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Research Article

Numerical methods for solving nonlinear Volterra integro-differential equations based on Hermite–Birkhoff interpolation

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Abstract

We introduce a new family of multivalue and multistage methods based on Hermite–Birkhoff interpolation for solving nonlinear Volterra integro-differential equations. The proposed methods that have high order and extensive stability region, use the approximated values of the first derivative of the solution in the m collocation points and the approximated values of the solution as well as its first derivative in the r previous steps. Convergence order of the new methods is determined and their linear stability is analyzed. Efficiency of the methods is shown by some numerical experiments.

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Keywords: Volterra integro-differential equations; Multistep collocation methods; Hermite–Birkhoff interpolation; Convergence; Linear stability.

1 Introduction

Consider Volterra integro-differential equations (VIDEs) of the form

$$y'(t) = g(t, y(t)) + \int_0^t K(t, \tau, y(\tau))d\tau, \qquad t \in I := [0, T], \qquad y(0) = y_0, \quad (1)$$

where $g \in C(S)$ and the kernel $K \in C(\Omega)$ and satisfies the Lipschitz condition with respect to y, with $S = \{(t, y) : \ t \in I, y \in \mathbb{R}\}$, $\Omega = \{(t, \tau, y) : \ 0 \leq \tau \leq t \leq T, \ y \in \mathbb{R}\}$. It is well known that, under the suitable conditions, (1) possesses a unique solution $y(t) \in C^1[0, T]$ (see [3]).

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VIDEs arise in a variety of applications, for example, in viscoelasticity, control theory, epidemiology, and population dynamics [10, 18, 26].

On the class of discretization methods for the numerical solution of VIDEs, Linz [19] proposed algorithms for the numerical solution of (1), which consist of linear multistep methods, of the type commonly used for the numerical solution of the initial value problem for an ordinary differential equation

$$y'(t) = f(t, y(t)), \qquad y(t_0) = y_0, \tag{2}$$

combined with a class of quadrature formulas. In [27], the construction of the quadrature rules generated by the backward differentiation formulas was discussed in detail and their linear stability properties were analyzed. Also, Brunner [2] introduced Runge–Kutta–Nyström methods, which are based on collocation techniques in certain polynomial spline spaces . One of the most popular numerical methods for solving this kind of equations is the class of collocation methods. In these methods, after discretization of the domain, the approximated solution in every subinterval depends on the fixed number m of the collocation points. These methods are of convergence order m for any choice of collocation parameters; see [5, 3]. Recently, a family of multistep collocation methods for (1) has also been introduced, which considers interpolation conditions in the previous r step points [7, 8]. Indeed, in [12] superimplicit multistep collocation methods (SIMCMs) are used for numerical solving VIDEs. In these methods, approximated values of the solution in the r previous steps and its first derivative in the m collocation points in the current and next subinterval are used, where the r step SIMCMs with m collocation points are of convergence order $2m+r$. Recently, a method based on general linear methods [6] for the numerical solution of (1) has been introduced and studied in [20, 21]. Using more of the derivative of the approximated solution has been successfully applied to construct methods with higher order and extensive region of stability for numerical solution of ODEs [17] and nonlinear VIEs [13], specially for stiff problems.

The purpose of this paper is to construct higher order methods with extensive region of stability for solving (1). To do this, we introduce a new class of multistep collocation methods, which approximate the solution in each subinterval depending on the values of approximated solution and its first derivative in the fixed number r of previous time steps, and also the values of the first derivative of the approximated solution in the m collocation points. These methods will enable us, in deal with stiff equations, to make a sensible choice for the steplength of the algorithm.

This paper is organized as follows: In Section 2, the construction of multistep Hermite collocation methods (MHCMs) is described, and in Section 3, the convergence orders of these methods are determined. The linear stability is analyzed in Section 4. The paper is closed in Section 5, by showing efficiency of the methods by some numerical examples.

2 Construction of the methods

In this section, we describe the construction of the MHCMs for solving (1). Let $I_h = \{t_n : 0 = t_0 < t_1 < \cdots < t_N = T\}$ be a partition of the interval $[0,T]$ with constant stepsize $h := t_{n+1} - t_n$, $n = 0,1,\ldots,N-1$. In these methods, to compute the approximated solution of (1) in the subinterval $[t_n, t_{n+1}]$, we use the approximated values of the solution and approximate values of the first derivative of the solution in the r previous steps and its first derivative in the m collocation points in the subinterval $[t_n, t_{n+1}]$. Denoting the fixed collocation parameters by $0 < c_1 < \cdots < c_m \le 1$ and the collocation points by $t_{n,j} = t_n + c_j h,\ j = 1,\ldots,m$, the collocation polynomials restricted to subinterval $[t_n, t_{n+1}]$ are defined by

$$u_n(t_n + sh) = \sum_{k=0}^{r-1} \varphi_k(s) y_{n-k} + h \sum_{j=1}^{m} \psi_j(s) U_{n,j} + h \sum_{k=0}^{r-1} \chi_k(s) y'_{n-k}, \tag{3}$$

where $s \in [0,1]$, $n = r-1,\ldots,N-1$ and $U_{n,j} = u'_n(t_{n,j})$. By differentiating from (3) with respect to s, an approximation for $y'(t)$ is in the form

$$hu'_n(t_n + sh) = \sum_{k=0}^{r-1} \varphi'_k(s) y_{n-k} + h \sum_{j=1}^{m} \psi'_j(s) U_{n,j} + h \sum_{k=0}^{r-1} \chi'_k(s) y'_{n-k}. \tag{4}$$

The functions $\varphi_k(s), \psi_j(s)$, and $\chi_k(s)$, $k = 0,1,\ldots,r-1$ and $j = 1,2,\ldots,m$ are polynomials of degree $2r+m-1$. By imposing the interpolation conditions at points $-k$, $k = 0,\ldots,r-1$, in the polynomial (3) and (4) and at points c_j, $j = 1,2,\ldots,m$, in the polynomial (4), we obtain

$$\begin{array}{lll} \varphi_i(-k) = \delta_{ik}, & \psi_j(-k) = 0, & \chi_i(-k) = 0, \\ \varphi'_k(c_j) = 0, & \psi'_l(c_j) = \delta_{lj}, & \chi'_i(c_j) = 0, \\ \varphi'_k(-i) = 0, & \psi'_j(-i) = 0, & \chi'_k(-i) = \delta_{ik}, \end{array} \tag{5}$$

where $i,k = 0,1,\ldots,r-1$ and $l,j = 1,2,\ldots,m$. The construction of these polynomials is obtained by Hermite–Birkhoff interpolation [24]. Let us assume the polynomials $\varphi_k(s)$, $\psi_j(s)$ and $\chi_j(s)$ in the form

$$\varphi_k(s) = \sum_{i=0}^{2r+m-1} \Phi_i^{[k]} \frac{s^i}{i!}, \quad \psi_j(s) = \sum_{i=0}^{2r+m-1} \Psi_i^{[j]} \frac{s^i}{i!}, \quad \chi_j(s) = \sum_{i=0}^{2r+m-1} \chi_i^{[j]} \frac{s^i}{i!}. \tag{6}$$

