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Research Article

Shifted Jacobi collocation scheme for multidimensional time-fractional order telegraph equation

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Abstract

We propose a numerical scheme to solve a general class of time-fractional order telegraph equation in multidimensions using collocation points nodes and approximating the solution using double shifted Jacobi polynomials. The main characteristic behind this approach is to investigate a time-space collocation approximation for temporal and spatial discretizations. The applicability and accuracy of the present technique have been examined by the given numerical examples in this paper. By means of these numerical examples, we ensure that the present technique is simple, applicable, and accurate.

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Keywords: Time-fractional order telegraph equation; Shifted Jacobi polynomials; Gauss-Jacobi nodes; Matrix equation.

1 Introduction

Fractional differential equations [2, 20] are exhibited as capable mathematical tools for factual and more precise depiction of various phenomena. They show up in different territories, counting mathematical chemistry [12, 19], viscoelasticity [25], biology [20], electrochemistry, physics [17], semiconductors, seismology, scattering theory, heat conduction, fluid flow, metallurgy, population dynamics, optimal control theory, mathematical economics, and

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chemical reaction. As the increasing of employing fractional partial differential equations [21, 10, 30] in many social and scientific fields, the principle challenge we defy is that getting answers for them. Unfortunately, for most of these fractional partial differential equations, no one able to achieve analytic solutions for such problems. There are an extraordinary number of demonstrating and fractional order differential equations, which have been illuminated numerically utilizing different methods, see [5, 6, 1, 8, 23, 24].

There are many applications of the telegraph equation such as signal analysis for transmission, propagation of electrical signals, and modeling of the reaction diffusion. Sinc-Legendre collocation method [26] has been used to solve the time-fractional order telegraph equation (T-FOTE). By means of radial basis functions, Hosseini, Chen, and Avazzadeh [18] treated with time-fractional telegraph equation. Wei et al. [27] applied the fully discrete local discontinuous Galerkin method to solve fractional telegraph equation. In [28], space- and time-fractional telegraph equations have been solved by using homotopy perturbation method. Furthermore, the authors in [16] transformed a wavelet method based on Haar wavelets to solve space and time fractional telegraph equations.

In this work, we mean to build up some successful and productive collocation schemes to comprehend time-fractional order telegraph equation. One great advantage of such schemes is that it reduces the problems under confederation to systems of algebraic equations by using combination of basis functions of shifted Jacobi polynomials and the Gauss-shifted Jacobi nodes as the collocation nodes. The collocation method has successfully been applied to many situations [9, 4, 3, 14, 13, 15]. The main advantage of the proposed method is that is easy to implement, and also, we obtain highly accurate semi-analytic solutions via few number of retained modes.

The framework of this paper is as per the following: In the following area, we present few relevant properties of fractional derivatives and shifted Jacobi polynomials in the coming section. Section 3 is assigned to the theoretical derivation of the shifted Jacobi collocation (SJC) method for one-dimensional T-FOTE with the homogeneous and nonhomogeneous conditions. Section 4 is assigned to applying the SJC method for two-dimensional T-FOTE with the homogeneous and nonhomogeneous conditions. Moreover, in Section 6, several numerical examples and simulations are presented to clarify the effectiveness and accuracy of the proposed underlying method. Finally, related conclusions and observations are introduced.

2 Mathematical preliminaries

Some definitions and preliminaries related to fractional calculus [25] are stated in this section. Also, we listed some properties related to the shifted Jacobi polynomials.

2.1 The fractional integration

For fractional integration of order $\mu > 0$, we can find different definitions, which are not necessarily equivalent; see [22]. The more used definitions are Riemann–Liouville and Caputo fractional definitions.

Definition 1. The Riemann–Liouville integral of order $\mu \geq 0$ is defined as

$$\begin{aligned} &J^{\mu} f(x) = \frac{1}{\Gamma(\mu)} \int_0^x (x-t)^{\mu-1} f(t) dt, \quad \mu > 0, \quad x > 0, \\ &J^0 f(x) = f(x), \end{aligned} \tag{1}$$

The operator J^{μ} satisfies the following properties:

$$\begin{aligned} J^{\mu} J^{\nu} f(x) =& J^{\mu+\nu} f(x), \quad J^{\mu} J^{\nu} f(x) = J^{\nu} J^{\mu} f(x), \\ & J^{\mu} x^{\beta} = \frac{\Gamma(\beta+1)}{\Gamma(\beta+1+\mu)} \, x^{\beta+\mu}. \end{aligned} \tag{2}$$

Definition 2. The Riemann–Liouville fractional derivatives of order ν is obtained by

$$D^{\mu} f(x) = \frac{1}{\Gamma(m-\mu)} \frac{d^m}{dx^m} \left(\int_0^x (x-t)^{m-\mu-1} f(t) dt \right), \quad m-1 < \mu \leq m, \; x > 0,$$

here m is used as the ceiling function of μ.

Definition 3. The Caputo fractional derivatives of order μ is defined as

$${}^{c}D^{\mu} f(x) = \frac{1}{\Gamma(m-\mu)} \int_0^x (x-t)^{m-\mu-1} \frac{d^m}{dt^m} f(t) dt, \quad m-1 < \mu \leq m, \; x > 0.$$

With simple calculations, we obtain

$$D^{\mu} C = 0, \qquad (C \text{ is a constant}) \tag{3}$$

$$D^{\mu} x^m = \begin{cases} 0, & \text{for } m \in N_0 \text{ and } m < \lceil \mu \rceil, \\ \dfrac{\Gamma(m+1)}{\Gamma(m+1-\mu)} \, x^{m-\mu}, & \begin{array}{c} \text{for } m \in N_0 \text{ and } m \geq \lceil \mu \rceil \\ \text{or } m \notin N \text{ and } m > \lfloor \mu \rfloor, \end{array} \end{cases} \tag{4}$$

where $\lceil \mu \rceil$ and $\lfloor \mu \rfloor$ are used as usual for the ceiling and floor functions, respectively.

2.2 Properties of shifted Jacobi polynomials

By means of the main properties of Jacobi polynomials, we conclude the following:

$$\mathcal{P}_{k+1}^{(\alpha,\vartheta)}(x) = (a_k^{(\alpha,\beta)}x - b_k^{(\alpha,\beta)})\mathcal{P}_k^{(\alpha,\beta)}(x) - c_k^{(\alpha,\beta)}\mathcal{P}_{k-1}^{(\alpha,\beta)}(x), \quad k \geq 1,$$
$$\mathcal{P}_0^{(\alpha,\beta)}(x) = 1, \quad \mathcal{P}_1^{(\alpha,\beta)}(x) = \frac{1}{2}(\alpha+\beta+2)x + \frac{1}{2}(\alpha-\beta),$$

$$\mathcal{P}_k^{(\alpha,\beta)}(-x) = (-1)^k\mathcal{P}_k^{(\alpha,\beta)}(x), \quad \mathcal{P}_k^{(\alpha,\beta)}(-1) = \frac{(-1)^k\Gamma(k+\beta+1)}{k!\Gamma(\beta+1)}, \tag{5}$$

where $\alpha,\ \beta > -1$, $x \in [-1,1]$, and

$$\begin{aligned} a_k^{(\alpha,\beta)} &= \frac{(2k+\alpha+\beta+1)(2k+\alpha+\beta+2)}{2(k+1)(k+\alpha+\beta+1)}, \\ b_k^{(\alpha,\beta)} &= \frac{(\beta^2-\alpha^2)(2k+\alpha+\beta+1)}{2(k+1)(k+\alpha+\beta+1)(2k+\alpha+\beta)}, \\ c_k^{(\alpha,\beta)} &= \frac{(k+\alpha)(k+\beta)(2k+\alpha+\beta+2)}{(k+1)(k+\alpha+\beta+1)(2k+\alpha+\beta)}. \end{aligned}$$

Furthermore, the rth derivative of $\mathcal{P}_j^{(\alpha,\beta)}(x)$, is computed as

$$D^r\mathcal{P}_j^{(\alpha,\beta)}(x) = \frac{\Gamma(j+\alpha+\beta+q+1)}{2^r\Gamma(j+\alpha+\beta+1)}\mathcal{P}_{j-r}^{(\alpha+r,\beta+r)}(x), \tag{6}$$

where r is an integer. For the shifted Jacobi polynomial $\mathcal{P}_{\mathcal{L},k}^{(\alpha,\beta)}(x) = \mathcal{P}_k^{(\alpha,\beta)}(\frac{2x}{\mathcal{L}}-1)$, $\mathcal{L} > 0$, the explicit analytic form is written as

$$\begin{aligned} P_{\mathcal{L},k}^{(\alpha,\beta)}(x) &= \sum_{j=0}^{k}(-1)^{k-j}\frac{\Gamma(k+\beta+1)\Gamma(j+k+\alpha+\beta+1)}{\Gamma(j+\beta+1)\Gamma(k+\alpha+\beta+1)(k-j)!j!\mathcal{L}^j}x^j \\ &= \sum_{j=0}^{k}\frac{\Gamma(k+\alpha+1)\Gamma(k+j+\alpha+\beta+1)}{j!(k-j)!\Gamma(j+\alpha+1)\Gamma(k+\alpha+\beta+1)\mathcal{L}^j}(x-\mathcal{L})^j. \end{aligned} \tag{7}$$

Thereby, we deduce the following:

$$\begin{aligned} P_{\mathcal{L},k}^{(\alpha,\beta)}(0) &= (-1)^k\frac{\Gamma(k+\beta+1)}{\Gamma(\beta+1)\ k!}, \\ \mathcal{P}_{\mathcal{L},k}^{(\alpha,\beta)}(\mathcal{L}) &= \frac{\Gamma(k+\alpha+1)}{\Gamma(\alpha+1)\ k!}, \end{aligned} \tag{8}$$

$$D^r\mathcal{P}_{\mathcal{L},k}^{(\alpha,\beta)}(0) = \frac{(-1)^{k-r}\Gamma(k+\beta+1)(k+\alpha+\beta+1)_r}{L^r\Gamma(k-r+1)\Gamma(r+\beta+1)}, \tag{9}$$

$$D^r\mathcal{P}^{(\alpha,\beta)}_{\mathcal{L},k}(\mathcal{L}) = \frac{\Gamma(k+\alpha+1)(k+\alpha+\beta+1)_r}{L^r\Gamma(k-r+1)\Gamma(r+\alpha+1)}, \tag{10}$$

$$D^r\mathcal{P}^{(\alpha,\beta)}_{\mathcal{L},k}(x) = \frac{\Gamma(r+k+\alpha+\beta+1)}{\mathcal{L}^r\Gamma(k+\alpha+\beta+1)}\mathcal{P}^{(\alpha+r,\beta+r)}_{\mathcal{L},k-r}(x). \tag{11}$$

Taking $w^{(\alpha,\beta)}_{\mathcal{L}}(x) = (\mathcal{L}-x)^\alpha x^\beta$, we list the following norm and inner product related to the weighted space: $L^2_{w^{(\alpha,\beta)}_{\mathcal{L}}}[0,\mathcal{L}]$ as

$$(u,v)_{w^{(\alpha,\beta)}_{\mathcal{L}}} = \int_0^{\mathcal{L}} u(x)\,v(x)\,w^{(\alpha,\beta)}_{\mathcal{L}}(x)\,dx, \quad \|v\|_{w^{(\alpha,\beta)}_{\mathcal{L}}} = (v,v)^{\frac{1}{2}}_{w^{(\alpha,\beta)}_{\mathcal{L}}}. \tag{12}$$

A complete $L^2_{w^{(\alpha,\beta)}_{\mathcal{L}}}[0,\mathcal{L}]$-orthogonal system is consisted of a set of shifted Jacobi polynomials, where

$$\|\mathcal{P}^{(\alpha,\beta)}_{\mathcal{L},k}\|^2_{w^{(\alpha,\beta)}_{\mathcal{L}}} = \left(\frac{\mathcal{L}}{2}\right)^{\alpha+\beta+1} h^{(\alpha,\beta)}_k = h^{(\alpha,\beta)}_{\mathcal{L},k}. \tag{13}$$

We used $x^{(\alpha,\beta)}_{\mathcal{N},j}$, and $\varpi^{(\alpha,\beta)}_{\mathcal{N},j}$, $0 \leqslant j \leqslant \mathcal{N}$, as the nodes and Christoffel numbers of the standard Jacobi-Gauss interpolation in the interval $[-1,1]$.

