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Strong approximation for Itô stochastic differential equations

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Abstract

In this paper, a class of semi-implicit two-stage stochastic Runge-Kutta methods (SRKs) of strong global order one, with minimum principal error constants are given. These methods are applied to solve Itô stochastic differential equations (SDEs) with a Wiener process. The efficiency of this method with respect to explicit two-stage Itô Runge-Kutta methods (IRKs), Itô method, Milstien method, semi-implicit and implicit two-stage Stratonovich Runge-Kutta methods are demonstrated by presenting some numerical results.

Keywords: Stochastic differential equations; Strong approximation; Runge-Kutta methods.

1 Introduction

In recent years, a great deal of concern has been raised regarding the study of SDEs as an important area of research. Many phenomena in science and engineering have been modeled by deterministic ordinary differential equations (DODEs). However, some of the parameters and initial data are not known with complete certainty due to lack of information. Therefore, to represent a more accurate model of the behavior of such phenomena they usually should be modeled by SDEs. Some areas where SDEs have been used extensively in modeling phenomena include chemistry, physics, engineering, mathematical biology and finance (see, for example, [5], [7]). Since explicit solutions are known only for a few equations, the study of numerical methods have become more important and these must be designed to be implemented with a certain order of accuracy. Consider the autonomous Itô SDE given by

$$dy(t) = g_0(y(t))\mathrm{d}t + g_1(y(t))\mathrm{d}W(t), \qquad y(t_0) = y_0, \qquad t \in [t_0, t_f], \qquad (1)$$

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where g_0 and g_1 are real-valued functions which are called the drift coefficient and the diffusion coefficient, respectively, and $W(t)$ is a one-dimensional standard Wiener process, whose increment $\Delta W(t) = W(t+h) - W(t)$ has a Gaussian distribution with mean 0 and variance h, i.e. $W(t+h) - W(t) \sim N(0,h) = \sqrt{h}\, N(0,1)$, and the solution $y(t)$ is an Itô process. A Wiener process (named after N. Wiener) is sometimes called Brownian Motion, which is a term used to describe the phenomenon of the erratic behaviour of a particle in a liquid, acted on by random impulses, in the absence of friction. Equation (1) can also be written as a stochastic integral equation

$$y(t) = y_0 + \int_{t_0}^{t} g_0(y(s))\mathrm{d}s + \int_{t_0}^{t} g_1(y(s))\mathrm{d}W(s),$$

where the first integral is a mean square Riemann-Stieltjes integral and the second integral is a stochastic integral which can be interpreted in many ways (see [10]). The two most studied interpretations are due to Itô and Stratonovich that depend on the points of the partitioning in which the integrand is evaluated. If the lower end point t_n is chosen, it leads to Itô integral and if midpoint $(t_n + t_{n+1})/2$ is chosen, it leads to Stratonovich integral. The Stratonovich interpretation follows the common rules of integral calculus, while the Itô formulation has the advantage of preserving the martingale property of Wiener process. It is always possible to switch from one interpretation to the other, because an Itô SDE can be converted to a Stratonovich SDE (and vice versa) by means of the following formula (see [5])

$$\bar{g}_0(y) = g_0(y) - \frac{1}{2} g_1'(y) g_1(y),$$

where equation (1) is in the Stratonovich form when $\bar{g}_0$ is used in place of g_0. There are different numerical methods for solving these kinds of differential equations (see, for example, [1], [6], [8]). Numerical methods for SDEs are recursive methods where trajectories, in other words, the sample paths of solution are computed at discrete time steps. These methods are classified to strong and weak. Only strong convergence will be considered in this paper. Strong convergence is required, when each trajectory of the numerical method must be closed to the exact solution. Formally, if y_N is the numerical approximation to $y(t_N)$ after N steps with constant stepsize $h = (t_f - t_0)/N$, then y_N is said to converge strongly to $y(t_N)$ with strong order p if there exists $C > 0$ (independent of h) and $\delta > 0$ such that

$$\mathbb{E}(|y_N - y(t_N)|) \leq Ch^p, \qquad h \in (0, \delta).$$

An outline of this paper is as follows: In Section 2, the semi-implicit SRKs for SDEs are introduced, moreover order conditions for a class of SRKs with order one are stated. In particular, the new class of semi-implicit two-stage SRKs for SDEs with minimum principal error constants is constructed and the fixed

point iteration algorithm will be used to improve the semi-implicit method. In Section 3 we have some numerical results which show the efficiency of this method.