Now by setting $s = c_i$ in (6) and differentiating from these polynomials and setting $s = c_i,\ i = 1,2,\ldots,m$ and $s = -k,\quad k = 0,1,\ldots,r-1$, a linear system for coefficient of these polynomials is obtained. The coefficient matrix $A \in \mathbb{R}^{(2r+m)\times(2r+m)}$ is in the form

$$A = \begin{bmatrix} 1 & 0 & 0 & 0 & \dots & 0 \\ 1 & \frac{(-1)^1}{1!} & \frac{(-1)^2}{2!} & \frac{(-1)^3}{3!} & \dots & \frac{(-1)^{2r+m-1}}{(2r+m-1)!} \\ \vdots & \vdots & \vdots & \vdots & \ddots & \vdots \\ 1 & \frac{(-r+1)^1}{1!} & \frac{(-r+1)^2}{2!} & \frac{(-r+1)^3}{3!} & \dots & \frac{(-r+1)^{2r+m-1}}{(2r+m-1)!} \\ & & & & & \\ 0 & 1 & 0 & 0 & \dots & 0 \\ 0 & 1 & \frac{-1^1}{1!} & \frac{-1^2}{2!} & \dots & \frac{-1^{2r+m-2}}{(2r+m-2)!} \\ \vdots & \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 1 & \frac{(-r+1)^1}{1!} & \frac{(-r+1)^2}{2!} & \dots & \frac{(-r+1)^{2r+m-2}}{(2r+m-2)!} \\ & & & & & \\ 0 & 1 & \frac{c_1^1}{1!} & \frac{c_1^2}{2!} & \dots & \frac{c_1^{2r+m-2}}{(2r+m-2)!} \\ \vdots & \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 1 & \frac{c_m^1}{1!} & \frac{c_m^2}{2!} & \dots & \frac{c_m^{2r+m-2}}{(2r+m-2)!} \end{bmatrix},$$

and the right-hand vectors are defined by $\mathbf{u}_r^{[k]} \in \mathbb{R}^r,\ k = 1, 2, \dots, r,\ \mathbf{v}_m^{[j]} \in \mathbb{R}^m,\ j = 1, 2, \dots, m$ as

$$(\mathbf{u}_r^{[k]})_i = \begin{cases} 0, & i \neq k, \\ 1, & i = k, \end{cases} \qquad (\mathbf{v}_m^{[j]})_i = \begin{cases} 0, & i \neq j, \\ 1, & i = j, \end{cases}$$

Their coefficients are obtained by solving the systems

$$\begin{array}{ll} A\Phi^{[k]} = [\mathbf{u}_r^{[k+1]}, \mathbf{0}_r, \mathbf{0}_m]^T, & k = 0, 1, \dots, r-1, \\ A\Psi^{[j]} = [\mathbf{0}_r, \mathbf{0}_r, \mathbf{v}_m^{[j]}]^T, & j = 1, 2, \dots, m, \\ A\chi^{[k]} = [\mathbf{0}_r, \mathbf{u}_r^{[k+1]}, \mathbf{0}_m]^T, & k = 0, 1, \dots, r-1. \end{array} \tag{7}$$

Now, we discuss on the uniqueness of the solution of (7). For this purpose, we briefly review some definitions and known theorems about Hermite–Birkhoff interpolation (see [15])

Let k and n be natural numbers and let

$$E = \|\epsilon_{ij}\|, \quad i = 1, \dots, k, \quad j = 0, 1, \dots, n-1,$$

be a matrix with k rows and n columns having elements

$$\epsilon_{ij} = 0 \ \ or \ \ 1,$$

which are such that

$$\sum_{i,j} \epsilon_{ij} = n.$$

We shall also assume that E has no row entirely composed of zeros. Let also

$$x_1 < x_2 < \cdots < x_k.$$

We shall also assume that E has no row entirely composed of zeros. Let also

$$x_1 < x_2 < \cdots < x_k,$$

be increasing reals. We also need the set of ordered pairs

$$e = \{(i, j) \mid \epsilon_{ij} = 1\}.$$

The reals x_i and the "incidence matrix" E describe the interpolation problem

$$f^{(j)}(x_i) = y_i^{(j)} \quad \textit{for} \quad (i, j) \in E. \tag{8}$$

It is appropriate to refer to (8) as a Hermite–Birkhoff interpolation problem, which we shall abbreviate to HB-problem.

Definition 1. We shall say that the HB-problem (8) is poised, provided that if

$$P(x) \in \pi_{n-1}, P^{(j)}(x_i) = 0, \quad \text{for all } (i, j) \in E,$$

then $P(x) \equiv 0$, in the other words, the matrix E is called poised if the associated interpolation problem is uniquely solvable for any set of constants $y_i^{(j)}$, regardless of the choice of the ordered points $x_l, x_2, \ldots, x_k$.

Define

$$\tilde{m}_j = \sum_{i=1}^{k} \epsilon_{i,j}, \qquad \tilde{M}_l = \sum_{j=0}^{l} \tilde{m}_l, \quad j, l = 0, 1, \ldots, n-1.$$

Schoenberg [24] showed that a necessary condition for E, to be poised is that

$$\tilde{M}_l \geq l + 1, \quad l = 0, 1, \ldots, n-1,$$

in which these inequalities are called the Polya condition.

Definition 2. Let the incidence matrix E have k rows. Let f_i be the column index of the first one that appears in the row i. Then E is called a pyramid matrix if, for each i, $\epsilon_{ij} = 1$ implies $\epsilon_{ij'} = 1$ for $f_i \leq j' \leq j$, and there is some value of $1 \leq i \leq k$ such that $f_1 \geq f_2 \geq \cdots \geq f_i$ and $f_i \leq f_{i+1} \leq \cdots \leq f_k$.

Then Ferguson [15] declared the following theorem for poising the matrix E with respect to the ordering $x_1 < x_2 < \cdots < x_k$.

Theorem 1. If E is a pyramid matrix with k rows, satisfying the Polya conditions, then E is poised with respect to the ordering $x_1 < x_2 < \cdots < x_k$.

Now, we show that our interpolation problems (5) have unique solution. For these problems, the interpolation points can be considered by

$$-r+1 < -r+2 < \cdots < -1 < 0 < c_1 < c_2 < \cdots < c_m.$$

Hence the matrix E with $m+r$ rows and $2r+m$ columns can be defined by

$$E = \begin{bmatrix} 1 & 1 & 0 & \cdots & 0 \\ 1 & 1 & 0 & \cdots & 0 \\ \vdots & \vdots & \vdots & & \vdots \\ 1 & 1 & 0 & \cdots & 0 \\ 0 & 1 & 0 & \cdots & 0 \\ \vdots & \vdots & \vdots & & \vdots \\ 0 & 1 & 0 & \cdots & 0 \end{bmatrix}.$$

Thus, we have

$$\tilde{m}_0 = r, \quad \tilde{m}_1 = m+r, \quad \tilde{m}_3 = 0, \ \ldots, \tilde{m}_{n-1} = 0,$$

$$\tilde{M}_0 = r, \quad \tilde{M}_1 = 2r+m, \quad \tilde{M}_3 = 2r+m, \ \ldots, \tilde{M}_{n-1} = 2r+m = n.$$

It can be easily seen that the the matrix E satisfies in the polya condition. Also by considering

$$f_1 = 0, \ f_2 = 0, \ \ldots, f_r = 0, \ f_{r+1} = 1, \ \ldots, f_{r+m} = 1,$$

one can see that the matrix E is a pyramid matrix. Thus by Theorem 1, we conclude that E is poised with respect to the ordering $-r+1 < -r+2 < \cdots < -1 < 0 < c_1 < c_2 < \cdots < c_m$ and the interpolation problems (5) have the unique solution. In the other word, the matrix A is nonsingular.