The corresponding nodes and corresponding Christoffel numbers of the shifted Jacobi-Gauss interpolation in the interval $[0,\mathcal{L}]$ can be given by

$$x^{(\alpha,\beta)}_{\mathcal{L},\mathcal{N},j} = \frac{\mathcal{L}}{2}(x^{(\alpha,\beta)}_{\mathcal{N},j}+1),$$

$$\varpi^{(\alpha,\beta)}_{\mathcal{L},\mathcal{N},j} = (\frac{\mathcal{L}}{2})^{\alpha+\beta+1}\varpi^{(\alpha,\beta)}_{\mathcal{N},j},\ 0 \leqslant j \leqslant \mathcal{N}.$$

For any positive integer N, $\phi \in S_{2\mathcal{N}+1}[0,\mathcal{L}]$ and by means of Jacobi–Gauss quadrature property, we obtain

$$\begin{aligned}\int_0^{\mathcal{L}} (\mathcal{L}-x)^\alpha x^\beta \phi(x)dx &= \left(\frac{\mathcal{L}}{2}\right)^{\alpha+\beta+1}\int_{-1}^{1}(1-x)^\alpha(1+x)^\beta\phi\left(\frac{\mathcal{L}}{2}(x+1)\right)dx\\ &= \left(\frac{\mathcal{L}}{2}\right)^{\alpha+\beta+1}\sum_{j=0}^{\mathcal{N}}\varpi^{(\alpha,\beta)}_{\mathcal{N},j}\phi\left(\frac{\mathcal{L}}{2}(x^{(\alpha,\beta)}_{\mathcal{N},j}+1)\right)\\ &= \sum_{j=0}^{\mathcal{N}}\varpi^{(\alpha,\beta)}_{\mathcal{L},\mathcal{N},j}\phi\left(x^{(\alpha,\beta)}_{\mathcal{L},\mathcal{N},j}\right).\end{aligned} \tag{14}$$

Use Caputo's fractional derivative given in (3) and (4) for the shifted Jacobi polynomials $\mathcal{P}^{(\alpha,\beta)}_{\mathcal{L},j}(t)$ to obtain its fractional derivative as

$$D^\mu_t\mathcal{P}^{(\alpha,\beta)}_{\mathcal{L},j}(t) = 0, \quad j = 0,1,\ldots,\lceil\mu\rceil-1,\ \mu>0, \tag{15}$$

$$\mathcal{P}_{\mathcal{L},j}^{(\alpha,\beta,\mu)}(t) = D_t^{\mu}\mathcal{P}_{\mathcal{L},j}^{(\alpha,\beta)}(t) = \sum_{n=\lceil\mu\rceil}^{j} \xi_{j,n}^{(\alpha,\beta,\mu)} t^{n-\mu}, \quad j = \lceil\mu\rceil, \lceil\mu\rceil+1, \dots, \tag{16}$$

where $\xi_{j,n}^{(\alpha,\beta,\mu)}$ is given by

$$\xi_{j,n}^{(\alpha,\beta,\mu)} = (-1)^{j-n} \frac{\Gamma(j+\beta+1)\Gamma(n+j+\alpha+\beta+1)t^{n-\mu}}{\Gamma(n+\beta+1)\Gamma(j+\alpha+\beta+1)(j-n)!\Gamma(n-\mu+1)\mathcal{L}^n}.$$

3 One-dimensional T-FOTE

3.1 One-dimensional T-FOTE with homogeneous conditions

Here, we are occupied with utilizing the SJC method to solve one-dimensional T-FOE [29]:

$$\frac{\partial^{\mu}u(x,t)}{\partial t^{\mu}} + \gamma_1 \frac{\partial^{\mu-1}u(x,t)}{\partial t^{\mu-1}} + \gamma_2 u(x,t) = \gamma_3 \frac{\partial^2 u(x,t)}{\partial x^2} + \mathcal{H}(x,t),\ 0 \le x \le \mathcal{L}, \quad 0 \le t \le \mathcal{T}, \tag{17}$$

where $1 < \mu \le 2$ and the term $\mathcal{H}(x,t)$ denotes the field variable. We also assume the homogeneous conditions:

$$u(0,t) = u(\mathcal{L},t) = u(x,0) = \frac{\partial u(x,t)}{\partial t}|_{t=0} = 0, \tag{18}$$

where $\mathcal{L}$ and $\mathcal{T}$ are given and $\dfrac{\partial^{\mu}u(x,t)}{\partial t^{\mu}}$ represents the Caputo fractional derivative. In addition, $\gamma_1,\ \gamma_2$ and γ_3 are given constant coefficients. The point of our method is to get solution can be extended, using combination of basis functions of shifted Jacobi polynomials, in the form

$$u(x,t) \backsimeq \sum_{i=0}^{N-2}\sum_{j=0}^{M-2} c_{i,j}\phi_{\mathcal{L},i}^{(\alpha_1,\beta_1)}(x)\psi_{\mathcal{T},j}^{(\alpha_2,\beta_2)}(t) = \varphi(x,t)C, \tag{19}$$

where, $c_{i,j},\ i = 0,1,\dots,N-2,\ j = 0,1,\dots,M-2$ are the unknown coefficients,

$$C = [c_{0,0}, c_{0,1}, \dots, c_{0,M-2}; c_{1,0}, c_{1,1}, \dots, c_{1,M-2}; \\ c_{N-2,0}, c_{N-2,1}, \dots, c_{N-2,M-2}]^T,$$

N and M are any arbitrary positive integers, and

$$\phi_{\mathcal{L},i}^{(\alpha_1,\beta_1)}(x) = \mathcal{P}_{\mathcal{L},i}^{(\alpha_1,\beta_1)}(x) + \epsilon_i \mathcal{P}_{\mathcal{L},i+1}^{(\alpha_1,\beta_1)}(x) + \varepsilon_i \mathcal{P}_{\mathcal{L},i+2}^{(\alpha_1,\beta_1)}(x), \tag{20}$$

$$\psi_{\mathcal{T},j}^{(\alpha_2,\beta_2)}(t) = \mathcal{P}_{\mathcal{T},j}^{(\alpha_2,\beta_2)}(t) + \rho_j \mathcal{P}_{\mathcal{T},j+1}^{(\alpha_2,\beta_2)}(t) + \varrho_j \mathcal{P}_{\mathcal{T},j+2}^{(\alpha_2,\beta_2)}(t). \tag{21}$$

From the boundary conditions $\phi_{\mathcal{L},i}^{(\alpha_1,\beta_1)}(0) = \phi_{\mathcal{L},i}^{(\alpha_1,\beta_1)}(\mathcal{L}) = 0$ and the two relations (8), we have the accompanying framework

$$-\epsilon_i \frac{(i+1+1)}{(i+1)} + \varepsilon_i \frac{(i+1+1)(i+1+2)}{(i+1)(i+2)} = -1, \tag{22}$$

$$\epsilon_i \frac{(i+1+1)}{(i+1)} + \varepsilon_i \frac{(i+1+1)(i+1+2)}{(i+1)(i+2)} = -1. \tag{23}$$

Thus ϵ_i and ε_i can be remarkably resolved to give

$$\begin{aligned} \epsilon_i &= -\frac{(i+1)(1-1)(2i+1+1+3)}{(i+1+1)(i+1+1)(2i+1+1+4)}, \\ \varepsilon_i &= -\frac{(i+1)(i+2)(2i+1+1+2)}{(i+1+1)(i+1+1)(2i+1+1+4)}. \end{aligned} \tag{24}$$

Also, one can without much of a stretch check that

$$\begin{aligned} \rho_j &= \frac{2(j+1)(2j+2+2+3)}{(j+2+1)(2j+2+2+4)}, \\ \varrho_j &= \frac{(j+1)(j+2)(2j+2+2+2)}{(j+2+1)(j+2+2)(2j+2+2+4)}. \end{aligned} \tag{25}$$

From (15) and (16), we have that

$$\begin{aligned} \psi_{\mathcal{T},j}^{(\alpha_2,\beta_2,\mu)}(t) = D_t^{\mu} \psi_{\mathcal{T},j}^{(\alpha_2,\beta_2)}(t) = \mathcal{P}_{\mathcal{T},j}^{(\alpha_2,\beta_2,\mu)}(t) + \rho_j \mathcal{P}_{\mathcal{T},j+1}^{(\alpha_2,\beta_2,\mu)}(t) \\ + \varrho_j \mathcal{P}_{\mathcal{T},j+2}^{(\alpha_2,\beta_2,\mu)}(t), \end{aligned} \tag{26}$$

and

$$\begin{aligned} \psi_{\mathcal{T},j}^{(\alpha_2,\beta_2,\mu-1)}(t) = D_t^{\mu-1} \psi_{\mathcal{T},j}^{(\alpha_2,\beta_2)}(t) = \mathcal{P}_{\mathcal{T},j}^{(\alpha_2,\beta_2,\mu-1)}(t) + \rho_j \mathcal{P}_{\mathcal{T},j+1}^{(\alpha_2,\beta_2,\mu-1)}(t) \\ + \varrho_j \mathcal{P}_{\mathcal{T},j+2}^{(\alpha_2,\beta_2,\mu-1)}(t). \end{aligned} \tag{27}$$

Also $\varphi(x,t)$ is the $1 \times (N-1)(M-1)$ matrix introduced as follows:

$$\begin{aligned} \varphi(x,t) = &[r_{0,0}(x,t), r_{0,1}(x,t), \dots, r_{0,M-2}(x,t); r_{1,0}(x,t), r_{1,1}(x,t), \\ &\dots, r_{1,M-2}(x,t); r_{N-2,0}(x,t), r_{N-2,1}(x,t), \dots, r_{N-2,M-2}(x,t)], \end{aligned}$$

where

$$r_{i,j}(x,t) = \phi^{(\alpha_1,\beta_1)}_{\mathcal{L},i}(x)\psi^{(\alpha_2,\beta_2)}_{\mathcal{T},j}(t), \quad i = 0, 1, \ldots, N-2,\ j = 0, 1, \ldots, M-2.$$