2 The semi-implicit Itô Runge-Kutta methods for SDEs

The most famous numerical method that can be obtained from a stochastic Taylor expansion is Milstein method. This method for the SDE problem (1) is given by

$$y_{n+1} = y_n + hg_0(y_n) + J_1 g_1(y_n) + \frac{1}{2}({J_1}^2 - h)g_1'(y_n)g_1(y_n),$$

where $J_1 = W(t_n+h) - W(t_n)$ with $h = (t_f - t_0)/N$ for some integer N. This method converges with strong order one as long as $\mathbb{E}(y_0^2) < \infty$, and g_0, g_0', g_1, g_1' and g_1'' satisfy a uniform Lipschitz condition. Higher order numerical methods can be obtained by truncating farther terms of the stochastic Taylor expansion. This technique involves considerable complexities in implementation because of the approximation of higher order stochastic integrals and the evaluation of high order derivatives of both the drift and diffusion coefficients. Thus, it is important to be able to derive derivatives free numerical methods and this leads to SRKs. For the SDE (1) SRKs is given by (see [2]):

$$\begin{aligned} Y_i &= y_n + \sum_{j=1}^{s} Z_{ij}^{(0)} g_0(Y_j) + \sum_{j=1}^{s} Z_{ij}^{(1)} g_1(Y_j), \qquad i = 1, 2, \ldots, s, \qquad (2) \\ y_{n+1} &= y_n + \sum_{j=1}^{s} z_j^{(0)} g_0(Y_j) + \sum_{j=1}^{s} z_j^{(1)} g_1(Y_j), \end{aligned}$$

which can be represented in tableau form as

$$\begin{array}{c|c} Z^{(0)} & Z^{(1)} \\ \hline {z^{(0)}}^T & {z^{(1)}}^T \end{array},$$

where $Z^{(k)} = (Z_{ij}^{(k)})$ for $i, j = 1, 2, \ldots, s$ and ${z^{(k)}}^T = (z_1^{(k)}, \ldots, z_s^{(k)})$ represents for $k = 0, 1$. Here $Y_1, \ldots, Y_s$ represent the internal stage of the method, and y_{n+1} is the update of the numerical solution at the end of the current step. Since (2) is a generalization of the class of Runge-Kutta methods in deterministic case, for consistency the stepsize will be included in the parameter matrix associated with the deterministic components, so $Z^{(0)} = hA$ and ${z^{(0)}}^T = h\alpha^T$, while $Z^{(1)}$ and ${z^{(1)}}^T$ have elements that are arbitrary random variables. In order to derive methods with strong global order one, the existence of stochastic Taylor series expansion of the SRK method in the Itô

case and the Itô Taylor series expansion of the exact solution is necessary. By comparing these two expansions, the local truncation error over one step with an exact initial value can be written as (see [3]):

$$L(t_0+h) = y(t_0+h) - Y(t_0+h) = \sum_{t\in T^\star} e(t)\ F(t)y_0,$$

where $e(t)$ and $F(t)y_0$ are called the local truncation error coefficients and the elementary differential for tree t, respectively and $T^\star$ is the set of bi-coloured rooted trees . Assuming certain conditions on the cofficients of the method and satisfying Lipschitz condition for the drift and diffusion cofficients SDE, a method will have strong global convergence of order one if it has strong local order one and mean local order one (see [3]). In [9] the order one conditions for a class of IRKs in the form

$$\begin{gathered} Z^{(0)} = hA, \qquad {z^{(0)}}^T = h\alpha^T, \qquad Z^{(1)} = \sqrt{h}B^{(1)} + J_1 B^{(2)}, \\ {z^{(1)}}^T = \sqrt{h}{\gamma^{(1)}}^T + J_1 {\gamma^{(2)}}^T, \end{gathered} \tag{3}$$

are given, where A, $B^{(1)}$ and $B^{(2)}$ are $s\times s$ real matrices, and $\alpha^T = (\alpha_1,\dots,\alpha_s)$, ${\gamma^{(1)}}^T = (\gamma_1^{(1)},\gamma_2^{(1)},\dots,\gamma_s^{(1)})$ and ${\gamma^{(2)}}^T = (\gamma_1^{(2)},\gamma_2^{(2)},\dots,\gamma_s^{(2)})$ are row s-dimensional vectors. In fact a SRK method of the form (3) will have strong global order one if (see [9])

