = 1-p$) is known, we seek to obtain an estimator $\hat{a}$ for the parameter a. Note that the parameter p is related to the probability behavior of the random shocks, but the threshold Δ depends on the system's resistance conditions. Thus, when the parameter p is assumed to be known, it means that we have already discovered the probability behavior of random shocks and intend to use inference to understand the resilient behavior of the system through estimating the parameter a of the probability distribution of Δ.

Let $t_1, t_2, \ldots, t_n$ be the observed values of a random sample of the lifetime data based on n independent systems, and let also $\bar{t} = \frac{1}{n}\sum_{i=1}^{n} t_i$ be the sample mean. Using the method of moments and using Theorem 4, we can obtain the moment estimate of the parameter a by solving the following equation:

$$E(T_\Delta) = \frac{\frac{1}{p}}{a+1}\sum_{\delta=0}^{a}\frac{1}{(1-q^{\delta+1})} = \bar{t}. \tag{4}$$

Since a and q are positive, we have

$$\sum_{\delta=0}^{a}\frac{1}{(1-q^{\delta+1})} = \frac{-\psi_q(a+2)+\psi_q(1)+(a+1)\log(q)}{\log(q)},$$

where $\psi_q(z)$ is the q-analogue of the digamma function, which is called the q-digamma function and is defined as $\psi_q(z) = \frac{1}{\Gamma_q(z)}\frac{\partial \Gamma_q(z)}{\partial z}$, where $\Gamma_q(z)$ is the q-gamma function. For more details about these functions, see, e.g., Singh et al. (2016).

Therefore, Eq. (4) can be written as follows:

$$E(T_\Delta) = \frac{-\psi_q(a+2)+\psi_q(1)}{(a+1)p\log(q)} + \frac{1}{p} = \bar{t}. \tag{5}$$

By solving Eq. (5), the estimate $\hat{a}$ of parameter a is obtained. However, it is clear that solving Eq. (5) is not possible through conventional methods and requires complex numerical calculations.

4 Markovian shock occurrences

In the previous section, the results were obtained based on the assumption that the occurrences of shocks are independent. In this section, we study the behavior of the system's lifetime by considering the shock occurrences as a Markov chain. We consider the two following patterns:

$$1\underbrace{\underbrace{0\ldots0}_{y_1>\Delta}1\underbrace{0\ldots0}_{y_2>\Delta}1\underbrace{0\ldots0}_{y_3>\Delta}1\ldots1\underbrace{0\ldots0}_{y_{n-1}>\Delta}1\underbrace{0\ldots0}_{y_n\le\Delta}1}_{t}, \tag{6}$$

and

$$\underbrace{\underbrace{0\ldots0}_{y_1\ge 0}1\underbrace{0\ldots0}_{y_2>\Delta}1\underbrace{0\ldots0}_{y_3>\Delta}1\ldots1\underbrace{0\ldots0}_{y_{n-1}>\Delta}1\underbrace{0\ldots0}_{y_n\le\Delta}1}_{t}. \tag{7}$$

In the next two theorems, we obtain the pmf of the system's lifetime under the above two situations.

Theorem 5. *Consider the model described in Section 2, and let the random critical threshold Δ take values from $\{0,1,2,\ldots,a\}$ with the pmf $\Pr(\Delta=\delta)=\frac{1}{a+1}$. Under the pattern in Eq. (6), the pmf of T_Δ is*

$$\Pr(T_\Delta=t)=\sum_{\delta=0}^{a}\Pr(\Delta=\delta)\sum_{n=2}^{\lfloor\frac{t+\delta+1}{\delta+2}\rfloor}\left[\binom{t-(n-1)(\delta+1)-1}{n-1}-\binom{t-n\delta-(n-2)}{n-1}\right]p_1p_{10}^{n-1}p_{00}^{t-2n+2}p_{01}^{n-1},$$

where $p_{ij}=\Pr(I_n=j|I_{n-1}=i)$ for $i,j=0,1$, and

$$I_n=\begin{cases}1 & \text{if a shock occurs at time } n,\\ 0 & \text{otherwise,}\end{cases}$$

and $p_1=\Pr(I_1=1)$.

Proof. In the pattern (6), we let the number of shocks be n and we have

$$\underset{y_1>\Delta,y_2>\Delta,\ldots,y_{n-1}>\Delta,y_n\le\Delta}{y_1+y_2+\cdots+y_{n-1}+y_n=t-n.}$$

This equation is equivalent to the equation

$$\underset{y_1\ge\Delta+1,y_2\ge\Delta+1,\ldots,y_{n-1}\ge\Delta+1,y_n\le\Delta}{y_1+y_2+\cdots+y_{n-1}=t-n}. \tag{8}$$