Substituting (19) into (17) yields

$$\begin{aligned}
&\frac{\partial^\mu}{\partial t^\mu}\Big(\sum_{i=0}^{N-2}\sum_{j=0}^{M-2} c_{i,j}\phi^{(\alpha_1,\beta_1)}_{\mathcal{L},i}(x)\psi^{(\alpha_2,\beta_2)}_{\mathcal{T},j}(t)\Big)\\
&+\gamma_1\frac{\partial^{\mu-1}}{\partial t^{\mu-1}}\Big(\sum_{i=0}^{N-2}\sum_{j=0}^{M-2} c_{i,j}\phi^{(\alpha_1,\beta_1)}_{\mathcal{L},i}(x)\psi^{(\alpha_2,\beta_2)}_{\mathcal{T},j}(t)\Big)\\
&+\gamma_2\sum_{i=0}^{N-2}\sum_{j=0}^{M-2} c_{i,j}\phi^{(\alpha_1,\beta_1)}_{\mathcal{L},i}(x)\psi^{(\alpha_2,\beta_2)}_{\mathcal{T},j}(t)\\
&\qquad=\gamma_3\frac{\partial^2}{\partial x^2}\Big(\sum_{i=0}^{N-2}\sum_{j=0}^{M-2} c_{i,j}\phi^{(\alpha_1,\beta_1)}_{\mathcal{L},i}(x)\psi^{(\alpha_2,\beta_2)}_{\mathcal{T},j}(t)\Big)+\mathcal{H}(x,t),
\end{aligned} \tag{28}$$

$$\begin{aligned}
&\sum_{i=0}^{N-2}\sum_{j=0}^{M-2} c_{i,j}\phi^{(\alpha_1,\beta_1)}_{\mathcal{L},i}(x)\frac{\partial^\mu}{\partial t^\mu}\psi^{(\alpha_2,\beta_2)}_{\mathcal{T},j}(t)\\
&+\gamma_1\sum_{i=0}^{N-2}\sum_{j=0}^{M-2} c_{i,j}\phi^{(\alpha_1,\beta_1)}_{\mathcal{L},i}(x)\frac{\partial^{\mu-1}}{\partial t^{\mu-1}}\psi^{(\alpha_2,\beta_2)}_{\mathcal{T},j}(t)\\
&+\gamma_2\sum_{i=0}^{N-2}\sum_{j=0}^{M-2} c_{i,j}\phi^{(\alpha_1,\beta_1)}_{\mathcal{L},i}(x)\psi^{(\alpha_2,\beta_2)}_{\mathcal{T},j}(t)\\
&\qquad=\gamma_3\frac{\partial^2}{\partial x^2}\Big(\sum_{i=0}^{N-2}\sum_{j=0}^{M-2} c_{i,j}\phi^{(\alpha_1,\beta_1)}_{\mathcal{L},i}(x)\psi^{(\alpha_2,\beta_2)}_{\mathcal{T},j}(t)\Big)+\mathcal{H}(x,t).
\end{aligned} \tag{29}$$

Therefore, adopting (26) and (27), enable one to write (29) in the following form:

$$
\begin{aligned}
&\sum_{i=0}^{N-2}\sum_{j=0}^{M-2} c_{i,j}\phi_{\mathcal{L},i}^{(\alpha_1,\beta_1)}(x)\psi_{\mathcal{T},j}^{(\alpha_2,\beta_2,\mu)}(t)\\
&+\gamma_1\sum_{i=0}^{N-2}\sum_{j=0}^{M-2} c_{i,j}\phi_{\mathcal{L},i}^{(\alpha_1,\beta_1)}(x)\psi_{\mathcal{T},j}^{(\alpha_2,\beta_2,\mu-1)}(t)\\
&+\gamma_2\sum_{i=0}^{N-2}\sum_{j=0}^{M-2} c_{i,j}\phi_{\mathcal{L},i}^{(\alpha_1,\beta_1)}(x)\psi_{\mathcal{T},j}^{(\alpha_2,\beta_2)}(t)\\
&\quad=\gamma_3\frac{\partial^2}{\partial x^2}(\sum_{i=0}^{N-2}\sum_{j=0}^{M-2} c_{i,j}\phi_{\mathcal{L},i}^{(\alpha_1,\beta_1)}(x)\psi_{\mathcal{T},j}^{(\alpha_2,\beta_2)}(t))+\mathcal{H}(x,t).
\end{aligned}
\tag{30}
$$

Assume that

$$
\begin{aligned}
f_{i,j}(x,t)=&\,\phi_{\mathcal{L},i}^{(\alpha_1,\beta_1)}(x)\psi_{\mathcal{T},j}^{(\alpha_2,\beta_2,\mu)}(t)+\gamma_1\phi_{\mathcal{L},i}^{(\alpha_1,\beta_1)}(x)\psi_{\mathcal{T},j}^{(\alpha_2,\beta_2,\mu-1)}(t)\\
&+\gamma_2\phi_{\mathcal{L},i}^{(\alpha_1,\beta_1)}(x)\psi_{\mathcal{T},j}^{(\alpha_2,\beta_2)}(t)-\gamma_3\frac{\partial^2}{\partial x^2}(\phi_{\mathcal{L},i}^{(\alpha_1,\beta_1)}(x)\psi_{\mathcal{T},j}^{(\alpha_2,\beta_2)}(t)),
\end{aligned}
$$

at that point, (30) can be modified as

$$
\sum_{i=0}^{N-2}\sum_{j=0}^{M-2} c_{i,j}f_{i,j}(x,t)=\mathcal{H}(x,t). \tag{31}
$$

Collocating (31) in $N-1$ and $M-1$ roots of the shifted Jacobi polynomials $\mathcal{P}_{\mathcal{L},N-1}^{(\alpha,\beta)}(x)$, the Gauss-shifted Jacobi nodes, we obtain

$$
\begin{aligned}
&\sum_{i=0}^{N-2}\sum_{j=0}^{M-2} c_{i,j}f_{i,j}(x_{\mathcal{L},n,i}^{(\alpha_1,\beta_1)},t_{\mathcal{T},m,j}^{(\alpha_2,\beta_2)})=\mathcal{H}(x_{\mathcal{L},n,i}^{(\alpha_1,\beta_1)},t_{\mathcal{T},m,j}^{(\alpha_2,\beta_2)}),\\
&\qquad\qquad for\ n=0,1,\dots,N-2,\ m=0,1,\dots,M-2,
\end{aligned}
\tag{32}
$$

which can be written in the following matrix form:

$$
\mathbf{F}^TC=\mathbf{R},
$$

where

$$
\begin{aligned}
\mathbf{R}=[&R_{0,0},R_{1,0},\dots,R_{N-2,0};R_{0,1},R_{1,1},\dots,R_{N-2,1};\\
&R_{0,M-2},R_{1,M-2},\dots,R_{N-2,M-2}]^T,
\end{aligned}
$$

$$
R_{i,j}=\mathcal{H}(x_{\mathcal{L},n,i}^{(\alpha_1,\beta_1)},t_{\mathcal{T},m,j}^{(\alpha_2,\beta_2)}),\ i=0,1,\dots,N-2,\ j=0,1,\dots,M-2,
$$

and

$$
\mathbf{F}=(f_{ijnm}),\quad i,n=0,1,\dots,N-2,\ j,m=0,1,\dots,M-2,
$$

in which the elements of the matrix $\mathbf{F}$ are determined as follows:

$$f_{ijnm} = f_{i,j}(x^{(\alpha_1,\beta_1)}_{\mathcal{L},n,i}, t^{(\alpha_2,\beta_2)}_{\mathcal{T},m,j}),\ i,n = 0,1,\dots,N-2,\ j,m = 0,1,\dots,M-2.$$

In our implementation, this system has been solved using the Mathematica function FindRoot with zero initial approximation. In this manner, the approximate solution of (17) is given by $u(x,t) = \varphi(x,t)C$.

3.2 One-dimensional T-FOTE with nonhomogeneous conditions

In the accompanying, we simplify alter the right-hand side to deal with the nonhomogeneous initial-boundary conditions. Give us a chance to consider, for example, one-dimensional time-fractional order telegraph equation (17) with the nonhomogeneous initial-boundary conditions:

$$\begin{aligned} u(0,t) =& q_0(t),\ u(\mathcal{L},t) = q_1(t), \quad 0 \le t \le \mathcal{T}, \\ u(x,0) =& p_0(x),\ \frac{\partial u(x,t)}{\partial t}|_{t=0} = p_1(x), \quad 0 \le x \le \mathcal{L}, \end{aligned} \tag{33}$$

where q_0, q_1, p_0, p_1 are known functions and the function u is unknown.

Presently, assume the accompanying transformation

$$V(x,t) = u(x,t) + a_0(t) + a_1(t)x + x(x-\mathcal{L})(b_0(x) + b_1(x)t), \tag{34}$$

where

$$\begin{aligned} a_0(t) = -q_0(t), \quad a_1(t) &= \frac{q_0(t) - q_1(t)}{\mathcal{L}}, \\ b_0(x) &= \frac{(\mathcal{L} - x)q_0(0) + q_1(0)x - \mathcal{L}p_0(x)}{\mathcal{L}x(x-\mathcal{L})}, \\ b_1(x) &= \frac{\frac{\partial q_0(t)}{\partial t}|_{t=0} - p_1(x) - \frac{x}{\mathcal{L}}(\frac{\partial q_0(t)}{\partial t}|_{t=0} - \frac{\partial q_1(t)}{\partial t}|_{t=0})}{x(x-\mathcal{L})}. \end{aligned}$$

The mapping (34) changes the nonhomogeneous conditions (33) into the following homogeneous conditions:

$$V(0,t) = V(\mathcal{L},t) = V(x,0) = \frac{\partial V(x,t)}{\partial t}|_{t=0} = 0, \quad 0 \le t \le \mathcal{T},\ 0 \le x \le \mathcal{L}. \tag{35}$$

Subsequently it suffices to solve the following time-fractional order telegraph equation:

$$\frac{\partial^{\mu}V(x,t)}{\partial t^{\mu}}+\gamma_1\frac{\partial^{\mu-1}V(x,t)}{\partial t^{\mu-1}}+\gamma_2V(x,t)=\gamma_3\frac{\partial^2V(x,t)}{\partial x^2}+\mathcal{G}(x,t),\quad 0\leq x\leq\mathcal{L},\ 0\leq t\leq\mathcal{T}, \tag{36}$$

with the homogeneous conditions (35), where $V(x,t)$ is obtained by (34) and

$$\begin{aligned}\mathcal{G}(x,t)=&\mathcal{H}(x,t)+(\frac{\partial^{\mu}a_0(t)}{\partial t^{\mu}}+\gamma_1\frac{\partial^{\mu-1}a_0(t)}{\partial t^{\mu-1}}+\gamma_2a_0(t))\\&+x(\frac{\partial^{\mu}a_1(t)}{\partial t^{\mu}}+\gamma_1\frac{\partial^{\mu-1}a_1(t)}{\partial t^{\mu-1}}+\gamma_2a_1(t))\\&+x(x-\mathcal{L})(b_1(x)\frac{\Gamma(2)}{\Gamma(2-\mu)}t^{1-\mu}+\gamma_1b_1(x)\frac{\Gamma(2)}{\Gamma(3-\mu)}t^{2-\mu}\\&+\gamma_2(b_0(x)+b_1(x)t)-\gamma_3(\frac{\partial^2b_0(x)}{\partial x^2}+\frac{\partial^2b_1(x)}{\partial x^2}t)\\&-2\gamma_3(2x-\mathcal{L})(\frac{\partial b_0(x)}{\partial x}+\frac{\partial b_1(x)}{\partial x}t)-2\gamma_3(b_0(x)+b_1(x)t))).\end{aligned}$$

4 Two-dimensional T-FOTE

4.1 Two-dimensional T-FOTE with homogeneous conditions

In this section, we test the following two-dimensional T-FOTE:

$$\begin{aligned}\frac{\partial^{\mu}u(x,y,t)}{\partial t^{\mu}}&+\gamma_1\frac{\partial^{\mu-1}u(x,y,t)}{\partial t^{\mu-1}}+\gamma_2u(x,y,t)\\&=\gamma_3(\frac{\partial^2u(x,y,t)}{\partial x^2}+\frac{\partial^2u(x,y,t)}{\partial y^2})+\mathcal{H}(x,y,t),\\&\qquad 0\leq x\leq\mathcal{L}_1,\ 0\leq y\leq\mathcal{L}_2,\ 0\leq t\leq\mathcal{T},\end{aligned} \tag{37}$$

with the homogeneous conditions

$$\begin{aligned}u(0,y,t)=u(\mathcal{L}_1,y,t)=u(x,0,t)=u(x,\mathcal{L}_2,t)=u(x,y,0)\\=\frac{\partial u(x,y,t)}{\partial t}|_{t=0}=0.\end{aligned} \tag{38}$$