$$\begin{cases} \alpha^T e = 1, \\ {\gamma^{(1)}}^T e = 0, \\ {\gamma^{(2)}}^T e = 1, \\ {\gamma^{(1)}}^T B^{(1)} e = -\frac{1}{2}, \\ {\gamma^{(1)}}^T B^{(2)} e + {\gamma^{(2)}}^T B^{(1)} e = 0, \\ {\gamma^{(2)}}^T B^{(2)} e = \frac{1}{2}, \\ \alpha^T B^{(1)} e = 0, \\ {\gamma^{(1)}}^T A e = 0, \\ {\gamma^{(1)}}^T (B^{(1)}e)^2 + {\gamma^{(1)}}^T (B^{(2)}e)^2 + 2{\gamma^{(2)}}^T (B^{(1)}e)(B^{(2)}e) = 0, \\ {\gamma^{(1)}}^T {B^{(1)}}^2 e + {\gamma^{(1)}}^T {B^{(2)}}^2 e + {\gamma^{(2)}}^T (B^{(1)}B^{(2)}e + B^{(2)}B^{(1)}e) = 0. \end{cases} \tag{4}$$

Here $e = (1,\dots,1)^T \in \mathbb{R}^s$ and multiplication of vectors are componentwise. If the matrices A, $B^{(1)}$ and $B^{(2)}$ are strictly lower triangular, then the method (3) is said to be explicit, while if A, $B^{(1)}$ and $B^{(2)}$ are lower triangular, then the method (3) is said to be semi-implicit. A family of two-stage explicit SRKs of the form (3) with minimum principal error terms can be presented by the following tableau (see [9]):

$$\begin{array}{c|cc|cc} 0 & 0 & 0 & 0 \\ 0 & 0 & -\frac{1}{2}(\sqrt{h}-J_1) & 0 \\ \hline h & 0 & -\sqrt{h} & \sqrt{h}+J_1 \end{array}$$

which is called '$EM1$', and has the principal error constants

$$\frac{1}{3}h^3,\ \frac{1}{3}h^3,\ \frac{1}{6}h^3,\ \frac{1}{16}h^3,$$

and the other family of two-stage explicit methods satisfying (4) with minimum principal error constants can be presented by (see [9]):

$$\begin{array}{c|cc|cc} 0 & 0 & 0 & 0 \\ 0 & 0 & \frac{1}{2}(\sqrt{h}+J_1) & 0 \\ \hline h & 0 & \sqrt{h} & -\sqrt{h}+J_1 \end{array}$$

which is called '$EM2$', and has the principal error constants

$$\frac{1}{3}h^3,\ \frac{1}{3}h^3,\ \frac{1}{6}h^3,\ \frac{1}{16}h^3.$$

Also the Itô method (see [2]) that is a derivative free version of the Milstein method with strong global order one, can be presented by the following tableau:

$$\begin{array}{c|cc|cc} 0 & 0 & 0 & 0 \\ 0 & 0 & \sqrt{h} & 0 \\ \hline h & 0 & J_1-\frac{\sqrt{h}}{2}((\frac{J_1}{\sqrt{h}})^2-1) & \frac{\sqrt{h}}{2}((\frac{J_1}{\sqrt{h}})^2-1) \end{array}$$

This method is called 'IRK' and has the principal error constants

$$\frac{1}{3}h^3,\ \frac{1}{3}h^3,\ \frac{1}{6}h^3,\ \frac{3}{8}h^3.$$

In [1] a class of semi-implicit and implicit Stratonovich Runge-Kutta methods of strong order one with minimum principal error constants for SDEs is constructed. More precisely, this class of semi-implicit and implicit two-stage Stratonovich Runge-Kutta methods with minimum principal error cofficients can be presented, by the following tableau (see [1])

$$\begin{array}{c|cc|cc} \frac{3+\sqrt{3}}{6}h & 0 & \frac{3+\sqrt{3}}{6}J_1 & 0 \\ -\frac{\sqrt{3}}{3}h & \frac{3+\sqrt{3}}{6}h & -\frac{\sqrt{3}}{3}J_1 & \frac{3+\sqrt{3}}{6}J_1 \\ \hline \frac{1}{2}h & \frac{1}{2}h & \frac{1}{2}J_1 & \frac{1}{2}J_1 \end{array}$$