The exact MHCM is then obtained by imposing the collocation conditions for equation (1), that is, the collocation polynomials (3) exactly satisfy (1) at the collocation points $t_{n,i}$, which leads to the system of m equations in the unknowns $U_{n,i}$ in the form

$$U_{n,i} = F_{n,i} + \Phi_{n,i}, \tag{9}$$

where

$$\begin{aligned} F_{n,i} &= g(t_{n,i}, u_n(t_{n,i})) + h \sum_{\nu=0}^{n-1} \int_0^1 K(t_{n,i}, t_\nu + sh, u_\nu(t_\nu + sh))ds, \\ \Phi_{n,i} &= h \int_0^{c_i} K(t_{n,i}, t_n + sh, u_n(t_n + sh))ds, \quad i = 1, \ldots, m. \end{aligned} \tag{10}$$

Then $y_{n+1} = u_n(t_{n+1})$ and $hy'_{n+1} = hu'_n(t_{n+1})$ are computed by

$$\begin{cases} y_{n+1} = \sum_{k=0}^{r-1} \varphi_k(1) y_{n-k} + h \sum_{j=1}^{m} \psi_j(1) U_{n,j} + h \sum_{k=0}^{r-1} \chi_k(1) y'_{n-k}, \\ h y'_{n+1} = \sum_{k=0}^{r-1} \varphi'_k(1) y_{n-k} + h \sum_{j=1}^{m} \psi'_j(1) U_{n,j} + h \sum_{k=0}^{r-1} \chi'_k(1) y'_{n-k}. \end{cases} \tag{11}$$

Also, the discretized MHCM is obtained by using suitable quadrature formulas for approximating $F_{n,i}$ and $\Phi_{n,i}$. The discretized multistep Hermite collocation polynomials for approximating $y(t_n + sh)$ and $y'(t_n + sh)$ take the forms

$$P_n(t_n + sh) = \sum_{k=0}^{r-1} \varphi_k(s) y_{n-k} + h \sum_{j=1}^{m} \psi_j(s) Y_{n,j} + h \sum_{k=0}^{r-1} \chi_k(s) y'_{n-k}, \tag{12}$$

$$h P'_n(t_n + sh) = \sum_{k=0}^{r-1} \varphi'_k(s) y_{n-k} + h \sum_{j=1}^{m} \psi'_j(s) Y_{n,j} + h \sum_{k=0}^{r-1} \chi'_k(s) y'_{n-k}, \tag{13}$$

where $n = r-1, \dots, N-1$ and the unknowns $Y_{n,j} = P'_n(t_{n,j})$ are determined by solving the nonlinear system

$$Y_{n,i} = \overline{F}_{n,i} + \overline{\Phi}_{n,i}. \tag{14}$$

By using quadrature formulas with the weights b_l and nodes $\xi_l,\ l = 1, \dots, \mu_1$, for integrating on $[0, 1]$, and the weights $w_{i,l}$ and nodes $d_{i,l},\ l = 1, \dots, \mu_0$, for integrating on $[0, c_i]$, with positive integers μ_0 and μ_1, one can write

$$\begin{aligned} \overline{F}_{n,i} =& g(t_{n,i}, P_n(t_{n,i})) + h \sum_{\nu=0}^{n-1} \sum_{l=1}^{\mu_1} b_l K(t_{n,i}, t_\nu + \xi_l h, P_\nu(t_\nu + \xi_l h)), \\ \overline{\Phi}_{n,i} =& h \sum_{l=1}^{\mu_0} w_{i,l} K(t_{n,i}, t_n + d_{i,l} h, P_n(t_n + d_{i,l} h)). \end{aligned} \tag{15}$$

Substituting from (12) in (15) yields

$$\begin{aligned} \overline{F}_{n,i} =& g(t_{n,i}, \sum_{k=0}^{r-1} \alpha_{ik} y_{n-k} + h \sum_{j=1}^{m} \beta ij Y_{n,j} + h \sum_{k=0}^{r-1} \rho_{ik} {y'}_{n-k}) \\ &+ h \sum_{\nu=0}^{r-2} \sum_{l=1}^{\mu_1} b_l K(t_{n,i}, t_\nu + \xi_l h, \overline{y}_\nu(t_\nu + \xi_l h)) \\ &+ h \sum_{\nu=r-1}^{n-1} \sum_{l=1}^{\mu_1} b_l K(t_{n,i}, t_\nu + \xi_l h, \sum_{k=0}^{r-1} \delta_{lk} y_{\nu-k} + h \sum_{j=1}^{m} \eta_{lj} Y_{\nu,j} + h \sum_{k=0}^{r-1} \zeta_{lk} {y'}_{\nu-k}), \end{aligned}$$

$$\overline{\Phi}_{n,i} = h \sum_{l=1}^{\mu_0} w_{i,l} K(t_{n,i}, t_n + d_{i,l} h, \sum_{k=0}^{r-1} \overline{\gamma}_{ilk} y_{n-k} + h \sum_{j=1}^{m} \overline{\beta}_{ilj} Y_{n,j} + h \sum_{k=0}^{r-1} \overline{\rho}_{ilk} {y'}_{n-k}),$$

where $\overline{y}_\nu(t_\nu+sh)$, $\nu=0,1,\dots,r-2$ are the starting approximated solutions, which are obtained by classical collocation method such that

$$\begin{array}{lll} \alpha_{ik}=\varphi_k(c_i), & \beta_{ij}=\psi_j(c_i), & \rho_{ik}=\chi_k(c_i), \\ \delta_{lk}=\varphi_k(\xi_l), & \eta_{lj}=\psi_j(\xi_l), & \zeta_{lk}=\chi_k(\xi_l), \\ \overline{\gamma}_{ilk}=\varphi_k(d_{i,l}), & \overline{\beta}_{ilj}=\psi_j(d_{i,l}), & \overline{\rho}_{ilk}=\chi_k(d_{i,l}). \end{array}$$

3 Convergence analysis

In this section, we study the zero-stability of the method. When the method is applied on the equation $y'=0$, the equations in (11) reduce to

$$\begin{aligned} y_{n+1} &= \sum_{k=0}^{r-1}\varphi_k(1)y_{n-k}, \\ hy'_{n+1} &= \sum_{k=0}^{r-1}\varphi'_k(1)y_{n-k}. \end{aligned}$$

Therefore, in analogy to [8, 22], zero-stability is defined as follows.

Definition 3. The MHCMs is said to be zero-stable if and only if all roots of the polynomials

$$p(\lambda)=\lambda^r-\sum_{k=0}^{r}\varphi_k(1)\lambda^{r-k-1}, \qquad q(\lambda)=\lambda^r-\sum_{k=0}^{r}\varphi'_k(1)\lambda^{r-k-1},$$

have module less than or equal to unity and those of modules unity are simple.

Example 1. In the case, $r=3$, $m=2$, and $c=[c_1,1]$, the method is zero-stable for all $c_1\in[0.345,1]$.

In what follows, we determine the convergence order of the exact and discretized MHCMs. We start by deriving local error estimate for the exact MHCMs solution $P(t)\in S^{(1)}_{2r+m-1}(I_h)$ for the linear VIDE

$$y'(t)=a(t)y(t)+g(t)+\int_0^t K(t,\tau)y(\tau)d\tau, \qquad t\in[0,T]. \tag{16}$$

Using the Peanos theorem [3, 15] for y and differentiating from it, the representation of the local error is obtained as follows.

Lemma 1. Suppose that $p=2r+m-1$ and that the given functions in (16) satisfy $a\in C^p(I)$, $g\in C^p(I)$, and $K\in C^p(D)$ with $D:=\{(t,\tau):\ 0\le\tau\le t\le T\}$. Then for any choice of collocation parameters $0<c_1<\cdots<c_m\le 1$, the exact solution $y(t)$ satisfies

$$y(t_n+sh)=\sum_{k=0}^{r-1}\varphi_k(s)y(t_{n-k})+h\sum_{j=1}^{m}\psi_j(s)y'(t_{n,j})+h\sum_{k=0}^{r-1}\chi_k(s)y'(t_{n-k})$$
$$+h^{2r+m}R_{m,r,n}(s), \tag{17}$$

and

$$hy'(t_n+sh)=\sum_{k=0}^{r-1}\varphi_k'(s)y(t_{n-k})+h\sum_{j=1}^{m}\psi_j'(s)y'(t_{n,j})$$
$$+h\sum_{k=0}^{r-1}\chi_k'(s)y'(t_{n-k})+h^{2r+m}R_{m,r,n}'(s), \tag{18}$$

with

$$R_{m,r,n}(s)=\int_{-r+1}^{1}K_{m,r}(s,\nu)y^{(2r+m)}(t_n+\nu h)d\nu,$$
$$K_{m,r}(s,\nu)=\frac{1}{(p-1)!}\Bigg\{(s-\nu)_+^{p-1}-\sum_{k=0}^{r-1}\varphi_k(s)(-k-\nu)_+^{p-1}$$
$$-h(p-1)\sum_{j=1}^{m}\psi_j(s)(-c_j-\nu)_+^{p-2}$$
$$-h(p-1)\sum_{k=0}^{r-1}\chi_k(s)(-k-\nu)_+^{p-2}\Bigg\},$$

and

$$(x-t)_+=\begin{cases}(x-t), & x\ge t,\\ 0, & x<t.\end{cases}$$

Theorem 2. Let $e(t)=y(t)-u(t)$ be the error of the exact MHCM and let $p=2r+m-1$. Suppose that