The point of our method is to get solution can be extended, using combination of basis functions of shifted Jacobi polynomials, in the form

$$\begin{aligned} u(x,y,t) &\simeq \sum_{i=0}^{N-2}\sum_{j=0}^{M-2}\sum_{k=0}^{K-2} c_{i,j,k}\phi_{\mathcal{L}_1,i}^{(\alpha_1,\beta_1)}(x)\varphi_{\mathcal{L}_2,j}^{(\alpha_2,\beta_2)}(y)\psi_{\mathcal{T},k}^{(\alpha 3,\beta 3)}(t) \\ &= \chi(x,y,t)C, \end{aligned} \tag{39}$$

where, $c_{i,j,k},\ i=0,1,\dots,N-2,\ j=0,1,\dots,M-2,\ k=0,1,\dots,K-2$ are the unknown coefficients,

$$\begin{aligned} C = [&c_{0,0,0}, c_{0,0,1}, \dots, c_{0,0,K-2}, c_{0,1,0}, c_{0,1,1}, \dots, c_{0,1,K-2}, \dots, c_{0,M-2,K-2}; \\ &c_{1,0,0}, c_{1,0,1}, \dots, c_{1,0,K-2}, c_{1,1,0}, c_{1,1,1}, \dots, c_{1,1,K-2}, \dots, c_{1,M-2,K-2}; \\ &c_{N-2,0,0}, c_{N-2,0,1}, \dots, c_{N-2,0,K-2}, c_{N-2,1,0}, c_{N-2,1,1}, \dots, c_{N-2,1,K-2}, \\ &\dots, c_{N-2,M-2,K-2}]^T, \end{aligned}$$

$N,\ M$, and K are any arbitrary positive integers, and

$$\phi_{\mathcal{L}_1,i}^{(\alpha_1,\beta_1)}(x) = \mathcal{P}_{\mathcal{L}_1,i}^{(\alpha_1,\beta_1)}(x) + \epsilon_i \mathcal{P}_{\mathcal{L}_1,i+1}^{(\alpha_1,\beta_1)}(x) + \varepsilon_i \mathcal{P}_{\mathcal{L}_1,i+2}^{(\alpha_1,\beta_1)}(x), \tag{40}$$

$$\varphi_{\mathcal{L}_2,j}^{(\alpha_2,\beta_2)}(y) = \mathcal{P}_{\mathcal{L}_2,j}^{(\alpha_2,\beta_2)}(y) + \epsilon_j \mathcal{P}_{\mathcal{L}_2,j+1}^{(\alpha_2,\beta_2)}(y) + \varepsilon_j \mathcal{P}_{\mathcal{L}_2,j+2}^{(\alpha_2,\beta_2)}(y), \tag{41}$$

$$\psi_{\mathcal{T},k}^{(\alpha 3,\beta 3)}(t) = \mathcal{P}_{\mathcal{T},k}^{(\alpha 3,\beta 3)}(t) + \rho_k \mathcal{P}_{\mathcal{T},k+1}^{(\alpha 3,\beta 3)}(t) + \varrho_k \mathcal{P}_{\mathcal{T},k+2}^{(\alpha 3,\beta 3)}(t), \tag{42}$$

where $\epsilon_i,\ \varepsilon_i,\ \epsilon_j,\ \varepsilon_j,\ \rho_k$ and ϱ_k are defined in (24) and (25).

Also $\chi(x,y,t)$ is the $1 \times (N-1)(M-1)(K-1)$ matrix introduced as follows:

$$\begin{aligned} \chi(x,y,t) =&[r_{0,0,0}(x,y,t), r_{0,0,1}(x,y,t), \dots, r_{0,0,K-2}(x,y,t), r_{0,1,0}(x,y,t), \\ &r_{0,1,1}(x,y,t), \dots, r_{0,M-2,K-2}(x,y,t); r_{1,0,0}(x,y,t), \\ &r_{1,0,1}(x,y,t), \dots, r_{1,0,K-2}(x,y,t), r_{1,1,0}(x,y,t), \\ &r_{1,1,1}(x,y,t), \dots, r_{1,1,K-2}(x,y,t), \dots, r_{1,M-2,K-2}(x,y,t); \\ &r_{N-2,0,0}(x,y,t), r_{N-2,0,1}(x,y,t), \dots, r_{N-2,0,K-2}(x,y,t), \\ &r_{N-2,1,0}(x,y,t), r_{N-2,1,1}(x,y,t), \dots, r_{N-2,1,K-2}(x,y,t), \dots, \\ &r_{N-2,M-2,K-2}(x,y,t)], \end{aligned}$$

where

$$\begin{aligned} r_{i,j,k}(x,y,t) =&\phi_{\mathcal{L}_1,i}^{(\alpha_1,\beta_1)}(x)\varphi_{\mathcal{L}_2,j}^{(\alpha_2,\beta_2)}(y)\ \psi_{\mathcal{T},k}^{(\alpha 3,\beta 3)}(t),\ i=0,1,\dots,N-2, \\ &j=0,1,\dots,M-2, \quad k=0,1,\dots,K-2. \end{aligned}$$

Substituting (39) into (37) yields

$$
\begin{aligned}
&\frac{\partial^{\mu}}{\partial t^{\mu}}(\sum_{i=0}^{N-2}\sum_{j=0}^{M-2}\sum_{k=0}^{K-2} c_{i,j,k}\phi_{\mathcal{L}_1,i}^{(\alpha_1,\beta_1)}(x)\varphi_{\mathcal{L}_2,j}^{(\alpha_2,\beta_2)}(y)\psi_{\mathcal{T},k}^{(\alpha 3,\beta 3)}(t))\\
&+\gamma_1\frac{\partial^{\mu-1}}{\partial t^{\mu-1}}(\sum_{i=0}^{N-2}\sum_{j=0}^{M-2}\sum_{k=0}^{K-2} c_{i,j,k}\phi_{\mathcal{L}_1,i}^{(\alpha_1,\beta_1)}(x)\varphi_{\mathcal{L}_2,j}^{(\alpha_2,\beta_2)}(y)\psi_{\mathcal{T},k}^{(\alpha 3,\beta 3)}(t))\\
&+\gamma_2\sum_{i=0}^{N-2}\sum_{j=0}^{M-2}\sum_{k=0}^{K-2} c_{i,j,k}\phi_{\mathcal{L}_1,i}^{(\alpha_1,\beta_1)}(x)\varphi_{\mathcal{L}_2,j}^{(\alpha_2,\beta_2)}(y)\psi_{\mathcal{T},k}^{(\alpha 3,\beta 3)}(t)\\
&\quad=\gamma_3\frac{\partial^2}{\partial x^2}(\sum_{i=0}^{N-2}\sum_{j=0}^{M-2}\sum_{k=0}^{K-2} c_{i,j,k}\phi_{\mathcal{L}_1,i}^{(\alpha_1,\beta_1)}(x)\varphi_{\mathcal{L}_2,j}^{(\alpha_2,\beta_2)}(y)\psi_{\mathcal{T},k}^{(\alpha 3,\beta 3)}(t))\\
&\qquad+\gamma_3\frac{\partial^2}{\partial y^2}(\sum_{i=0}^{N-2}\sum_{j=0}^{M-2}\sum_{k=0}^{K-2} c_{i,j,k}\phi_{\mathcal{L}_1,i}^{(\alpha_1,\beta_1)}(x)\varphi_{\mathcal{L}_2,j}^{(\alpha_2,\beta_2)}(y)\psi_{\mathcal{T},k}^{(\alpha 3,\beta 3)}(t))\\
&\qquad+\mathcal{H}(x,y,t),
\end{aligned}
\tag{43}
$$

$$
\begin{aligned}
&\sum_{i=0}^{N-2}\sum_{j=0}^{M-2}\sum_{k=0}^{K-2} c_{i,j,k}\phi_{\mathcal{L}_1,i}^{(\alpha_1,\beta_1)}(x)\varphi_{\mathcal{L}_2,j}^{(\alpha_2,\beta_2)}(y)\frac{\partial^{\mu}}{\partial t^{\mu}}\psi_{\mathcal{T},k}^{(\alpha 3,\beta 3)}(t)\\
&+\gamma_1\sum_{i=0}^{N-2}\sum_{j=0}^{M-2}\sum_{k=0}^{K-2} c_{i,j,k}\phi_{\mathcal{L}_1,i}^{(\alpha_1,\beta_1)}(x)\varphi_{\mathcal{L}_2,j}^{(\alpha_2,\beta_2)}(y)\frac{\partial^{\mu-1}}{\partial t^{\mu-1}}\psi_{\mathcal{T},k}^{(\alpha 3,\beta 3)}(t)\\
&+\gamma_2\sum_{i=0}^{N-2}\sum_{j=0}^{M-2}\sum_{k=0}^{K-2} c_{i,j,k}\phi_{\mathcal{L}_1,i}^{(\alpha_1,\beta_1)}(x)\varphi_{\mathcal{L}_2,j}^{(\alpha_2,\beta_2)}(y)\psi_{\mathcal{T},k}^{(\alpha 3,\beta 3)}(t)\\
&\quad=\gamma_3\frac{\partial^2}{\partial x^2}(\sum_{i=0}^{N-2}\sum_{j=0}^{M-2}\sum_{k=0}^{K-2} c_{i,j,k}\phi_{\mathcal{L}_1,i}^{(\alpha_1,\beta_1)}(x)\varphi_{\mathcal{L}_2,j}^{(\alpha_2,\beta_2)}(y)\psi_{\mathcal{T},k}^{(\alpha 3,\beta 3)}(t))\\
&\qquad+\gamma_3\frac{\partial^2}{\partial y^2}(\sum_{i=0}^{N-2}\sum_{j=0}^{M-2}\sum_{k=0}^{K-2} c_{i,j,k}\phi_{\mathcal{L}_1,i}^{(\alpha_1,\beta_1)}(x)\varphi_{\mathcal{L}_2,j}^{(\alpha_2,\beta_2)}(y)\psi_{\mathcal{T},k}^{(\alpha 3,\beta 3)}(t))\\
&\qquad+\mathcal{H}(x,y,t).
\end{aligned}
\tag{44}
$$

Therefore, adopting (40), (41), and (42), enables one to write (44) in the form

$$\begin{aligned}
&\sum_{i=0}^{N-2}\sum_{j=0}^{M-2}\sum_{k=0}^{K-2} c_{i,j,k}\phi_{\mathcal{L}_1,i}^{(\alpha_1,\beta_1)}(x)\varphi_{\mathcal{L}_2,j}^{(\alpha_2,\beta_2)}(y)\psi_{\mathcal{T},k}^{(\alpha 3,\beta 3,\mu)}(t)\\
&+\gamma_1\sum_{i=0}^{N-2}\sum_{j=0}^{M-2}\sum_{k=0}^{K-2} c_{i,j,k}\phi_{\mathcal{L}_1,i}^{(\alpha_1,\beta_1)}(x)\varphi_{\mathcal{L}_2,j}^{(\alpha_2,\beta_2)}(y)\psi_{\mathcal{T},k}^{(\alpha 3,\beta 3,\mu-1)}(t)\\
&+\gamma_2\sum_{i=0}^{N-2}\sum_{j=0}^{M-2}\sum_{k=0}^{K-2} c_{i,j,k}\phi_{\mathcal{L}_1,i}^{(\alpha_1,\beta_1)}(x)\varphi_{\mathcal{L}_2,j}^{(\alpha_2,\beta_2)}(y)\psi_{\mathcal{T},k}^{(\alpha 3,\beta 3)}(t)\\
&\quad=\gamma_3\frac{\partial^2}{\partial x^2}(\sum_{i=0}^{N-2}\sum_{j=0}^{M-2}\sum_{k=0}^{K-2} c_{i,j,k}\phi_{\mathcal{L}_1,i}^{(\alpha_1,\beta_1)}(x)\varphi_{\mathcal{L}_2,j}^{(\alpha_2,\beta_2)}(y)\psi_{\mathcal{T},k}^{(\alpha 3,\beta 3)}(t))\\
&\qquad+\gamma_3\frac{\partial^2}{\partial y^2}(\sum_{i=0}^{N-2}\sum_{j=0}^{M-2}\sum_{k=0}^{K-2} c_{i,j,k}\phi_{\mathcal{L}_1,i}^{(\alpha_1,\beta_1)}(x)\varphi_{\mathcal{L}_2,j}^{(\alpha_2,\beta_2)}(y)\psi_{\mathcal{T},k}^{(\alpha 3,\beta 3)}(t))\\
&\qquad+\mathcal{H}(x,y,t).
\end{aligned}\tag{45}$$