and

$$\begin{array}{c|cc|cc} \frac{1}{4}h & \frac{3-2\sqrt{3}}{12}h & \frac{1}{4}J_1 & \frac{3-2\sqrt{3}}{12}J_1 \\ \frac{3-2\sqrt{3}}{12}h & \frac{1}{4}h & \frac{3-2\sqrt{3}}{12}J_1 & \frac{1}{4}J_1 \\ \hline \frac{1}{2}h & \frac{1}{2}h & \frac{1}{2}J_1 & \frac{1}{2}J_1 \end{array}$$

which are called '$SIM1$' and 'IM', respectively. In order to generalize the above explicit IRKs to semi-implicit case, consider $s=2$, hence the matrices $A,\ B^{(1)}$ and $B^{(2)}$ will have the following forms:

$$A=\begin{pmatrix} a_{11} & 0 \\ a_{21} & a_{22} \end{pmatrix},\quad B^{(1)}=\begin{pmatrix} b_{11}^{(1)} & 0 \\ b_{21}^{(1)} & b_{22}^{(1)} \end{pmatrix},\quad B^{(2)}=\begin{pmatrix} b_{11}^{(2)} & 0 \\ b_{21}^{(2)} & b_{22}^{(2)} \end{pmatrix}.$$

Now by system equations (4) and according to the structure of matrices A, $B^{(1)}$ and $B^{(2)}$ of the above form and by MAPLE, we have the following system equations of ten equations with fifteen unknowns:

$$\begin{cases} \alpha_1+\alpha_2=1, \\ \gamma_1^{(1)}+\gamma_2^{(1)}=0, \\ \gamma_1^{(2)}+\gamma_2^{(2)}=1, \\ \gamma_1^{(1)}b_{11}^{(1)}+\gamma_2^{(1)}b_{21}^{(1)}+\gamma_2^{(1)}b_{22}^{(1)}=-\frac{1}{2}, \\ \gamma_1^{(1)}b_{11}^{(2)}+\gamma_2^{(1)}b_{21}^{(2)}+\gamma_2^{(1)}b_{22}^{(2)}+\gamma_1^{(2)}b_{11}^{(1)}+\gamma_2^{(2)}b_{21}^{(1)}+\gamma_2^{(2)}b_{22}^{(1)}=0, \\ \gamma_1^{(2)}b_{11}^{(2)}+\gamma_2^{(2)}b_{21}^{(2)}+\gamma_2^{(2)}b_{22}^{(2)}=\frac{1}{2}, \\ \alpha_1 b_{11}^{(1)}+\alpha_2 b_{21}^{(1)}+\alpha_2 b_{22}^{(1)}=0, \\ \gamma_1^{(1)}a_{11}+\gamma_2^{(1)}a_{21}+\gamma_2^{(1)}a_{22}=0, \\ \gamma_1^{(1)}{b_{11}^{(1)}}^2+\gamma_2^{(1)}\left(b_{21}^{(1)}+b_{22}^{(1)}\right)^2+\gamma_1^{(1)}{b_{11}^{(2)}}^2+\gamma_2^{(1)}\left(b_{21}^{(2)}+b_{22}^{(2)}\right)^2+2\gamma_1^{(2)}b_{11}^{(1)}b_{11}^{(2)} \\ +2\gamma_2^{(2)}\left(b_{21}^{(2)}+b_{22}^{(2)}\right)\left(b_{21}^{(1)}+b_{22}^{(1)}\right)=0, \\ 2\gamma_1^{(2)}b_{11}^{(1)}b_{11}^{(2)}+\gamma_2^{(2)}\left(b_{21}^{(1)}b_{11}^{(2)}+b_{22}^{(1)}b_{21}^{(2)}+2b_{22}^{(1)}b_{22}^{(2)}+b_{21}^{(2)}b_{11}^{(1)}+b_{22}^{(2)}b_{21}^{(1)}\right) \\ +\gamma_1^{(1)}{b_{11}^{(1)}}^2+\gamma_2^{(1)}\left(b_{21}^{(1)}b_{11}^{(1)}+b_{22}^{(1)}b_{21}^{(1)}\right)+\gamma_2^{(1)}{b_{22}^{(1)}}^2+\gamma_1^{(1)}{b_{11}^{(2)}}^2 \\ +\gamma_2^{(1)}\left(b_{21}^{(2)}b_{11}^{(2)}+b_{22}^{(2)}b_{21}^{(2)}\right)+\gamma_2^{(1)}{b_{22}^{(2)}}^2=0. \end{cases} \tag{5}$$