(i) $K\in C^p(D\times\mathbb{R})$, $g\in C^p(I)$,

(ii) the starting error is $\|e\|_{\infty,[0,t_r]}=O(h^p)$,

(iii)$\rho(\mathbf{H})<1$, where ρ denotes the spectral radius and

$$\mathbf{H}=\begin{bmatrix}\mathbf{A} & \tilde{\mathbf{A}}\\ \hat{\mathbf{A}} & \bar{\mathbf{A}}\end{bmatrix} \tag{19}$$

with

$$\mathbf{A}=\left[\begin{array}{c|c}\mathbf{0}_{r-1,1} & \mathbf{I}_{r-1}\\ \hline \varphi_{r-1}(1) & \varphi_{r-2}(1),\ldots,\varphi_0(1)\end{array}\right],\ \tilde{\mathbf{A}}=\left[\begin{array}{c|c}\mathbf{0}_{r-1,1} & \mathbf{I}_{r-1}\\ \hline \chi_{r-1}(1) & \chi_{r-2}(1),\ldots,\chi_0(1)\end{array}\right],$$
$$\hat{\mathbf{A}}=\left[\begin{array}{c|c}\mathbf{0}_{r-1,1} & \mathbf{I}_{r-1}\\ \hline \varphi_{r-1}'(1) & \varphi_{r-2}'(1),\ldots,\varphi_0'(1)\end{array}\right],\ \bar{\mathbf{A}}=\left[\begin{array}{c|c}\mathbf{0}_{r-1,1} & \mathbf{I}_{r-1}\\ \hline \chi_{r-1}'(1) & \chi_{r-2}'(1),\ldots,\chi_0'(1)\end{array}\right]. \tag{20}$$

Then

$$\|e\|_\infty = O(h^{2r+m-1}).$$

Proof. We point out to the important parts of the proof for MHCMs in the case of linear VIDE. The proof can be straightforwardly extended to the case of a nonlinear VIDE (1) by using the mean value theorem [3]. Suppose that $Z_{n,j} = y'(t_{n,j})$. By subtracting (3) from (17), representation of the local error may be written as

$$\begin{aligned} e(t_n + sh) = &\sum_{k=0}^{r-1} \varphi_k(s)\mathcal{E}_{n-k} + h\sum_{j=1}^{m} \psi_j(s)\mathcal{E}'_{n,j} + h\sum_{k=0}^{r-1} \chi_k(s)\mathcal{E}'_{n-k} \\ &+ h^{2r+m} R_{m,r,n}(s), \end{aligned} \tag{21}$$

with $\mathcal{E}_{n-k} = y(t_{n-k}) - y_{n-k}$ and $\mathcal{E}'_{n,j} = Z_{n,j} - U_{n,j}$. Differentiating from (21) yields

$$\begin{aligned} he'(t_n + sh) = &\sum_{k=0}^{r-1} \varphi'_k(s)\mathcal{E}_{n-k} + h\sum_{j=1}^{m} \psi'_j(s)\mathcal{E}'_{n,j} + h\sum_{k=0}^{r-1} \chi'_k(s)\mathcal{E}'_{n-k} \\ &+ h^{2r+m} R'_{m,r,n}(s). \end{aligned} \tag{22}$$

Replacing n by $l-1$ in (21) and (22), and $s = 1$, lead to

$$\mathcal{E}_l^{(1)} = \mathbf{A}\mathcal{E}_{l-1}^{(1)} + h\tilde{\mathbf{A}}\mathcal{E}'^{(1)}_{l-1} + h\tilde{\mathbf{S}}\mathcal{E}'^{(2)}_{l-1} + h^p \tilde{\boldsymbol{\rho}}_{m,r,l-1} \tag{23}$$

and

$$h\mathcal{E}'^{(1)}_l = \hat{\mathbf{A}}\mathcal{E}_{l-1}^{(1)} + h\bar{\mathbf{A}}\mathcal{E}'^{(1)}_{l-1} + h\bar{\mathbf{S}}\mathcal{E}'^{(2)}_{l-1} + h^p \bar{\boldsymbol{\rho}}_{m,r,l-1}, \tag{24}$$

where $\mathbf{A}$, $\hat{\mathbf{A}}$, $\bar{\mathbf{A}}$, and $\tilde{\mathbf{A}}$ are given in (20) and

$$\tilde{\mathbf{S}} = \begin{pmatrix} \mathbf{0}_{r-1,m} \\ \boldsymbol{\psi}(1)^T \end{pmatrix}, \; \tilde{\boldsymbol{\rho}}_{m,r,j} = \begin{pmatrix} \mathbf{0}_{r-1,1} \\ R_{m,r,j}(1) \end{pmatrix},$$

$$\bar{\mathbf{S}} = \begin{pmatrix} \mathbf{0}_{r-1,m} \\ \boldsymbol{\psi'}(1)^T \end{pmatrix}, \; \bar{\boldsymbol{\rho}}_{m,r,j} = \begin{pmatrix} \mathbf{0}_{r-1,1} \\ R'_{m,r,j}(1)^T \end{pmatrix},$$

$$\varepsilon_l^{(1)} = [\varepsilon_{l-r+1}, \ldots, \varepsilon_l]^T \in \mathbb{R}^r,$$

$$\varepsilon'^{(1)}_l = [\varepsilon'_{l-r+1}, \ldots, \varepsilon'_l]^T \in \mathbb{R}^r, \quad \varepsilon'^{(2)}_l = [\varepsilon'_{l,1}, \ldots, \varepsilon'_{l,m}]^T \in \mathbb{R}^m,$$

$$\boldsymbol{\psi}(1) = [\psi_1(1), \ldots, \psi_m(1)]^T, \quad \boldsymbol{\psi'}(1) = [\psi'_1(1), \ldots, \psi'_m(1)]^T.$$

Combining (23) and (24) gives the following matrix equation:

$$\begin{pmatrix} \mathcal{E}_l^{(1)} \\ h\mathcal{E}'^{(1)}_l \end{pmatrix} = \mathbf{H} \begin{pmatrix} \mathcal{E}_{l-1}^{(1)} \\ h\mathcal{E}'^{(1)}_{l-1} \end{pmatrix} + \mathbf{G}\mathcal{E}_{l-1}^{(2)} + \mathbf{Q}_{m,r,l-1}h^p, \tag{25}$$

where $\mathbf{H}$ is given by (19) and

$$\mathbf{G} = \begin{pmatrix} h\tilde{\mathbf{S}} \\ h\bar{\mathbf{S}} \end{pmatrix}, \; \mathbf{Q}_{m,r,j} = \begin{pmatrix} \tilde{\boldsymbol{\rho}}_{m,r,j} \\ \bar{\boldsymbol{\rho}}_{m,r,j} \end{pmatrix}.$$

The solution of difference equation (25) is (see [16])

$$\mathcal{E}_l^{(1)} = \mathbf{H}^{l-r+1}\mathcal{E}_{r-1}^{(1)} + \sum_{j=r-1}^{l-1} \mathbf{H}^{l-j-1}(\mathbf{G}\mathcal{E}_j^{(2)} + h^p\mathbf{Q}_{m,r,j}). \tag{26}$$

On the other hand, setting $s = c_i$, $i = 1, 2, \ldots, m$ in (22), by using (4), leads to

$$e'(t_{n,i}) = \mathcal{E}'_{n,i} + h^{2r+m-1}R'_{m,r,n}(c_i). \tag{27}$$

On the other hand, equation (9) for the linear form of VIDE is equivalent to

$$\begin{aligned} U_{n,i} =& a(t_{n,i})u_n(t_{n,i}) + g(t_{n,i}) + h\sum_{l=0}^{n-1}\int_0^1 K(t_{n,i}, t_l + sh)u_l(t_l + sh)ds \\ &+ h\int_0^{c_i} K(t_{n,i}, t_n + sh)u_n(t_n + sh)ds \end{aligned} \tag{28}$$

and (16) at points $t_{n,i}$ can be written as

$$\begin{aligned} y'(t_{n,i}) =& a(t_{n,i})y(t_{n,i}) + g(t_{n,i}) + h\sum_{l=0}^{n-1}\int_0^1 K(t_{n,i}, t_l + sh)y(t_l + sh)ds \\ &+ h\int_0^{c_i} K(t_{n,i}, t_n + sh)y(t_n + sh)ds. \end{aligned} \tag{29}$$