Assume that

$$\begin{aligned}
f_{i,j,k}(x,y,t)=&\phi_{\mathcal{L}_1,i}^{(\alpha_1,\beta_1)}(x)\varphi_{\mathcal{L}_2,j}^{(\alpha_2,\beta_2)}(y)\psi_{\mathcal{T},k}^{(\alpha 3,\beta 3,\mu)}(t)\\
&+\gamma_1\phi_{\mathcal{L}_1,i}^{(\alpha_1,\beta_1)}(x)\varphi_{\mathcal{L}_2,j}^{(\alpha_2,\beta_2)}(y)\psi_{\mathcal{T},k}^{(\alpha 3,\beta 3,\mu-1)}(t)\\
&+\gamma_2\phi_{\mathcal{L}_1,i}^{(\alpha_1,\beta_1)}(x)\varphi_{\mathcal{L}_2,j}^{(\alpha_2,\beta_2)}(y)\psi_{\mathcal{T},k}^{(\alpha 3,\beta 3)}(t)\\
&-\gamma_3\frac{\partial^2}{\partial x^2}(\phi_{\mathcal{L}_1,i}^{(\alpha_1,\beta_1)}(x)\varphi_{\mathcal{L}_2,j}^{(\alpha_2,\beta_2)}(y)\psi_{\mathcal{T},k}^{(\alpha 3,\beta 3)}(t))\\
&-\gamma_3\frac{\partial^2}{\partial y^2}(\phi_{\mathcal{L}_1,i}^{(\alpha_1,\beta_1)}(x)\varphi_{\mathcal{L}_2,j}^{(\alpha_2,\beta_2)}(y)\psi_{\mathcal{T},k}^{(\alpha 3,\beta 3)}(t)),
\end{aligned}$$

at that point, (45) can be modified as:

$$\sum_{i=0}^{N-2}\sum_{j=0}^{M-2}\sum_{k=0}^{K-2} c_{i,j,k}f_{i,j,k}(x,y,t)=\mathcal{H}(x,y,t).\tag{46}$$

Collocating (46) in $N-1,\ M-1$ and $K-1$ roots of the shifted Jacobi polynomials $\mathcal{P}_{\mathcal{L},N-1}^{(\alpha,\beta)}(x)$, the Gauss-shifted Jacobi nodes, we obtain

$$\begin{aligned}
&\sum_{i=0}^{N-2}\sum_{j=0}^{M-2}\sum_{k=0}^{K-2} c_{i,j,k}f_{i,j,k}(x_{\mathcal{L}_1,n,i}^{(\alpha_1,\beta_1)},y_{\mathcal{L}_2,m,j}^{(\alpha_2,\beta_2)},t_{\mathcal{T},l,k}^{(\alpha 3,\beta 3)})\\
&\qquad=\mathcal{H}(x_{\mathcal{L}_1,n,i}^{(\alpha_1,\beta_1)},y_{\mathcal{L}_2,m,j}^{(\alpha_2,\beta_2)},t_{\mathcal{T},l,k}^{(\alpha 3,\beta 3)}),\ \mathit{for}\ n=0,1,\dots,N-2,\\
&\qquad\qquad m=0,1,\dots,M-2,\ l=0,1,\dots,K-2,
\end{aligned}\tag{47}$$

which can be written in the matrix form $\mathbf{F}^T C=\mathbf{R}$, where

$$\mathbf{R} = [R_{0,0,0}, R_{0,0,1}, \ldots, R_{0,0,K-2}, R_{0,1,0}, R_{0,1,1}, \ldots, R_{0,1,K-2}, \ldots,$$
$$R_{0,M-2,K-2}; R_{1,0,0}, R_{1,0,1}, \ldots, R_{1,0,K-2}, R_{1,1,0}, R_{1,1,1}, \ldots,$$
$$R_{1,1,K-2}, \ldots, R_{1,M-2,K-2}; R_{N-2,0,0}, R_{N-2,0,1}, \ldots, R_{N-2,0,K-2},$$
$$R_{N-2,1,0}, R_{N-2,1,1}, \ldots, R_{N-2,1,K-2}, \ldots, R_{N-2,M-2,K-2}]^T,$$

$$\begin{aligned} R_{i,j,k} =& \mathcal{H}(x^{(\alpha_1,\beta_1)}_{\mathcal{L}_1,n,i}, y^{(\alpha_2,\beta_2)}_{\mathcal{L}_2,m,j}, t^{(\alpha 3,\beta 3)}_{\mathcal{T},l,k}), \quad i = 0,1,\ldots,N-2, \\ & j = 0,1,\ldots,M-2,\ k = 0,1,\ldots,K-2, \end{aligned}$$

and

$$\begin{aligned} \mathbf{F} =& (f_{ijknml}),\ i,n = 0,1,\ldots,N-2,\ j,m = 0,1,\ldots,M-2, \\ & k,l = 0,1,\ldots,K-2, \end{aligned}$$

in which the elements of the matrix $\mathbf{F}$ are determined as follows:

$$\begin{aligned} f_{ijknml} =& f_{i,j,k}(x^{(\alpha_1,\beta_1)}_{\mathcal{L}_1,n,i}, y^{(\alpha_2,\beta_2)}_{\mathcal{L}_2,m,j}, t^{(\alpha 3,\beta 3)}_{\mathcal{T},l,k}), \\ & i,n = 0,1,\ldots,N-2, \\ & j,m = 0,1,\ldots,M-2,\ k,l = 0,1,\ldots,K-2. \end{aligned}$$

In our implementation, this system has been solved using the Mathematica function FindRoot with zero initial approximation. In this manner, the approximate solution of (37) is given by $u(x,y,t) = \chi(x,y,t)C$.

4.2 Two-dimensional T-FOTE with nonhomogeneous conditions

In the accompanying, we simplify alter the right-hand side to deal with the nonhomogeneous conditions. Give us a chance to treat with two-dimensional (2D) T-FOTE (37) with the nonhomogeneous initial-boundary conditions:

$$\begin{aligned} u(0,y,t) =& q_0(y,t),\ u(\mathcal{L}_1,y,t) = q_1(y,t), \quad 0 \le y \le \mathcal{L}_2, 0 \le t \le \mathcal{T}, \\ u(x,0,t) =& q_2(x,t),\ u(x,\mathcal{L}_2,t) = q_3(x,t), \quad 0 \le x \le \mathcal{L}_1, 0 \le t \le \mathcal{T}, \\ u(x,y,0) =& q_4(x,y),\ \frac{\partial u(x,y,t)}{\partial t}|_{t=0} = q_5(x,y), \quad 0 \le x \le \mathcal{L}_1, 0 \le y \le \mathcal{L}_2, \end{aligned} \tag{48}$$

where $q_0(y,t),\ q_1(y,t),\ q_2(x,t),\ q_3(x,t),\ q_4(x,y)$ and $q_5(x,y)$ are given functions.

Presently, assume the accompanying transformation

$$\begin{aligned} V(x,y,t) =& u(x,y,t) + a_0(y,t) + x a_1(y,t) \\ & + x(x-\mathcal{L}_1)(b_0(x,t) + y b_1(x,t)) \\ & + x(x-\mathcal{L}_1)y(y-\mathcal{L}_2)(c_0(x,y) + t c_1(x,y)), \end{aligned} \tag{49}$$

where

$$a_0(y,t) = -q_0(y,t), \quad a_1(y,t) = \frac{q_0(y,t) - q_1(y,t)}{\mathcal{L}_1},$$
$$b_0(x,t) = \frac{(\mathcal{L}_1 - x)q_0(0,t) - \mathcal{L}_1 q_2(x,t) + x q_1(0,t)}{x(x-\mathcal{L}_1)\mathcal{L}_1},$$

$$b_1(x,t) = \frac{\mathcal{L}_1(q_2(x,t) - q_3(x,t)) + (\mathcal{L}_1 - x)(q_0(\mathcal{L}_2,t) - q_0(0,t)) + x(q_1(\mathcal{L}_2,t) - q_1(0,t))}{\mathcal{L}_1\mathcal{L}_2 x(x-\mathcal{L}_1)},$$
$$\begin{aligned} c_0(x,y) &= \frac{\mathcal{L}_1(q_2(x,0) + q_0(y,0) - q_4(x,y)) + x(q_1(y,0) - q_1(0,0) - q_0(y,0)) - q_0(0,0)(\mathcal{L}_1 - x)}{xy\mathcal{L}_1(x-\mathcal{L}_1)(y-\mathcal{L}_2)} \\ &\quad + \frac{y\mathcal{L}_1(q_3(x,0) - q_2(x,0)) + y(\mathcal{L}_1 - x)(q_0(0,0) - q_0(\mathcal{L}_2,0)) + xy(q_1(0,0) - q_1(\mathcal{L}_2,0))}{xy\mathcal{L}_1\mathcal{L}_2(x-\mathcal{L}_1)(y-\mathcal{L}_2)}, \end{aligned}$$

$$\begin{aligned} c_1(x,y) &= \left[\frac{\mathcal{L}_1(\frac{\partial q_0(y,t)}{\partial t} - q_5(x,y) - \frac{\partial q_0(0,t)}{\partial t} + \frac{\partial q_2(x,t)}{\partial t}) + x(\frac{\partial q_1(y,t)}{\partial t} - \frac{\partial q_0(y,t)}{\partial t} + \frac{\partial q_0(0,t)}{\partial t} - \frac{\partial q_1(0,t)}{\partial t})}{xy\mathcal{L}_1(x-\mathcal{L}_1)(y-\mathcal{L}_2)}\right]_{t=0} \\ &\quad + \left[\frac{y\mathcal{L}_1(\frac{\partial q_3(x,t)}{\partial t} - \frac{\partial q_2(x,t)}{\partial t}) + y(\mathcal{L}_1 - x)(\frac{\partial q_0(0,t)}{\partial t} - \frac{\partial q_0(\mathcal{L}_2,t)}{\partial t}) + xy(\frac{\partial q_1(0,t)}{\partial t} - \frac{\partial q_1(\mathcal{L}_2,t)}{\partial t})}{xy\mathcal{L}_1\mathcal{L}_2(x-\mathcal{L}_1)(y-\mathcal{L}_2)}\right]_{t=0}. \end{aligned}$$

The mapping (49) changes the nonhomogeneous conditions (48) into the following homogeneous conditions:

$$\begin{aligned} V(0,y,t) = V(\mathcal{L}_1,y,t) = V(x,0,t) = V(x,\mathcal{L}_2,t) = V(x,y,0) \\ = \frac{\partial V(x,y,t)}{\partial t}|_{t=0} = 0, \end{aligned} \tag{50}$$