Moreover, by system equations (4), since $\alpha^T B^{(1)}e=0$ and ${\gamma^{(1)}}^T Ae=0$, hence we can minimize the error constants corresponding to trees $[\tau_1]_0$ and $[\tau_0]_1$, that are given by

$$\begin{cases} E[I_{10}-{z^{(0)}}^T Z^{(1)}e]^2=\left(\frac{1}{3}-\left(\alpha^T B^{(2)}e\right)+\left(\alpha^T B^{(2)}e\right)^2+\left(\alpha^T B^{(1)}e\right)^2\right)h^3 \\ \qquad\qquad\qquad\qquad =\left(\frac{1}{3}-\left(\alpha^T B^{(2)}e\right)+\left(\alpha^T B^{(2)}e\right)^2\right)h^3, \\ E[I_{01}-{z^{(1)}}^T Z^{(0)}e]^2=\left(\frac{1}{3}-\left({\gamma^{(2)}}^T Ae\right)+\left({\gamma^{(2)}}^T Ae\right)^2+\left({\gamma^{(1)}}^T Ae\right)^2\right)h^3 \\ \qquad\qquad\qquad\qquad =\left(\frac{1}{3}-\left({\gamma^{(2)}}^T Ae\right)+\left({\gamma^{(2)}}^T Ae\right)^2\right)h^3. \end{cases}$$

These error constants are minimized with the minimum value $\frac{1}{12}$ if

$$\alpha^T B^{(2)}e=\frac{1}{2},\qquad {\gamma^{(2)}}^T Ae=\frac{1}{2},$$

or equivalently, if

$$\begin{cases} \alpha_1 b_{11}^{(2)}+\alpha_2 b_{21}^{(2)}+\alpha_2 b_{22}^{(2)}=\frac{1}{2}, \\ \gamma_1^{(2)}a_{11}+\gamma_2^{(2)}a_{21}+\gamma_2^{(2)}a_{22}=\frac{1}{2}. \end{cases} \tag{6}$$

By augmenting equations (6) to system (5) and solving the new system by MAPLE it is observed that the new system has a three parameters solution

that are given by

$$\begin{cases} a_{11}=\frac{1}{2}, \quad a_{21}=\frac{1}{2}-a_{22}, \quad \alpha_1=\alpha_2=\frac{1}{2}, \\ b_{11}^{(1)}=\frac{1}{4\gamma_2^{(1)}}, \quad b_{21}^{(1)}=\frac{2b_{22}^{(2)}-1}{4\gamma_2^{(1)}}, \quad b_{22}^{(1)}=-\frac{b_{22}^{(2)}}{2\gamma_2^{(1)}}, \quad \gamma_1^{(1)}=-\gamma_2^{(1)}, \\ b_{11}^{(2)}=\frac{1}{2}, \quad b_{21}^{(2)}=\frac{1}{2}-b_{22}^{(2)}, \quad \gamma_1^{(2)}=\gamma_2^{(2)}=\frac{1}{2}, \quad \gamma_2^{(1)}\neq 0. \end{cases} \tag{7}$$

In order to determine the free parameter of the deterministic part, i.e. a_{22}, we choose the deterministic part of SRK method (2) to be the Runge –Kutta method given by

$$\begin{array}{c|cc} \frac{1}{2} & \frac{1}{2} & 0 \\ \frac{1}{2} & \frac{1}{4} & \frac{1}{4} \\ \hline & \frac{1}{2} & \frac{1}{2} \end{array}$$

that is, it has order 2 (see [4]). This ensures that the semi-implicit method works well in the case of small stochastic influence. From (7) we can assume $A=B^{(2)}$, and consequently for $\gamma_2^{(1)}\neq 0$, a one-parameter solution can be represented by the following tableau

$$\begin{array}{cc|cc} \frac{1}{2}h & 0 & \frac{\sqrt{h}}{4\gamma_2^{(1)}}+\frac{1}{2}J_1 & 0 \\ \frac{1}{4}h & \frac{1}{4}h & -\frac{\sqrt{h}}{8\gamma_2^{(1)}}+\frac{1}{4}J_1 & -\frac{\sqrt{h}}{8\gamma_2^{(1)}}+\frac{1}{4}J_1 \\ \hline \frac{1}{2}h & \frac{1}{2}h & -\gamma_2^{(1)}\sqrt{h}+\frac{1}{2}J_1 & \gamma_2^{(1)}\sqrt{h}+\frac{1}{2}J_1 \end{array}$$