Now by subtracting (28) from (29), we get

$$\begin{aligned} e'(t_{n,i}) =& a(t_{n,i})e(t_{n,i}) + h\sum_{l=0}^{n-1}\int_0^1 K(t_{n,i}, t_l + sh)e(t_l + sh)ds \\ &+ h\int_0^{c_i} K(t_{n,i}, t_n + sh)e(t_n + sh)ds. \end{aligned} \tag{30}$$

Also, by substituting (30) in the (27), we obtain

$$\mathcal{E}'_{n,i} = a(t_{n,i})e(t_{n,i}) + h\sum_{l=0}^{n-1}\int_0^1 K(t_{n,i}, t_l + sh)e(t_l + sh)ds$$

$$+ h\int_0^{c_i} K(t_{n,i}, t_n + sh)e(t_n + sh)ds$$
$$- h^{2r+m-1}R'_{m,r,n}(c_i). \tag{31}$$

By the hypothesis on the starting error and by substituting (21) in (31), we have

$$\begin{aligned}\mathcal{E}'_{n,i} =& a(t_{n,i})\left(\sum_{k=0}^{r-1}\alpha_{i,k}\mathcal{E}_{n-k} + h\sum_{j=1}^{m}\beta_{i,j}\mathcal{E}'_{n,j}\right.\\ &\left. +h\sum_{k=0}^{r-1}\rho_{i,k}\mathcal{E}'_{n-k} + h^{2r+m}R_{m,r,n}(c_i)\right)\\ &+ h\sum_{l=0}^{n-1}\int_0^1 K(t_{n,i}, t_l + sh)\left(\sum_{k=0}^{r-1}\varphi_k(s)\mathcal{E}_{n-k}\right.\\ &\left. +h\sum_{j=1}^{m}\psi_j(s)\mathcal{E}'_{n,j} + h\sum_{k=0}^{r-1}\chi_k(s)\mathcal{E}'_{n-k} + h^{2r+m}R_{m,r,n}(s)\right) ds\\ &+ h\int_0^{c_i} K(t_{n,i}, t_n + sh)\left(\sum_{k=0}^{r-1}\varphi_k(s)\mathcal{E}_{n-k}\right.\\ &\left. +h\sum_{j=1}^{m}\psi_j(s)\mathcal{E}'_{n,j} + h\sum_{k=0}^{r-1}\chi_k(s)\mathcal{E}'_{n-k} + h^{2r+m}R_{m,r,n}(s)\right) ds\\ &- h^{2r+m-1}R'_{m,r,n}(c_i).\end{aligned}$$

By the hypothesis on the starting error, it follows that

$$e(t_l + sh) = h^p q_l(s), \quad l = 0, 1, \dots, r-1, \tag{32}$$

with $\|q_l\|_\infty \le C_1$ independent of h. Hence, we obtain

$$\begin{aligned}(\mathbf{I}_m - hC_n^{(n)})\mathcal{E}'^{(2)}_n =& h^{p+1}\sum_{l=0}^{n}\bar{R}_n^{(l)} + h\sum_{l=r}^{n-1}C_n^{(l)}\mathcal{E}'^{(2)}_n\\ &+ h\sum_{l=r}^{n}B_n^{(l)}\mathcal{E}_l^{(1)} + h^2\sum_{l=r}^{n}D_n^{(l)}\mathcal{E}'^{(1)}_l,\end{aligned} \tag{33}$$

where $\bar{R}_n^{(l)} \in \mathbb{R}^m$, $C_n^{(l)}$, and $B_n^{(l)},\ D_n^{(l)} \in \mathbb{R}^{m\times r}$ are defined as

$$(\bar{R}_n^{(l)})_i := \begin{cases} \int_0^1 K(t_{n,i}, t_l + sh)q_l(s)ds, & l = 0, 1, \dots, r-1,\\ \int_0^1 K(t_{n,i}, t_l + sh)R_{m,r,l}(s)ds, & l = r, \dots, n-1,\\ \int_0^1 K(t_{n,i}, t_n + sh)R_{m,r,n}(s)ds + a(t_{n,i})R_{m,r,n}(c_i), & l = n,\end{cases}$$

$$(B_n^{(l)})_{ik} := \begin{cases} \int_0^1 K(t_{n,i}, t_l + sh)\varphi_k(s)ds, & l = r, \dots, n-1, \\ \int_0^{c_i} K(t_{n,i}, t_n + sh)\varphi_k(s)ds + a(t_{n,i})\varphi_k(c_i), & l = n, \end{cases}$$

$$(C_n^{(l)})_{ij} := \begin{cases} \int_0^1 K(t_{n,i}, t_l + sh)\psi_j(s)ds, & l = r, \dots, n-1, \\ \int_0^{c_i} K(t_{n,i}, t_n + sh)\psi_j(s)ds + a(t_{n,i})\psi_j(c_i), & l = n, \end{cases}$$

$$(D_n^{(l)})_{ik} := \begin{cases} \int_0^1 K(t_{n,i}, t_l + sh)\chi_k(s)ds, & l = r, \dots, n-1, \\ \int_0^{c_i} K(t_{n,i}, t_n + sh)\chi_k(s)ds + a(t_{n,i})\chi_k(c_i), & l = n. \end{cases}$$

Substituting (26) in (33), a recurrence formula for ${\mathcal{E}'}_n^{(2)}$ is obtained. Then by the same way as described in [9, Theorem 4.2] and by considering the starting errors and the fact that $\rho(H) < 1$ lead to the estimate

$$\|\mathcal{E}_n^{(2)}\| \leq M_2 h^p,$$

and then from (26), a bound for $\|\mathcal{E}_n^{(1)}\|$ is obtained in the form

$$\|\mathcal{E}_n^{(1)}\| \leq M_1 h^p.$$

Using the local error representation and two above inequalities together, completes the proof. □

Theorem 3. Suppose that the hypotheses of Theorem 2 hold. If the quadrature formulas defined in (15) have order $2r+m$ and $2r+m-1$, respectively, then the uniform order of the discretized MHCMs is equal to $2r+m-1$.

Now, in the table 1, we give a comparison between the methods MCMs [8], SIMCMs [12], and the new proposed method (MHCMs) in view of computational costs with respect to convergence order.

Table 1: Comparison with multistep and superimplicit multistep collocation methods.

	MCMs	SIMCMs (type 1)	SIMCMs (type 2)	MHCM
collocation points	m	m	m	m
dimensional of system	m	m	$2m$	m
order	$m+r-1$	$2m+r-1$	$2m+r-1$	$2r+m-1$

4 Linear stability analysis

In this section, we analyze the stability properties of the introduced methods with respect to the basic test equation (see [5, 3, 4, 23])

$$y'(t) = g(t) + \xi y(t) + \eta \int_0^t y(\tau)d\tau, \quad t > 0, \qquad y(0) = y_0, \tag{34}$$

where $\xi, \eta \in \mathbb{C}$. The solution of (34) is stable if $Re(\lambda_1) < 0$ and $Re(\lambda_2) < 0$, where $\lambda_{1,2} = (\xi \pm \sqrt{\xi^2 + 4\eta})/2$ (see [1]). We observe that, particularly for real ξ and η, these conditions reduce to $\xi < 0$ and $\eta < 0$. As usual, we look for sufficient conditions for the stability of the numerical solution of (34).