Subsequently it suffices to solve the following 2D T-FOTE:

$$\begin{aligned} \frac{\partial^\mu V(x,y,t)}{\partial t^\mu} + \gamma_1 \frac{\partial^{\mu-1} V(x,y,t)}{\partial t^{\mu-1}} + \gamma_2 V(x,y,t) \\ = \gamma_3(\frac{\partial^2 V(x,y,t)}{\partial x^2} + \frac{\partial^2 V(x,y,t)}{\partial y^2}) + \mathcal{G}(x,y,t), \\ 0 \le x \le \mathcal{L}_1,\ 0 \le y \le \mathcal{L}_2,\ 0 \le t \le \mathcal{T}, \end{aligned} \tag{51}$$

subject to the homogeneous initial-boundary conditions (50), where $V(x,y,t)$ is given by (49), and

$$
\begin{aligned}
\mathcal{G}(x,y,t) =&\mathcal{H}(x,y,t) + \frac{\partial^{\mu} a_0(y,t)}{\partial t^{\mu}} + \gamma_1 \frac{\partial^{\mu-1} a_0(y,t)}{\partial t^{\mu-1}} + \gamma_2 a_0(y,t) \\
& - 2\gamma_3 (b_0(x,t) + y b_1(x,t)) - \gamma_3 \frac{\partial^2 a_0(y,t)}{\partial y^2} + x(\frac{\partial^{\mu} a_1(y,t)}{\partial t^{\mu}} \\
& + \gamma_1 \frac{\partial^{\mu-1} a_1(y,t)}{\partial t^{\mu-1}} + \gamma_2 a_1(y,t) - \gamma_3 \frac{\partial^2 a_1(y,t)}{\partial y^2}) \\
& + x(x-\mathcal{L}_1)(\frac{\partial^{\mu} b_0(x,t)}{\partial t^{\mu}} + y \frac{\partial^{\mu} b_1(x,t)}{\partial t^{\mu}} \\
& + \gamma_1 (\frac{\partial^{\mu-1} b_0(x,t)}{\partial t^{\mu-1}} + y \frac{\partial^{\mu-1} b_1(x,t)}{\partial t^{\mu-1}}) \\
& + \gamma_2 (b_0(x,t) + y b_1(x,t)) \\
& - \gamma_3 (\frac{\partial^2 b_0(x,t)}{\partial x^2} + y \frac{\partial^2 b_1(x,t)}{\partial x^2} + \frac{\partial^2 c_0(x,y)}{\partial y^2} + t \frac{\partial^2 c_1(x,y)}{\partial y^2} \\
& + 2 \frac{\partial c_0(x,y)}{\partial y} + 2t \frac{\partial c_1(x,y)}{\partial y})) \\
& + x(x-\mathcal{L}_1)(y-\mathcal{L}_2)(c_1(x,y) \frac{\Gamma(2) t^{1-\mu}}{\Gamma(2-\mu)} \\
& + \gamma_1 c_1(x,y) \frac{\Gamma(2) t^{2-\mu}}{\Gamma(3-\mu)} + \gamma_2 (c_0(x,y) + t c_1(x,y)) \\
& - \gamma_3 (\frac{\partial^2 c_0(x,y)}{\partial x^2} + t \frac{\partial^2 c_1(x,y)}{\partial x^2})) \\
& - 2\gamma_3 (2x - \mathcal{L}_1)(\frac{\partial b_0(x,t)}{\partial x} + y \frac{\partial b_1(x,t)}{\partial x}) \\
& - 2\gamma_3 (y-\mathcal{L}_2)(c_0(x,y) + t c_1(x,y)) \\
& - 2\gamma_3 (2x-\mathcal{L}_1)(y-\mathcal{L}_2)(\frac{\partial c_0(x,y)}{\partial x} + t \frac{\partial c_1(x,y)}{\partial x}).
\end{aligned}
$$

5 Convergence and error analysis

In this section, following the analysis given in [11], we have the following theorems. It is worthy to note here that the proofs of the next theorems are similar to the proofs given in [11].

Lemma 1. The Caputo fractional derivative of the shifted Jacobi polynomials satisfies the following estimate:

$$
|\,{}^{c}D^{\mu}\,\mathcal{P}^{(\alpha,\beta)}_{\mathcal{L},k}(x)| \leq \mathcal{C}\, i^{\mu+q},
$$

where $\mathcal{C}$ is a positive generic constant and $q = \max(\alpha, \beta, -\frac{1}{2})$.

Theorem 1. If $\xi_i = \dfrac{(2i+\lambda_1+1)_2}{\mathcal{L}^2(i+\alpha_1+1)(i+\beta_1+1)}$ and $\eta_j = \dfrac{(2j+\lambda_2+1)_2}{\tau^2(j+\beta_2+1)_2}$, then we have the following two connection formulas:

$$\phi_i(x) = \xi_i\, x(\mathcal{L}-x)P_{i,\mathcal{L}}^{(\alpha_1+1,\beta_1+1)}(x),$$

$$\psi_j(t) = \eta_j\, t^2 P_{j,\tau}^{(\alpha_2,\beta_2+2)}(x).$$

Proof. The two formulas are easily obtained by application of the moment formula given in Doha [7]. □

Theorem 2. The following two orthogonality relations are valid

$$\int_0^{\mathcal{L}} \phi_i(x)\phi_j(x)\, x^{\beta_1-1}\,(\mathcal{L}-x)^{\beta_1-1}dx$$
$$= \delta_{ij}\frac{L^{\lambda_1-2}\,(2i+\lambda_1+1)^2\,(2i+\lambda_1+2)\,\Gamma(i+\alpha_1+1)\,\Gamma(i+\beta_1+1)}{(i+\alpha_1+1)\,(i+\beta_1+1)\,i!\,\Gamma(i+\lambda_1+2)},$$

$$\int_0^{\tau} \psi_m(t)\psi_n(t)\, t^{\beta_2-2}\,(\tau-t)^{\alpha_2}dt$$
$$= \delta_{mn}\frac{L^{\lambda_2-2}\,(2m+\lambda_2+1)^2\,(2m+\lambda_2+2)\,\Gamma(m+\alpha_2+1)\,\Gamma(m+\beta_2+1)}{(m+\beta_2+1)_2\,m!\,\Gamma(m+\lambda_2+2)}.$$

Proof. The proof is a direct consequence from the orthogonality relation (13). □

Theorem 3. (Convergence) If $u(x,t)$ the exact solution of (17) is separable in the sense that $u(x,t) = x(\mathcal{L}-x)tf(x)g(t)$ and f, g are C^3 functions with $\mid f^{'''}(x) \mid\leq a$, $\mid g^{'''}(t) \mid\leq b$, where a, b are positive constants, then the expansion coefficients c_{ij} in (19) satisfy the following estimate:

$$\mid c_{ij} \mid= \mathcal{O}(i^{-\frac{5}{2}}j^{-\frac{5}{2}}) \quad \text{for all } i,j>3.$$

Moreover the series in (19) converges absolutely as $N, M \to \infty$.

Theorem 4. If $u, u_{N,M}$ are the exact and approximate solutions of (17), respectively, and under the assumptions of Theorem 3, then we have the following truncation error estimate

$$\|u-u_{N,M}\|_2 = \mathcal{O}(N^{-\frac{3}{2}}M^{-\frac{3}{2}}).$$

Theorem 5. If $e_{N,M} = u - u_{N,M}$ is the truncation error of the solution of (17) and under the assumptions of Theorem 3, then we have the following global error estimate:

$$
\begin{aligned}
&\| {}^{c}D_t^{\mu}\, e_{N,M} + \gamma_1\ {}^{c}D_t^{\mu-1}\, e_{N,M} + \gamma_2\, e_{N,M} - \gamma_3\, D_x^2\, e_{N,M} \|_2 \\
&\qquad \le \rho_1\, M^{\mu+q-\frac{3}{2}}\, N^{-\frac{3}{2}} + \rho_2\, |\gamma_1|\, M^{\mu+q-\frac{5}{2}}\, N^{-\frac{3}{2}} \\
&\qquad\quad + \rho_3\, |\gamma_2|\, M^{-\frac{3}{2}}\, N^{-\frac{3}{2}} + \rho_4\, |\gamma_3|\, M^{-\frac{3}{2}}\, N^{q+\frac{1}{2}}.
\end{aligned}
$$

Theorem 6. [Convergence] If $u(x,t)$ the exact solution of (37) is separable in the sense that $u(x,y,t) = x(\mathcal{L}_\infty - x)y(\mathcal{L}_\in - y)tf(x)\, g(y)\, h(t)$ and f, g, h are C^3 functions with $\mid f^{'''}(x) \mid \le a,\ \mid g^{'''}(x) \mid \le b,\ \mid h^{'''}(t) \mid \le c$, where a, b, c are positive constants, then the expansion coefficients c_{ijk} in (39) satisfy the following estimate:

$$
\mid c_{ijk} \mid = \mathcal{O}(i^{-\frac{5}{2}} j^{-\frac{5}{2}} k^{-\frac{5}{2}}) \ \text{ for all } i, j, k > 3.
$$

Moreover the series in (39) converges absolutely as $N, M, K \to \infty$.

Theorem 7. If $u, u_{N,M,K}$ are the exact and approximate solutions of (37), respectively, and under the assumptions of Theorem 6, then we have the following truncation error estimate:

$$
\|u - u_{N,M,K}\|_2 = \mathcal{O}(N^{-\frac{3}{2}} M^{-\frac{3}{2}} K^{-\frac{3}{2}}).
$$

Theorem 8. If $e_{N,M,K} = u - u_{N,M,K}$ is the truncation error of the solution of (37) and under the assumptions of Theorem 6, then we have the following global error estimate:

$$
\begin{aligned}
&\| {}^{c}D_t^{\mu}\, e_{N,M,K} + \gamma_1\ {}^{c}D_t^{\mu-1}\, e_{N,M,K} + \gamma_2\, e_{N,M,K} - \gamma_3\, (D_x^2 + D_y^2)\, e_{N,M,K} \|_2 \\
&\qquad \le \tilde{\rho}_1\, K^{\mu+q-\frac{3}{2}}\, M^{-\frac{3}{2}}\, N^{-\frac{3}{2}} + \tilde{\rho}_2\, |\gamma_1|\, K^{\mu+q-\frac{5}{2}}\, M^{-\frac{3}{2}}\, N^{-\frac{3}{2}} \\
&\qquad\quad + \tilde{\rho}_3\, |\gamma_2|\, M^{-\frac{3}{2}}\, N^{-\frac{3}{2}}\, K^{-\frac{3}{2}} \\
&\qquad\quad + |\gamma_3| \left(\tilde{\rho}_4\ M^{-\frac{3}{2}}\, N^{q+\frac{1}{2}}\, K^{-\frac{3}{2}} + \tilde{\rho}_5\ N^{-\frac{3}{2}}\, M^{q+\frac{1}{2}}\, K^{-\frac{3}{2}} \right).
\end{aligned}
$$

6 Numerical results

In this section, a few examples are outlined to demonstrate the pertinence and proficiency of the novel technique in homogeneous and nonhomogeneous conditions. The calculations are executed by utilizing Mathematica of Version 8, and all counts are completed in a PC of CPU Intel(R) Core(TM) i3-2350M 2 Duo CPU 2.30 GHz, 6.00 GB of RAM.

The distinction between the measured value of approximate solution and its actual value (absolute error), are given by

$$
E(x,t) = |u(x,t) - \widetilde{u}(x,t)|, \tag{52}
$$

$$E(x,y,t)=|u(x,y,t)-\widetilde{u}(x,y,t)|, \tag{53}$$

where $u(x,t)$ and $\widetilde{u}(x,t)$ are the exact solution and the numerical solution at the point (x,t), respectively. Also $u(x,y,t)$ and $\widetilde{u}(x,y,t)$ are the exact solution and the numerical solution at the point (x,y,t), respectively.