In order to choose $\gamma_2^{(1)}$, one can use the minimum of the error constants corresponding to trees $[[\tau_1]_1]_1$ and $[\tau_1,\tau_1]_1$, that are given by

$$\begin{cases} E[I_{111}-{z^{(1)}}^T{Z^{(1)}}^2e]^2=\left(\frac{3+96{\gamma_2^{(1)}}^2+560{\gamma_2^{(1)}}^4}{3072{\gamma_2^{(1)}}^4}\right)h^3, \\ E[I_{111}+\frac{1}{2}I_{01}-\frac{1}{2}{z^{(1)}}^T\left(Z^{(1)}e\right)^2-\frac{1}{2}{z^{(1)}}^TZ^{(0)}e]^2=\left(\frac{1+8{\gamma_2^{(1)}}^2+48{\gamma_2^{(1)}}^4}{1024{\gamma_2^{(1)}}^4}\right)h^3. \end{cases}$$

By introducing two functions f and g in the following form

$$f(\lambda)=\frac{3+96\lambda^2+560\lambda^4}{3072\lambda^4}, \qquad g(\lambda)=\frac{1+8\lambda^2+48\lambda^4}{1024\lambda^4}$$

it can be shown that these are decreasing functions on the interval $(0,+\infty)$, and moreover

$$\lim_{\lambda\to+\infty}f(\lambda)=\frac{35}{192}, \qquad \lim_{\lambda\to+\infty}g(\lambda)=\frac{3}{64}.$$

Now by choosing $\gamma_2^{(1)}=3$, this class of methods can be represented by the following tableau

$$\begin{array}{cc|cc} \frac{1}{2}h & 0 & \frac{\sqrt{h}}{12}+\frac{1}{2}J_1 & 0 \\ \frac{1}{4}h & \frac{1}{4}h & -\frac{\sqrt{h}}{24}+\frac{1}{4}J_1 & -\frac{\sqrt{h}}{24}+\frac{1}{4}J_1 \\ \hline \frac{1}{2}h & \frac{1}{2}h & -3\sqrt{h}+\frac{1}{2}J_1 & 3\sqrt{h}+\frac{1}{2}J_1 \end{array}$$

which is named '$SIM2$', and has principal error constants

$$\frac{1}{12}h^3,\ \frac{1}{12}h^3,\ \frac{15409}{82944}h^3,\ \frac{3961}{82944}h^3.$$

Note that the principal error coefficients corresponding to trees $[[\tau_1]_1]_1$ and $[\tau_1,\tau_1]_1$, are very close to the limits of f and g as $\lambda \to +\infty$, respectively. Since f and g are even functions, the above analysis shows that the choice $\gamma_2^{(1)}=3$ is suitable. If we use the 1-norm to estimate the contribution of all error terms to the principal error term, then, Table1 represents the following values for methods 'IRK' , '$EM1$' , '$EM2$' and '$SIM2$'.

Table 1: 1-norm of principal error coefficients

	IRK	$EM1$	$EM2$	$SIM2$
$\|principal\ \ error\|_1$	1.2083	0.89583	0.89583	0.40019

From Table1, it follows that the 1-norm principal error of the method '$SIM2$' is less than the 1-norm principal error of '$EM1$' and '$EM2$' methods. In order to improve the results of employing the '$SIM2$' method at each step, we can solve the system for stage-variables Y_1 and Y_2 by the fixed-point iteration scheme with starting values for these variables coming from the '$EM1$' or '$EM2$' methods. In fact, for the stage-variable Y_1 in the '$SIM2$' method let

$$G_1(Y_1) \equiv y_n + \frac{1}{2}h\ g_0(Y_1) + \frac{1}{12}(\sqrt{h}+6J_1)\ g_1(Y_1),$$

and hence the fixed-point iteration for solving Y_1 is given by

$$Y_1{}^{[s+1]} = G_1(Y_1{}^{[s]}), \qquad s=0,1,2,\ldots, \tag{8}$$

with stopping criteria

$$|Y_1^{[s+1]} - Y_1^{[s]}| < \epsilon, \tag{9}$$

where ϵ is a positive known tolerance value. In order to consider the convergence property of fixed point iterations (8), it is sufficient to have

$$|\ G_1'(Y)\ | = |\ \frac{1}{2}h\ g_0'(Y) + \frac{1}{12}(\sqrt{h}+6J_1)\ g_1'(Y)\ | < 1.$$