Definition 4. Absolute stability region of the method, $\mathcal{R}$, is the set of all $(z := \xi h, w := \eta h^2) \in \mathbb{C} \times \mathbb{C}$, such that the numerical solution y_n of test equation (34) with a constant stepsize h, tends to zero as $n \to \infty$. The method is A_0-stable if $\mathcal{R} \supseteq \mathbb{R}^- \times \mathbb{R}^-$ and is A-stable if it is stable for any value of (z, w) such that $Re(\lambda_1) < 0$ and $Re(\lambda_2) < 0$. An A-stable method is A_0-stable too.

To state the main results of stability properties of MHCM, let us define

$$\alpha_k = \int_0^1 \varphi_k(s)ds, \quad \beta_j = \int_0^1 \psi_j(s)ds, \quad \gamma_k = \int_0^1 \chi_k(s)ds,$$

$$\Omega_{ik} = \int_0^{c_i} \varphi_k(s)ds, \quad \Gamma_{ij} = \int_0^{c_i} \psi_j(s)ds, \quad \Delta_{ik} = \int_0^{c_i} \chi_k(s)ds,$$

$$(\boldsymbol{\phi}(\boldsymbol{c}))_{ik} = \varphi_k(c_i), \quad (\boldsymbol{\psi}(\boldsymbol{c}))_{ij} = \psi_j(c_i), \quad (\boldsymbol{\chi}(\boldsymbol{c}))_{ik} = \chi_k(c_i),$$

and consider the vectors and matrices

$$\begin{aligned}
\boldsymbol{\varphi}(1) &= [\varphi_0(1), \dots, \varphi_{r-1}(1)]^T, & \boldsymbol{\psi}(1) &= [\psi_1(1), \dots, \psi_m(1)]^T, \\
\boldsymbol{\chi}(1) &= [\chi_0(1), \dots, \chi_{r-1}(1)]^T, & \boldsymbol{\chi}'(1) &= [\chi_0'(1), \dots, \chi_{r-1}'(1)]^T, \\
\boldsymbol{\varphi}'(1) &= [\varphi_0'(1), \dots, \varphi_{r-1}'(1)]^T, & \boldsymbol{\psi}'(1) &= [\psi_1'(1), \dots, \psi_m'(1)]^T, \\
\mathbf{y}_n^{(r)} &= [y_n, \dots, y_{n-r+1}]^T, & \mathbf{y'}_n^{(r)} &= [y_n', \dots, y_{n-r+1}']^T, \\
\mathbf{U}_n &= [U_{n,1}, \dots, U_{n,m}]^T, & u &= [1, 1, \dots, 1]^T \in \mathbb{R}^m,
\end{aligned}$$

$$\mathbf{E}_1 = \begin{bmatrix} \varphi(1)^T \\ \mathbf{I}_{r-1} \ \mathbf{0}_{r-1,1} \end{bmatrix}, \ \mathbf{F}_1 = \begin{bmatrix} \psi(1)^T \\ \mathbf{0}_{r-1,m} \end{bmatrix}, \quad \mathbf{G}_1 = \begin{bmatrix} \chi(1)^T \\ \mathbf{0}_{r-1,r} \end{bmatrix}, \tag{35}$$

$$\mathbf{E}_2 = \begin{bmatrix} \varphi'(1)^T \\ \mathbf{I}_{r-1} \ \mathbf{0}_{r-1,1} \end{bmatrix}, \ \mathbf{F}_2 = \begin{bmatrix} \psi'(1)^T \\ \mathbf{0}_{r-1,m} \end{bmatrix}, \ \mathbf{G}_2 = \begin{bmatrix} \chi'(1)^T \\ \mathbf{0}_{r-1,r} \end{bmatrix}. \tag{36}$$

Theorem 4. Applying the exact MHCM on the test equation (34), leads to the following recurrence relation:

$$\begin{bmatrix} \mathbf{y}_n^{(r)} \\ h\mathbf{U}_n \\ \mathbf{y'}_n^{(r)} \end{bmatrix} = R(z, w) \begin{bmatrix} \mathbf{y}_{n-1}^{(r)} \\ h\mathbf{U}_{n-1} \\ \mathbf{y'}_{n-1}^{(r)} \end{bmatrix} + h\overline{\mathbf{G}}_n,$$

where $z := \xi h$, $w = \eta h^2$,

$$R(z,w) = [Q(z,w)]^{-1}M(z,w),$$

and

$$\overline{\mathbf{G}}_n = \begin{bmatrix} \mathbf{0}_{r,m} \\ \mathbf{0}_{r,m} \\ \boldsymbol{\mathcal{G}}_n - \boldsymbol{\mathcal{G}}_{n-1} \end{bmatrix},$$

$$Q(z,w) = \begin{bmatrix} \mathbf{I}_r & \mathbf{0}_{r,m} & \mathbf{0}_{r,r} \\ \mathbf{0}_{r,r} & \mathbf{0}_{r,m} & \mathbf{I}_r \\ -z\boldsymbol{\varphi}(\boldsymbol{c}) - w\boldsymbol{\Omega} & \mathbf{I}_m - z\boldsymbol{\psi}(\boldsymbol{c}) - w\boldsymbol{\Gamma} & -z\boldsymbol{\chi}(\boldsymbol{c}) - w\boldsymbol{\Delta} \end{bmatrix},$$

$$M(z,w) = \begin{bmatrix} \mathbf{E}_1 & \mathbf{F}_1 & \mathbf{G}_1 \\ \mathbf{E}_2 & \mathbf{F}_2 & \mathbf{G}_2 \\ M_1 & M_2 & M_3 \end{bmatrix}$$

with

$$\begin{aligned} M_1 =& -z\boldsymbol{\varphi}(\boldsymbol{c}) - w\boldsymbol{\Omega} + w\boldsymbol{\alpha}, \\ M_2 =& \mathbf{I}_m - z\boldsymbol{\psi}(\boldsymbol{c}) - w\boldsymbol{\Gamma} + w\boldsymbol{\beta}, \\ M_3 =& -z\boldsymbol{\chi}(\boldsymbol{c}) - w\boldsymbol{\Delta} + w\boldsymbol{\gamma}. \end{aligned}$$

Proof. By the notations of this section, we rewrite the first relation in (11) in the form

$$y_{n+1} = \boldsymbol{\varphi}(1)^T y_n^{(r)} + \boldsymbol{\psi}(1)^T h\mathbf{U}_n + \boldsymbol{\chi}(1)^T h{\mathbf{y}'}_n^{(r)}, \tag{37}$$

or equivalently

$$\mathbf{y}_n^{(r)} = \mathbf{E}_1 y_{n-1}^{(r)} + \mathbf{F}_1 h\mathbf{U}_{n-1} + \mathbf{G}_1 h{\mathbf{y}'}_{n-1}^{(r)}. \tag{38}$$

Also, the second relation in (11) can be written in the form

$$h{\mathbf{y}'}_n^{(r)} = \mathbf{E}_2 y_{n-1}^{(r)} + \mathbf{F}_2 h\mathbf{U}_{n-1} + \mathbf{G}_2 h{\mathbf{y}'}_{n-1}^{(r)}. \tag{39}$$

Now we apply (10) on the test equation to get

$$U_{n,i} = g(t_{n,i}) + \xi u_n(t_{n,i}) + F_n(t_{n,i}) + \Phi_n(t_{n,i}), \tag{40}$$

where it can be written in the matrix form

$$\mathbf{U}_n = \boldsymbol{\mathcal{G}}_n + \xi\left(\boldsymbol{\phi}(\boldsymbol{c}) y_n^{(r)} + \boldsymbol{\psi}(\boldsymbol{c}) h\mathbf{U}_n + \boldsymbol{\chi}(\boldsymbol{c}) h{y'}_n^{(r)}\right) + \mathbf{F}_{n-1} + \boldsymbol{\Phi}_n, \tag{41}$$

with $(\boldsymbol{\mathcal{G}}_n)_i = g(t_{n,i})$ and

$$(\mathbf{F}_{n-1})_i := F_n(t_{n,i}) = \eta h \sum_{l=0}^{n-1} \int_0^1 u_l(t_l + sh)$$

$$= \eta h \sum_{l=0}^{n-1} \int_0^1 \left(\sum_{k=0}^{r-1} \varphi_k(s) y_{l-k} + h \sum_{j=1}^{m} \psi_j(s) U_{l,j} + h \sum_{k=0}^{r-1} \chi_j(s)\ {y'}_{l-k} \right) ds,$$

$$(\boldsymbol{\Phi}_n)_i := \Phi_n(t_{n,i}) = \eta h \int_0^{c_i} u_n(t_n + sh) ds$$
$$= \eta h \int_0^{c_i} \left(\sum_{k=0}^{r-1} \varphi_k(s) y_{n-k} + h \sum_{j=1}^{m} \psi_j(s) U_{n,j} + h \sum_{j=1}^{m} \chi_j(s) y'_{n-k} \right) ds.$$