Moreover, the maximum absolute errors (MAEs) is given by

$$\text{MAEs}=\text{Max}\{E(x,t):\forall(x,t)\in[0,\mathcal{L}]\times[0,\mathcal{T}]\}=L^{\infty}, \tag{54}$$

$$\text{MAEs}=\text{Max}\{E(x,y,t):\forall(x,y,t)\in[0,\mathcal{L}_1]\times[0,\mathcal{L}_2]\times[0,\mathcal{T}]\}=L^{\infty}. \tag{55}$$

Also we can denote to the root mean square error (RMSE) by

$$RMSE=\sqrt{\frac{\sum\limits_{i=0}^{N-2}(u(x_{\mathcal{L},n,i}^{(\alpha_1,\beta_1)},t)-\widetilde{u}(x_{\mathcal{L},n,i}^{(\alpha_1,\beta_1)},t))^2}{N}}, \tag{56}$$

or

$$RMSE=\sqrt{\frac{\sum\limits_{i=0}^{N-2}\sum\limits_{j=0}^{M-2}(u(x_{\mathcal{L}_1,n,i}^{(\alpha_1,\beta_1)},y_{\mathcal{L}_2,m,j}^{(\alpha_2,\beta_2)},t)-\widetilde{u}(x_{\mathcal{L}_1,n,i}^{(\alpha_1,\beta_1)},y_{\mathcal{L}_2,m,j}^{(\alpha_2,\beta_2)},t))^2}{NM}}. \tag{57}$$

Example 1. Give us initial a chance to consider the accompanying T-FOTE (17) in the domain $0\le x\le\mathcal{L}$, with taking after initial and boundary conditions,

$$u(0,t)=u(\mathcal{L},t)=0,\quad 0\le x\le\mathcal{L},$$
$$u(x,0)=\frac{\partial u(x,t)}{\partial t}|_{t=0}=0,\quad t\ge 0.$$

The exact solution is given by

$$u(x,t)=t^2(\mathcal{L}-x)\sin^2(x),$$

and

$$\begin{aligned}\mathcal{H}(x,t)=&\frac{2t^{2-\mu}(x-\mathcal{L})\sin^2(x)\,(\mu-\gamma_1 t-3)}{\Gamma(4-\mu)}+t^2\gamma_2(\mathcal{L}-x)\sin^2(x)\\&+2t^2\gamma_3((x-\mathcal{L})\cos(2x)+\sin(2x))\end{aligned}.$$

RMSE is shown in Table 1 for different values of $\alpha_1,\ \beta_1,\ \alpha_2,\ \beta_2,\ \gamma_1,\ \gamma_2$ and γ_3 and $\mathcal{H}(x,t)$. Figure 1 shows the exact solution and numerical solutions of Example 1, where $N=14,\ M=4,\ \gamma_1=\gamma_2=\gamma_3=1,\ \mathcal{L}=\mathcal{T}=5,\ \mu=$

Table 1: Comparison of RMSE for Example 1, where $\mathcal{L} = \mathcal{T} = 1$ and $N = 12,\ M = 4$.

$\gamma_1, \gamma_2, \gamma_3$	Our method ($\mu = 1.25$)		Our method ($\mu = 1.75$)	
	$\alpha_1 = \beta_1 = -\frac{1}{2}$, $\alpha_2 = \beta_2 = -\frac{1}{2}$	$\alpha_1 = \beta_1 = \frac{1}{2}$, $\alpha_2 = \beta_2 = \frac{1}{2}$	$\alpha_1 = \beta_1 = -\frac{1}{2}$, $\alpha_2 = \beta_2 = -\frac{1}{2}$	$\alpha_1 = \beta_1 = \frac{1}{2}$, $\alpha_2 = \beta_2 = \frac{1}{2}$
$\gamma_1 = 0, \gamma_2 = \gamma_3 = 1$	$5.8845.10^{-11}$	$2.6905.10^{-10}$	$6.0047.10^{-11}$	$8.1272.10^{-11}$
$\gamma_2 = 0, \gamma_1 = \gamma_3 = 1$	$1.7662.10^{-10}$	$4.6050.10^{-10}$	$6.3574.10^{-11}$	$1.2146.10^{-10}$
$\gamma_3 = 0, \gamma_1 = \gamma_2 = 1$	$1.8565.10^{-11}$	$2.4882.10^{-11}$	$9.0080.10^{-12}$	$7.2685.10^{-12}$
$\gamma_1 = 1, \gamma_2 = \gamma_3 = 0$	$1.7330.10^{-11}$	$2.2806.10^{-11}$	$1.0194.10^{-11}$	$1.1746.10^{-11}$
$\gamma_2 = 1, \gamma_1 = \gamma_3 = 0$	$2.2531.10^{-11}$	$2.3049.10^{-11}$	$6.4857.10^{-12}$	$8.7682.10^{-12}$
$\gamma_3 = 1, \gamma_1 = \gamma_2 = 0$	$3.3257.10^{-11}$	$2.6122.10^{-10}$	$8.2929.10^{-11}$	$2.6548.10^{-10}$

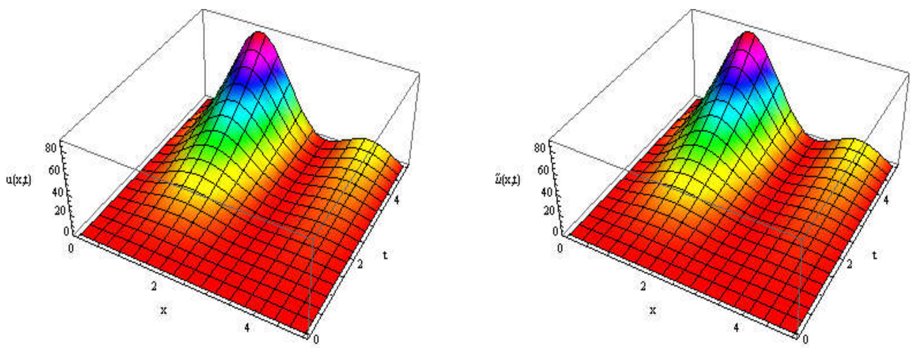


Figure 1: The exact $u(x,t)$ and numerical $\widetilde{u}(x,t)$ solutions for Example 1, where $N = 14,\ M = 4,\ \gamma_1 = \gamma_2 = \gamma_3 = 1,\ \mathcal{L} = \mathcal{T} = 5$, and $\mu = 1.95$.

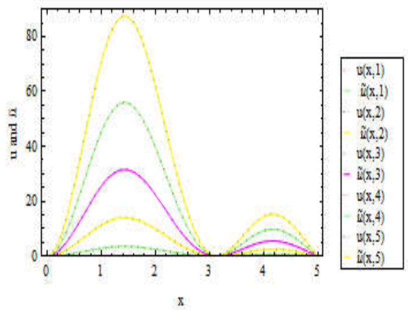
1.95, and $\alpha_1 = \beta_1 = \alpha_2 = \beta_2 = 0$. In Figures 2 and 3, we depicted the following:

Fig. 2 The curves-graph of exact $u(x,t)$ and numerical $\widetilde{u}(x,t)$ solutions for Example 1, where $N = 14,\ M = 4,\ \gamma_1 = \gamma_2 = \gamma_3 = 1,\ \mathcal{L} = \mathcal{T} = 5,\ \mu = 1.95$ and $\alpha_1 = \beta_1 = \alpha_2 = \beta_2 = 0$ at five different values of t,

Fig. 3 The curves-graph of exact $u(x,t)$ and numerical $\widetilde{u}(x,t)$ solutions for Example 1, where $N = 14,\ M = 4,\ \gamma_1 = \gamma_2 = \gamma_3 = 1,\ \mathcal{L} = \mathcal{T} = 5,\ \mu = 1.95$ and $\alpha_1 = \beta_1 = \alpha_2 = \beta_2 = 0$ at five different values of x.

It should be noted here that the CPU time τ used to implement our algorithm was computed using the *TimeUsed* command in Mathematica and was found to be $15 < \tau < 22$, $17 < \tau < 28$ in seconds for $N = 12,\ M = 4$, $N = 14,\ M = 4$, respectively.

Example 2. In this example, we considered the T-FOTE (17) with the coefficients $\gamma_1 = \gamma_2 = 1,\ \gamma_3 = \pi$ and the following initial and boundary conditions:



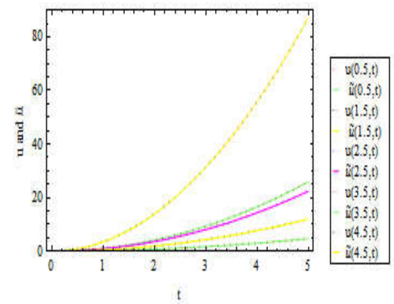


Figure 2: The curves-graph of exact $u(x,t)$ and numerical $\widetilde{u}(x,t)$ solutions for Example 1, where $N = 14,\ M = 4,\ \gamma_1 = \gamma_2 = \gamma_3 = 1,\ \mathcal{L} = \mathcal{T} = 5$, and $\mu = 1.95$ at five different values of t.

Figure 3: The curves-graph of exact $u(x,t)$ and numerical $\widetilde{u}(x,t)$ solutions for Example 1 where $N = 14,\ M = 4,\ \gamma_1 = \gamma_2 = \gamma_3 = 1,\ \mathcal{L} = \mathcal{T} = 5$ and $\mu = 1.95$ at five different values of x.

$$
\begin{aligned}
&u(0,t) = 0, \quad u(\mathcal{L},t) = t^3 \sin^2(\mathcal{L}), \quad 0 \le t \le \mathcal{T},\\
&u(x,0) = 0, \quad \frac{\partial u(x,t)}{\partial t}|_{t=0} = 0, \quad 0 \le x \le \mathcal{L}.
\end{aligned}
$$

where the source term $\mathcal{H}(x,t)$ is chosen to be consistent with the exact solution $u(x,t) = t^3 \sin^2(x)$.

The results of Example 2 are reported in Table 2 and Figures 4, 5, and 6. Table 2 compares RMSE with [26] at different values of $\mu,\ N,\ M,\ \alpha_1,\ \beta_1,\ \alpha_2$, and β_2. The space-time graphs of the exact and approximate solutions at $N = 20,\ M = 3,\ \mathcal{L} = \mathcal{T} = 10$, and $\mu = 1.05$ are plotted in Figure 4. We plotted in Figures 5 and 6.

Fig. 5 The curves-graph of exact $u(x,t)$ and numerical $\widetilde{u}(x,t)$ solutions for Example 2, where $N = 20,\ M = 3,\ \mathcal{L} = \mathcal{T} = 10$, and $\mu = 1.05$ at five different values of t,

Fig. 6 The curves-graph of exact $u(x,t)$ and numerical $\widetilde{u}(x,t)$ solutions for Example 2, where $N = 20,\ M = 3,\ \mathcal{L} = \mathcal{T} = 10$ and $\mu = 1.05$ at five different values of x.

It should be noted here that the CPU time τ used to implement our algorithm was computed using the *TimeUsed* command in Mathematica and was found to be $33 < \tau < 41$ for $N = 20,\ M = 3$.

Example 3. Let us test the T-FOTE (17) with the coefficients $\gamma_1 = \gamma_2 = \gamma_3 = 1$, and $\mathcal{H}(x,t)$ according to the exact solution $u(x,t) = x\cos(x^2 + t^2)$ and the following initial and boundary conditions:

Table 2: RMSE for Example 2 at $\mathcal{L} = \mathcal{T} = 1$.