Also for the stage-variable Y_2, let

$$G_2(Y_2) \equiv y_n + \frac{1}{4}h\ (g_0({Y_1}^{[s+1]}) + g_0(Y_2)) + \frac{1}{24}(-\sqrt{h} + 6J_1)\ (g_1({Y_1}^{[s+1]}) + g_1(Y_2)),$$

such that $Y_1^{[s+1]}$ satisfy condition (9). Consequently the fixed-point iteration for solving Y_2 is given by

$$ {Y_2}^{[t+1]} = G_2({Y_2}^{[t]}), \qquad t = 0, 1, 2, \ldots, \tag{10}$$

with stopping criteria

$$|Y_2^{[t+1]} - Y_2^{[t]}| < \epsilon. \tag{11}$$

Note that iterations (10) is convergent if

$$|\ G_2'(Y)\ | = |\ \frac{1}{4}h\ g_0'(Y) + \frac{1}{24}(-\sqrt{h} + 6J_1)g_1'(Y)\ | < 1.$$

Finally y_{n+1} for the '$SIM2$' method will be evaluated by

$$y_{n+1} = y_n + \frac{1}{2}h\left(g_0\left(Y_1^{[s+1]}\right) + g_0\left(Y_2^{[t+1]}\right)\right) + \left(-3\sqrt{h} + \frac{1}{2}J_1\right) g_1\left(Y_1^{[s+1]}\right) + \left(3\sqrt{h} + \frac{1}{2}J_1\right) g_1\left(Y_2^{[t+1]}\right),$$

where $Y_1^{[s+1]}$ and $Y_2^{[t+1]}$ satisfy conditions (9) and (11).

3 Numerical results and conclusion

In this section, the numerical results from the implementation of the above seven methods are compared. These methods are 'IRK', '$Milstein$', '$EM1$', '$EM2$', '$SIM1$', 'IM' and '$SIM2$'. They will be implemented with constant stepsize on two problems taken from [5], for which the exact solution in terms of a Wiener process is known. Since $J_1 \sim N(0, h)$, hence for generating the Wiener increments J_1 in MATLAB environment of random numbers generator `randn` $(\#traj, \#step)$ is used, such that each call to `randn` $(\#traj, \#step)$ creates a $\#traj \times \#step$ matrix of independent $N(0, 1)$ samples. When these methods are simulated, the same sequence of random numbers for the Wiener increment J_1 are used for the stepsize under consideration. The average error for each stepsize at the end of the interval of integration is defined by

$$AE = \frac{1}{K}\sum_{i=1}^{K} |\ y_N^{(i)} - y^{(i)}(t_N)\ |,$$

where $y_N^{(i)}$ is the numerical approximation and $y^{(i)}(t_N)$ is the exact solution of SDE at t_N in the i-th simulation over all K simulations. All of the numerical results are based on 1000 simulated trajectories. The results appear in Tables 2-4.

Test problem 1. Consider

$$dy(t) = -a^2 y(t)(1-y^2(t))dt + a(1-y^2(t))dW(t), \quad y(0) = 0, \quad t \in [0,1]$$

with the exact solution

$$y(t) = \tan h(aW(t) + \arctan h(y_0)).$$

This problem is solved numerically with the choice of parameter $a = 1$.

Table 2: Global errors for Test problem 1, with $a = 1$, $K = 1000$ and $\epsilon = 0.001$

h	$\frac{1}{25}$	$\frac{1}{50}$	$\frac{1}{100}$	$\frac{1}{200}$	$\frac{1}{400}$
IRK	0.21400e-1	0.10299e-1	0.51948e-2	0.24299e-2	0.12254e-2
$Milstein$	0.16276e-1	0.82454e-2	0.42156e-2	0.19930e-2	0.10127e-2
$EM1$	0.12121e-1	0.59344e-2	0.30475e-2	0.14587e-2	0.70585e-3
$EM2$	0.12043e-1	0.57056e-2	0.29270e-2	0.13901e-2	0.71060e-3
$SIM1$	0.55857e-2	0.21190e-2	0.96207e-3	0.45136e-3	0.22157e-3
IM	0.13035e-3	0.64121e-4	0.34962e-4	0.17710e-4	0.81462e-5
$SIM2$	0.80715e-4	0.44013e-4	0.21736e-4	0.10551e-4	0.51995e-5