Thus

$$\mathbf{F}_{n-1} = \eta h \sum_{l=0}^{n-1} \boldsymbol{\alpha} y_l^{(r)} + \boldsymbol{\beta} h \mathbf{U}_l + \boldsymbol{\gamma} h {y'}_l^{(r)},$$
$$\boldsymbol{\Phi}_n = \eta h \left(\boldsymbol{\Omega} y_n^{(r)} + \boldsymbol{\Gamma} h \mathbf{U}_n + \boldsymbol{\Delta} h {y'}_n^{(r)} \right).$$

Therefore, we can write

$$\mathbf{F}_{n-1} - \mathbf{F}_{n-2} = \boldsymbol{\alpha} y_{n-1}^{(r)} + \boldsymbol{\beta} h \mathbf{U}_{n-1} + \boldsymbol{\gamma} h {y'}_{n-1}^{(r)}. \tag{42}$$

Now for obtaining a recurrence relation, substituting (42) in (41) and multiplying it by h yield

$$\begin{aligned} &(\mathbf{I}_m - z\boldsymbol{\psi}(\boldsymbol{c}) - w\boldsymbol{\Gamma})\, h\mathbf{U}_n + (-z\boldsymbol{\varphi}(\boldsymbol{c}) - w\Omega) y_n^{(r)} + (-z\boldsymbol{\chi}(\boldsymbol{c}) - w\boldsymbol{\Delta}) h {y'}_n^{(r)} \\ &= (\mathbf{I}_m - z\boldsymbol{\psi}(\boldsymbol{c}) - w\boldsymbol{\Gamma} + w\boldsymbol{\beta})\, h\mathbf{U}_{n-1} + (-z\boldsymbol{\varphi}(\boldsymbol{c}) - w\Omega + w\boldsymbol{\alpha}) y_{n-1}^{(r)} \\ &\quad + (-z\boldsymbol{\chi}(\boldsymbol{c}) - w\boldsymbol{\Delta} + w\boldsymbol{\gamma}) h {y'}_{n-1}^{(r)} + h(\boldsymbol{\mathcal{G}}_n - \boldsymbol{\mathcal{G}}_{n-1}). \end{aligned} \tag{43}$$

Hence the relations (38), (39), and (43) complete the proof. □

Let us define

$$\widetilde{\alpha}_k = \sum_{l=1}^{\mu_1} b_l \varphi_k(\xi_l), \quad \widetilde{\beta}_j = \sum_{l=1}^{\mu_1} b_l \psi_j(\xi_l), \quad \widetilde{\gamma}_k = \sum_{l=1}^{\mu_1} b_l \chi_k(\xi_l),$$

$$\widetilde{\Omega}_{ik} = \sum_{l=1}^{\mu_0} w_{i,l} \varphi_k(d_{i,l}), \quad \widetilde{\Gamma}_{ij} = \sum_{l=1}^{\mu_0} w_{i,l} \psi_j(d_{i,l}), \quad \widetilde{\Delta}_{ik} = \sum_{l=1}^{\mu_0} w_{i,l} \chi_k(d_{i,l}).$$

Theorem 5. The discretized MHCM, applied to the test equation (34), leads to the following recurrence relation:

$$\begin{bmatrix} \mathbf{y}_n^{(r)} \\ h\mathbf{Y}_n \\ \mathbf{y'}_n^{(r)} \end{bmatrix} = \widetilde{R}(z,w) \begin{bmatrix} \mathbf{y}_{n-1}^{(r)} \\ h\mathbf{Y}_{n-1} \\ \mathbf{y'}_{n-1}^{(r)} \end{bmatrix} + h\overline{\mathbf{G}}_n,$$

where $z := \xi h$, $w = \eta h^2$,

$$\widetilde{R}(z,w) = [\widetilde{Q}(z,w)]^{-1}\widetilde{M}(z,w),$$

and

$$\widetilde{Q}(z,w) = \begin{bmatrix} \mathbf{I}_r & \mathbf{0}_{r,m} & \mathbf{0}_{r,r} \\ \mathbf{0}_{r,r} & \mathbf{0}_{r,m} & \mathbf{I}_r \\ -z\boldsymbol{\varphi}(\boldsymbol{c}) - w\widetilde{\boldsymbol{\Omega}} & \mathbf{I}_m - z\boldsymbol{\psi}(\boldsymbol{c}) - w\widetilde{\boldsymbol{\Gamma}} & -z\boldsymbol{\chi}(\boldsymbol{c}) - w\widetilde{\boldsymbol{\Delta}} \end{bmatrix},$$

$$\widetilde{M}(z,w) = \begin{bmatrix} \mathbf{E}_1 & \mathbf{F}_1 & \mathbf{G}_1 \\ \mathbf{E}_2 & \mathbf{F}_2 & \mathbf{G}_2 \\ \widetilde{M}_1 & \widetilde{M}_2 & \widetilde{M}_3 \end{bmatrix}$$

with

$$\begin{aligned} \widetilde{M}_1 &= -z\boldsymbol{\varphi}(\boldsymbol{c}) - w\widetilde{\boldsymbol{\Omega}} + w\widetilde{\boldsymbol{\alpha}}, \\ \widetilde{M}_2 &= \mathbf{I}_m - z\boldsymbol{\psi}(\boldsymbol{c}) - w\widetilde{\boldsymbol{\Gamma}} + w\widetilde{\boldsymbol{\beta}}, \\ \widetilde{M}_3 &= -z\boldsymbol{\chi}(\boldsymbol{c}) - w\widetilde{\boldsymbol{\Delta}} + w\widetilde{\boldsymbol{\gamma}}. \end{aligned}$$

Proof. It is similar to the proof of Theorem 4. □

The $R(z,w)$ is called the stability matrix of the method. Now, the method is stable if $\rho(R(z,w) < 1$ and the stability region of the method is $\mathcal{R} = \{(z,w) \in \mathbb{C} \times \mathbb{C} : \rho(R(z,w) < 1\}$. Here, the term G_n does not influence stability. The stability function of the method with respect to (34) is defined as

$$p(z,w;\lambda) = \det(\lambda \mathbf{I}_{2r+m} - R(z,w)). \tag{44}$$

To investigate the stability properties of the exact MHCM, it is more convenient to work with the polynomial obtained by multiplying the stability function (44) by its denominator. The resulting polynomial will be denoted by the same symbol $p(z,w;\lambda)$. This polynomial takes the form

$$p(z,w;\lambda) = \sum_{i=0}^{2r+m} p_i(z,w)\lambda^i, \tag{45}$$

where $p_i(z,w)$, $i = 0, 1, \ldots, 2r+m$ are polynomials of degree less than or equal to m. Denoting the roots of the polynomial $p(z,w;\lambda)$ by $\lambda_1, \lambda_2, \ldots,$ λ_{2r+m}, the absolute stability region of the method is then defined by

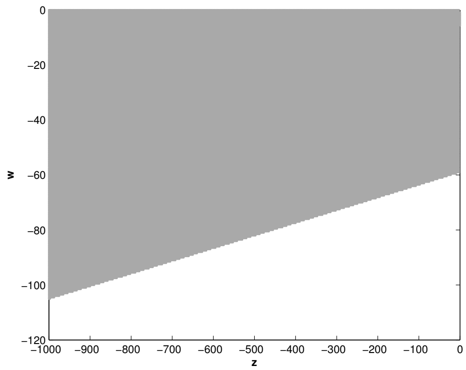

Figure 1: Stability region with $r = 2,\ m = 2,\ c = [\frac{8}{10}, 1]$.