μ	Sweilam et al. $(m = 20,\ n = 3)$ [26]	Our method $(N = 12,\ M = 3)$ $\alpha_1 = \beta_1 = -\frac{1}{2}$, $\alpha_2 = \beta_2 = -\frac{1}{2}$	$\alpha_1 = \beta_1 = 0$, $\alpha_2 = \beta_2 = 0$	$\alpha_1 = \beta_1 = \frac{1}{2}$, $\alpha_2 = \beta_2 = \frac{1}{2}$
1.75	$1.0474.10^{-5}$	$4.2800.10^{-11}$	$7.4066.10^{-11}$	$7.8044.10^{-11}$
1.95	$1.0080.10^{-5}$	$1.8994.10^{-11}$	$2.7463.10^{-11}$	$9.4081.10^{-11}$
μ	Sweilam et al. $(m = 20,\ n = 4)$ [26]	Our method $(N = 12,\ M = 4)$ $\alpha_1 = \beta_1 = -\frac{1}{2}$, $\alpha_2 = \beta_2 = -\frac{1}{2}$	$\alpha_1 = \beta_1 = 0$, $\alpha_2 = \beta_2 = 0$	$\alpha_1 = \beta_1 = \frac{1}{2}$, $\alpha_2 = \beta_2 = \frac{1}{2}$
1.75	$1.0429.10^{-5}$	$8.2456.10^{-11}$	$7.9494.10^{-11}$	$2.2020.10^{-10}$
1.95	$9.8685.10^{-6}$	$3.5018.10^{-11}$	$1.3740.10^{-10}$	$3.8586.10^{-10}$

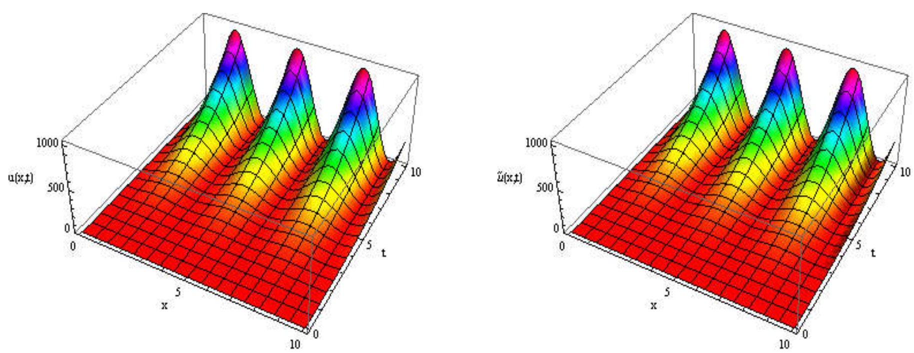
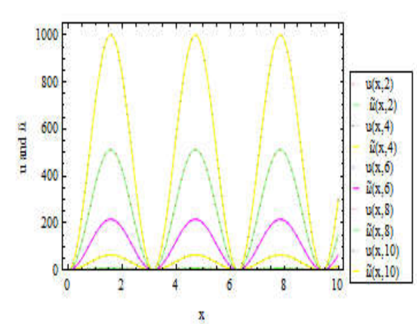


Figure 4: The exact $u(x,t)$ and numerical $\widetilde{u}(x,t)$ solutions for Example 2, where $N = 20,\ M = 3,\ \mathcal{L} = \mathcal{T} = 10$, and $\mu = 1.05$.



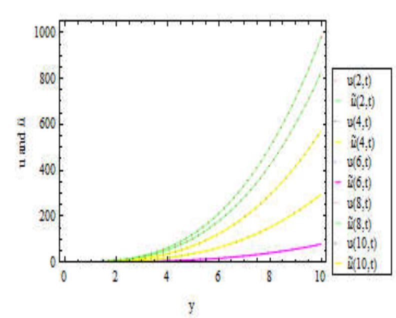


Figure 5: The curves-graph of exact $u(x,t)$ and numerical $\widetilde{u}(x,t)$ solutions for Example 2, where $N = 20,\ M = 3,\ \mathcal{L} = \mathcal{T} = 10$, and $\mu = 1.05$ at five different values of t.

Figure 6: The curves-graph of exact $u(x,t)$ and numerical $\widetilde{u}(x,t)$ solutions for Example 2, where $N = 20,\ M = 3,\ \mathcal{L} = \mathcal{T} = 10$, and $\mu = 1.05$ at five different values of x.

Table 3: TRMSE for Example 3 at $\mathcal{L} = \mathcal{T} = 1$.

μ	Hosseini et al. $(N = n = 50)$ [18]	Our method $(N = 12 = M = 3)$		
		$\alpha_1 = \beta_1 = -\frac{1}{2}$, $\alpha_2 = \beta_2 = -\frac{1}{2}$	$\alpha_1 = \beta_1 = 0$ $\alpha_2 = \beta_2 = 0$	$\alpha_1 = \beta_1 = \frac{1}{2}$, $\alpha_2 = \beta_2 = \frac{1}{2}$
1.25	$2.6551.10^{-3}$	$5.7636.10^{-8}$	$5.9213.10^{-8}$	$5.8662.10^{-8}$
1.5	$4.1743.10^{-3}$	$3.4483.10^{-8}$	$5.8363.10^{-8}$	$1.0047.10^{-7}$
1.75	$5.4164.10^{-3}$	$4.2018.10^{-8}$	$9.7424.10^{-8}$	$2.2745.10^{-7}$
1.95	$3.8800.10^{-3}$	$1.4632.10^{-8}$	$4.6185.10^{-8}$	$4.3415.10^{-8}$

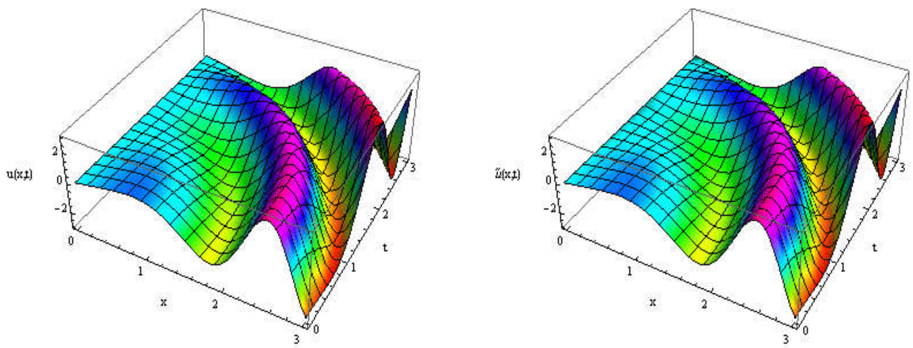


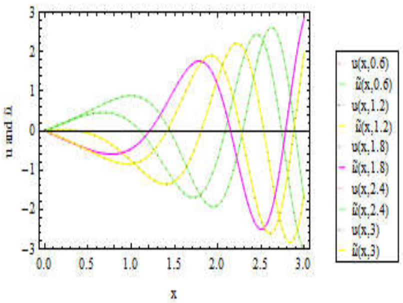
Figure 7: The exact $u(x,t)$ and numerical $\widetilde{u}(x,t)$ solutions for Example 3, where $N = 20$, $M = 20$, $\mathcal{L} = \mathcal{T} = 3$, and $\mu = 1.05$.

$$u(0,t) = 0, \quad u(\mathcal{L},t) = \mathcal{L}\sin(\mathcal{L}^2 + t^2), \quad 0 \le t \le \mathcal{T},$$
$$u(x,0) = x\cos(x^2), \quad \frac{\partial u(x,t)}{\partial t}|_{t=0} = 0, \quad 0 \le x \le \mathcal{L}.$$

In Table 3, we give the RMSE with various choices of μ, α_1, β_1, α_2, β_2, and $\mathcal{L} = \mathcal{T} = 1$. The outcomes are contrasted with the outcome of radial basis function [18]. It is clear from this table that the solution gotten by our technique is great in examination with radial basis function [18]. The numerical and exact solutions are compared in Figure 7, where $N = M = 20$, $\alpha_1 = \beta_1 = \alpha_2 = \beta_2 = 1$, $\mu = 1.05$, and $\mathcal{L} = \mathcal{T} = 3$ Moreover, in Figures 8 and 9, we sketched the following:

Fig. 8 The curves-graph of exact $u(x,t)$ and numerical $\widetilde{u}(x,t)$ solutions for Example 3, where $N = M = 20$, $\mathcal{L} = \mathcal{T} = 3$, $\alpha_1 = \beta_1 = \alpha_2 = \beta_2 = 1$, and $\mu = 1.05$ at five different values of t,

Fig. 9 The curves-graph of exact $u(x,t)$ and numerical $\widetilde{u}(x,t)$ solutions for Example 3, where $N = M = 20$, $\mathcal{L} = \mathcal{T} = 3$, $\alpha_1 = \beta_1 = \alpha_2 = \beta_2 = 1$, and $\mu = 1.05$ at five different values of x.



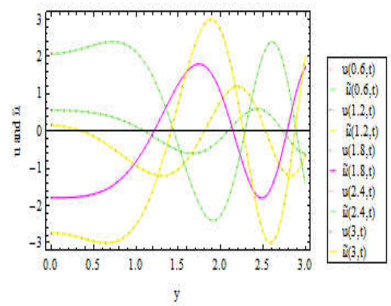


Figure 8: The curves-graph of exact $u(x,t)$ and numerical $\widetilde{u}(x,t)$ solutions for Example 3, where $N = 20,\ M = 20,\ \mathcal{L} = \mathcal{T} = 3$, and $\mu = 1.05$ at five different values of t.

Figure 9: The curves-graph of exact $u(x,t)$ and numerical $\widetilde{u}(x,t)$ solutions for Example 3, where $N = 20,\ M = 20,\ \mathcal{L} = \mathcal{T} = 3$, and $\mu = 1.05$ at five different values of x.

Table 4: RMSE for Example 4 at $\mu = 1.5$.

(N, M, K)	$_1 =_1= -\frac{1}{2}$, $_2 =_2= -\frac{1}{2}$, $_3 =_3= -\frac{1}{2}$	$_1 =_1= 0$, $_2 =_2= 0$, $_3 =_3= 0$	$_1 =_1= \frac{1}{2}$, $_2 =_2= \frac{1}{2}$, $_3 =_3= \frac{1}{2}$
(2,2,2)	4.8245×10^{-5}	1.2683×10^{-4}	1.9055×10^{-4}
(4,4,4)	2.3555×10^{-5}	2.1954×10^{-5}	2.0665×10^{-5}
(6,6,6)	1.5245×10^{-7}	1.6220×10^{-7}	3.9732×10^{-7}
(8,8,8)	2.9602×10^{-10}	3.4822×10^{-10}	3.9732×10^{-10}
(10,10,10)	3.7346×10^{-13}	5.1405×10^{-13}	7.0877×10^{-13}

Example 4. We consider the 2D T-FOTE (37) with the coefficients $\gamma_1 = \gamma_2 = \gamma_3 = 1$, and $\mathcal{H}(x,y,t)$ according to the analytical solution $u(x,y,t) = t^2 \sin(x+y)$ and the following initial and boundary conditions:

$$u(0,y,t) = t^2 \sin(y),\ u(1,y,t) = t^2 \sin(y+1),\ 0 \le y \le 1,\ 0 \le t \le 1,$$
$$u(x,0,t) = t^2 \sin(x),\ u(x,1,t) = t^2 \sin(x+1),\ 0 \le x \le 1,\ 0 \le t \le 1,$$
$$u(x,y,0) = 0,\ \frac{\partial u(x,y,t)}{\partial t}|_{t=0} = 0,\ 0 \le x \le 1,\ 0 \le y \le 1.$$

Tables 4 and 5 show the RMSE and the MAEs, respectively, with different values of $\alpha,\ \beta,\ N,\ M$, and K.

Table 5: MAEs for Example 4 at $\mu = 1.5$.

(N, M, K)	$_1 =_1= -\frac{1}{2}$, $_2 =_2= -\frac{1}{2}$, $_3 =_3= -\frac{1}{2}$	$_1 =_1= 0$, $_2 =_2= 0$, $_3 =_3= 0$	$_1 =_1= \frac{1}{2}$, $_2 =_2= \frac{1}{2}$, $_3 =_3= \frac{1}{2}$
(2,2,2)	4.0454×10^{-3}	3.7132×10^{-3}	3.4097×10^{-3}
(4,4,4)	4.1167×10^{-5}	3.7308×10^{-5}	3.4409×10^{-5}
(6,6,6)	2.1882×10^{-7}	2.2566×10^{-7}	2.2929×10^{-7}
(8,8,8)	4.5656×10^{-10}	5.5665×10^{-10}	6.4892×10^{-10}
(10,10,10)	5.6932×10^{-13}	9.0227×10^{-13}	1.3389×10^{-13}

7 Conclusions

The SJC method is effectively connected for finding the agreement of T-FOTEs. We have accomplished a decent understanding between the approximate solution acquired by SJC and the exact one. The aftereffects of the examples demonstrate that the shifted Jacobi collocation method is dependable and effective technique for solving time fractional telegraph equation and also other equations. The key feature of the proposed method is to obtain highly accurate semi-analytic solutions via few number of retained modes.

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