Test problem 2. Consider

$$dy(t) = -(\alpha+\beta^2 y(t))(1-y^2(t))dt + \beta(1-y^2(t))dW(t), \quad y(0) = 0.5, \quad t \in [0,1]$$

with the exact solution

$$y(t) = \frac{(1+y_0)\exp(-2\alpha t + 2\beta W(t)) + y_0 - 1}{(1+y_0)\exp(-2\alpha t + 2\beta W(t)) + 1 - y_0}.$$

This problem is solved numerically with $\alpha = -1$ and for $\beta = 1$ and 0.01. Comparing the numerical results in Tables 2-4, it follows that the '$SIM2$' method is more accurate than the '$EM1$', '$EM2$', '$SIM1$' and 'IM' methods. Also for problems in which the deterministic term dominates (Test problem 2 with $\beta = 0.01$) the improvement of the '$SIM2$' method becomes noticeable as the stepsize is reduced. This is because the deterministic component of the '$SIM2$' method is the second order Runge-Kutta method. On the other hand, for problems in which deterministic term dominates (Test problem 2 with $\beta = 0.01$) the global errors for two-stage explicit methods are the same. This is because these methods the deterministic components are the same. The future work should be based on the construction of implicit IRKs for SDEs with two or more Wiener processes.

Table 3: Global errors for test problem 2, with $\alpha = -1,\ \beta = 1,\ K = 1000$ and $\epsilon = 0.001$.

h	$\frac{1}{25}$	$\frac{1}{50}$	$\frac{1}{100}$	$\frac{1}{200}$	$\frac{1}{400}$
IRK	0.12763e-1	0.58682e-2	0.29961e-2	0.15034e-2	0.74495e-3
$Milstein$	0.11513e-1	0.51633e-2	0.27770e-2	0.13806e-2	0.68995e-3
$EM1$	0.96413e-2	0.41781e-2	0.21225e-2	0.10660e-2	0.54324e-3
$EM2$	0.93988e-2	0.42298e-2	0.20985e-2	0.10210e-2	0.52317e-3
$SIM1$	0.65238-3	0.32108e-3	0.15186-3	0.65537e-4	0.30367e-4
IM	0.79517e-4	0.42130e-4	0.21167e-4	0.10561e-4	0.51995e-5
$SIM2$	0.57845e-4	0.30499e-4	0.15806e-4	0.79504e-5	0.37761e-5

Table 4: Global errors for test problem 2, with $\alpha = -1,\ \beta = 0.01,\ K = 1000$ and $\epsilon = 0.001$

h	$\frac{1}{25}$	$\frac{1}{50}$	$\frac{1}{100}$	$\frac{1}{200}$	$\frac{1}{400}$
IRK	0.50778e-2	0.25193e-2	0.12544e-2	0.62592e-3	0.31264e-3
$Milstein$	0.50778e-2	0.25193e-2	0.12544e-2	0.62592e-3	0.31264e-3
$EM1$	0.50778e-2	0.25193e-2	0.12544e-2	0.62592e-3	0.31264e-3
$EM2$	0.50778e-2	0.25193e-2	0.12544e-2	0.62592e-3	0.31264e-3
$SIM1$	0.70238e-5	0.34182e-5	0.17423e-5	0.52895e-6	0.25624e-6
IM	0.55102e-5	0.26426e-5	0.13193e-5	0.58242e-6	0.29121e-6
$SIM2$	0.62103e-6	0.15741e-6	0.40441e-7	0.10595e-7	0.28838e-8

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تقریب قوی برای معادلات دیفرانسیل تصادفی ایتو

مهران نامجو

دانشگاه ولی عصر رفسنجان، دانشکده علوم ریاضی، گروه ریاضی

چکیده : در این مقاله خانواده ای از روشهای رانگ–کوتا تصادفی دو مرحله ا ی نیمه ضمنی از مرتبه قوی یک با مینیمم ثابتهای خطا که برای حل معادلات دیفرانسیل تصادفی ایتو با یک فرایند وینر بکار می رود، معرفی شده است. کارایی این روش نسبت به روشهای رانگ–کوتا ایتو دو مرحله ای صریح، روش ایتو، روش مایلشتن و روشهای رانگ–کوتا استراتنویچ دو مرحله ای نیمه ضمنی و ضمنی با استفاده از نتایج عددی نشان داده شده است.

کلمات کلیدی : معادلات دیفرانسیل تصادفی؛ تقریب قوی؛ روشهای رانگ–کوتا.