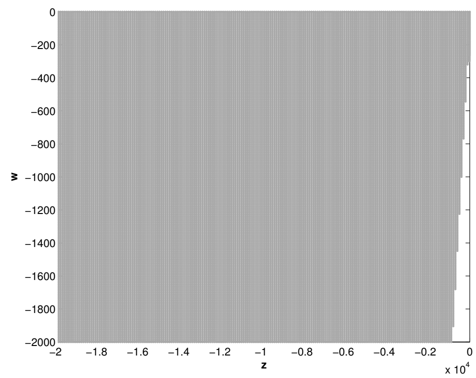

Figure 2: Stability region with $r = 3,\ m = 2,\ c = [\frac{19}{20}, 1]$.

$$\mathcal{R} = \{(z, w) \in \mathbb{C}^- \times \mathbb{C}^- : |\lambda_i(z, w)| < 1, \quad i = 1, 2, \ldots, 2r + m\}.$$

We did not find A_0-stable methods within this class, but wide stability regions exist. For example, in the cases $r = 2$, $m = 2$ with collocation parameters $c = [\frac{8}{10}\ \ 1]$, and $r = 3$, $m = 2$ with collocation parameters $c = [\frac{19}{20}\ \ 1]$, the regions of stability are unbounded. These regions are given in Figures 1 and 2, respectively.

5 Numerical results

In this section, to check the numerical performance of the method, we have considered a variety of problems. Here, the starting values $y_1, \ldots, y_{r-1}$ are obtained by a one step MHCMs of the same order of the present method.

In practice, we need quadrature rules to obtain the numerical solutions. For this propose, we have to apply the rules that preserve order of the main

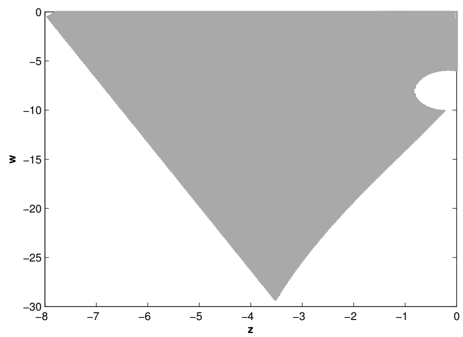

Figure 3: Stability region with $r = 3,\ m = 2,\ c = [\frac{2}{3}, 1]$.

method. A suitable choice is the 3-times Romberg quadrature formula with the Simpson rule [25]. In this rule that is of order 8, the nodes and weights for integration in the interval $[0, h]$ are

$$\zeta_i = \frac{ih}{8}, \quad i = 0, 1, \dots, 8$$

and

$$b = \frac{1}{5760}[217\ \ 1024\ \ 352\ \ 1024\ \ 436\ \ 1024\ \ 352\ \ 1024\ \ 217].$$

In what follows, we describe details of the implemented methods:

- **Method 1:** MHCM of convergence order 5 with $r = 2,\ m = 2$, and collocation parameters $c_1 = \frac{2}{3}$ and $c_2 = 1$.
- **Method 2:** MHCM of convergence order 7 with $r = 3,\ m = 2$, and collocation parameters $c_1 = \frac{2}{3}$ and $c_2 = 1$, with bounded stability region. (see Figure 3)
- **Method 3:** MHCM of convergence order 7 with $r = 3,\ m = 2$, and collocation parameters $c_1 = \frac{19}{20}$, and $c_2 = 1$, with unbounded stability region.
- **Method 4:** Multistep collocation method [8] of convergence order 4 with $r = 3,\ m = 2$, and collocation parameters $c_1 = \frac{7}{10}$ and $c_2 = 1$.

Computational experiments are done by applying the Methods 1–4 on the following problems:

I. The linear VIDE

$$y'(t) = 1 + 2t - y(t) + \int_0^t \tau(1 + 2\tau)e^{\tau(t-\tau)}y(\tau)d\tau, \quad t \in [0, 2],$$

Table 2: The results for problem **I**.

	k	4	5	6	7	8	9
Method 1	cd	2.74	4.20	5.69	7.18	8.68	10.19
	$p(h)$		4.86	4.93	4.97	4.98	4.99
Method 3	cd	4.02	6.06	8.14	10.23	12.33	14.44
	$p(h)$		6.77	6.90	6.96	6.98	6.99
Method 4	cd	2.94	4.07	5.22	6.41	7.60	8.81
	$p(h)$		3.71	4.84	3.92	3.96	3.98

$$y(0) = 1,$$

with the exact solution $y(t) = e^{t^2}$.

II. The nonlinear VIDE

$$\begin{aligned} y'(t) =& -t - \frac{1}{(1+t)^2} + \frac{1}{y(t)}\ln(\frac{2+2t}{2+t}) + \int_0^t \frac{d\tau}{1+(1+t)y(\tau)}, \qquad t \in [0,4], \\ y(0) =& 1, \end{aligned}$$

with the exact solution $y(t) = \frac{1}{1+t}$.

III. The stiff nonlinear VIDE

$$\begin{aligned} y'(t) =& -\frac{\lambda}{2}y(t) + 1 + \frac{\lambda}{2}t + \frac{\lambda}{2}te^{-t^2} - \lambda\int_0^t t\tau e^{-y^2(\tau)}d\tau, \qquad t \in [0,4], \\ y(0) =& 0, \end{aligned}$$

with the exact solution $y(t) = t$.

We have implemented the methods with a fixed stepsize $h = \frac{T}{2^k}$, with several integer values of k. In the following tables, the maximal end point error is written as 10^{-cd}, where cd is the number of correct significant digits. Also, a numerical estimation of the order of convergence of the methods is computed by the formula $p(h) = \log_2(\frac{e(2h)}{e(h)})$, where $e(h)$ is the maximal absolute end point error.

The results in Tables 2 and 3 confirm the proved convergence order. In Table 4, we show the effect of linear stability properties of the methods in solving stiff problems. For Method 2, the interval of absolute stability on real line is bounded. When $z = \lambda h$ or $w = \lambda h^2$ lies out of this interval, increasing of absolute error is evidently seen while for Method 3, with unbounded stability region, better results are obtained.

Table 3: The results for problem **II**.

	k	5	6	7	8	9
Method 1	cd	6.45	7.85	9.31	10.79	12.28
	$p(h)$		4.67	4.84	4.92	4.96
Method 3	cd	7.47	9.33	11.31	13.36	15.43
	$p(h)$		6.17	6.57	6.78	6.89
Method 4	cd	4.61	5.67	6.77	7.92	9.06
	$p(h)$		3.47	3.65	3.82	3.91

Table 4: Comparison of the methods (absolute errors of the methods for problem **III** with $\lambda = 500$).

	Method 2		Method 3	
t	$k = 7$	$k = 8$	$k = 7$	$k = 8$
0.25	1.44E-20	1.28E-22	3.25E-20	1.38E-22
0.5	5.69E-19	8.35E-22	2.14E-19	8.58E-22
0.75	2.20E-17	1.89E-21	4.79E-19	1.90E-21
1	6.21E-16	2.93E-21	7.34E-19	2.90E-21
1.5	4.86E-13	7.34E-21	1.82E-18	7.34E-21
2	4.24E-10	1.71E-20	4.28E-18	1.72E-20
3	3.47E-04	3.09E-20	7.72E-18	3.09E-20
3.5	3.13E-01	3.57E-20	9.03E-18	3.61E-20
4	2.73E+02	4.13E-20	1.03E-17	4.13E-20

6 Conclusion

The introduced methods for VIDEs, based on Hermite–Birkhoff interpolation, use of the approximated values of the solution in the m collocation points and the approximated values of the solution as well as its first derivative in the r previous steps. Using of this technique caused to get methods of higher orders and also with extensive stability regions. As we showed in Table 4, the extensive stability region of the method let us to make a sensible choice for the steplength of the algorithm for solving stiff equations.

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