We can rewrite Eq. (8) as follows:

$$\underset{y_1\ge0,y_2\ge0,\ldots,y_{n-1}\ge0,y_n\le\Delta}{y_1-(\Delta+1)+y_2-(\Delta+1)+\cdots+y_{n-1}-(\Delta+1)+y_n=t-n-(n-1)(\Delta+1),}$$

that is equivalent to the equation

$$\underset{y'_1\ge0,y'_2\ge0,\ldots,y'_{n-1}\ge0,y_n\le\Delta}{y'_1+y'_2+\cdots+y'_{n-1}+y_n=t-n-(n-1)(\Delta+1).} \tag{9}$$

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The number of integer solutions of Eq. (9) is equal to the difference between the number of integer solutions of the following two equations:

$$\underset{y'_1\geq 0, y'_2\geq 0,\ldots,y'_{n-1}\geq 0, y_n\geq 0}{y'_1+y'_2+\cdots+y'_{n-1}+y_n = t-n-(n-1)(\Delta+l),} \tag{10}$$

and

$$\underset{y'_1\geq 0, y'_2\geq 0,\ldots,y'_{n-1}\geq 0, y_n\geq \Delta}{y'_1+y'_2+\cdots+y'_{n-1}+y_n = t-n-(n-1)(\Delta+1).} \tag{11}$$

The number of integer solutions of Eq. (10) is

$$\binom{t-(n-1)(\Delta+1)-1}{n-1},$$

and the number of integer solutions of Eq. (11) is

$$\binom{t-n\Delta-(n-2)}{n-1}.$$

Thus, the number of integer solutions of Eq. (9) is

$$\binom{t-(n-1)(\Delta+1)-1}{n-1}-\binom{t-n\Delta-(n-2)}{n-1}.$$

This completes the proof. □

Theorem 6. *Consider the model described in Section 2, and let the random critical threshold Δ take values from $\{0,1,2,\ldots,a\}$ with the pmf $\Pr(\Delta=\delta)=\frac{1}{a+1}$. Under the pattern in Eq. (7), the pmf of T_Δ is*

$$\Pr(T_\Delta=t)=\sum_{\delta=0}^{a}\Pr(\Delta=\delta)\sum_{n=2}^{\lfloor \frac{t}{\delta+2}+2\rfloor}\left[\binom{t-(n-2)(\delta+1)+1}{n-1}-\binom{t-(n-1)(\delta+1)+1}{n-1}\right]p_0p_{10}^{n-1}p_{00}^{t-2n+1}p_{01}^{n},$$

where $p_{ij}=\Pr(I_n=j|I_{n-1}=i)$ for $i,j=0,1$, and

$$I_n=\begin{cases}1 & \textit{if a shock occurs at time } n,\\ 0 & \textit{otherwise,}\end{cases}$$

and $p_0=\Pr(I_1=0)$.

Proof. In the pattern (7), we let the number of shocks is n and we have

$$\underset{y_1\geq 0, y_2>\Delta,\ldots,y_{n-1}>\Delta, y_n\leq \Delta}{y_1+y_2+\cdots+y_{n-1}+y_n = t-n.}$$

This equation is equivalent to the equation

$$\underset{y_1\geq 0, y_2\geq \Delta+1,\ldots,y_{n-1}\geq \Delta+1, y_n\leq \Delta}{y_1+y_2-(\Delta+1)+\cdots+y_{n-1}-(\Delta+1)+y_n = t-n-(n-2)(\Delta+1),}$$

that is,

$$\underset{y_1\geq 0, y'_2\geq 0,\ldots,y'_{n-1}\geq 0, y_n\leq \Delta}{y_1+y'_2+\cdots+y'_{n-1}+y_n = t-n-(n-2)(\Delta+1).}$$

The number of integer solutions of the above equation is

$$\binom{t-(n-2)(\Delta+1)+1}{n-1}-\binom{t-(n-1)(\Delta+1)+1}{n-1}.$$

The proof is complete. □

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5 Conclusions and future directions

We have introduced a new discrete-time shock model in which the system fails whenever the inter-arrival time between successive shocks is less than or equal to a random critical threshold Δ. In fact, this model is a randomized-threshold version of the classical $\boldsymbol{\delta}$-shock model. Moreover, we have shown that when Δ is supported on the set $\{\delta\}$ with $\delta > 0$, that is, when Δ has a degenerate distribution, the proposed model reduces to the classical $\boldsymbol{\delta}$-shock model. We have derived the pmf of both the system's stopping time and the system's lifetime under the proposed model by assuming that the inter-arrival times follow a geometric distribution and that Δ is uniformly distributed. We have also considered the case in which shock occurrences are dependent within a Markovian framework.

From a theoretical perspective, introducing a random threshold into $\boldsymbol{\delta}$-shock modeling provides a more advanced framework for studying this class of problems and enables the modeling of systems with complex failure mechanisms. Therefore, based on this study, future research can be directed toward more complex shock environments in which system operating conditions are more intricate than before. One important source of this complexity may be the randomness of critical thresholds against shocks. This idea can also be extended to settings in which both time and threshold are modeled as continuous random variables.

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