Application of Newton–Cotes quadrature rule for nonlinear Hammerstein integral equations

A. Shahsavaran

Abstract

A numerical method for solving Fredholm and Volterra integral equations of the second kind is presented. The method is based on the use of the Newton–Cotes quadrature rule and Lagrange interpolation polynomials. By the proposed method, the main problem is reduced to solve some nonlinear algebraic equations that can be solved by Newton’s method. Also, we prove some statements about the convergence of the method. It is shown that the approximated solution is uniformly convergent to the exact solution. In addition, to demonstrate the efficiency and applicability of the proposed method, several numerical examples are included, which confirms the convergence results.

AMS subject classifications (2020): 45B05; 45D05; 65R20; 45G10.

Keywords: Fredholm integral equation; Volterra integral equation; Newton–Cotes quadrature rule; Lagrange interpolation; Convergence.

1 Introduction

Integral equations have lots of applications in science and engineering. Fredholm integral equations arising in the theory of signal processing, which is a subfield of mathematics, information, and electrical engineering. In physics, the solution of such integral equations allows for experimental spectra to be related to various underlying distributions, for example, the mass distribution of polymers in a polymeric melt. They also prevalently appear in linear forward modeling and inverse problems. Also, Volterra integral equations arise in many scientific applications such as the population dynamics, the spread of epidemics, and semi-conductor devices. It was also shown that

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Ahmad Shahsavaran
Department of Mathematics, Borujerd Branch, Islamic Azad University, Borujerd, Iran.
e-mail: a.shahsavaran@iaub.ac.ir
Volterra integral equations can be derived from initial value problems; see [21]. Therefore, providing effective methods for solving such equations is consequential. Fredholm and Volterra integral equations have been extensively studied in many papers, and numerical methods have been widely used to solve such equations, for instance, Newton–Kantorovich and Haar wavelets for nonlinear Fredholm and Volterra integral equations [1], hat basis functions for the system of linear and nonlinear integral equations [2], Haar wavelets for nonlinear Fredholm integral equations [3], alternative Legendre polynomials for nonlinear Volterra–Hammerstein integral equations [5], operational matrix approach for nonlinear Volterra-Fredholm integral equations [4], cubic spline method for Fredholm integral equations [6], implicitly linear collocation method for nonlinear Volterra equations [7], Chebyshev approximation for nonlinear Fredholm-Volterra integral equations [9], Chebyshev collocation method for the class of Fredholm integral equations with highly oscillatory kernels [10], collocation-type method for Hammerstein integral equations [12], Sinc-collocation method for nonlinear Fredholm integral equations with weakly singular kernel [15], triangular functions for nonlinear Volterra-Fredholm integral equations [13], wavelets-Galerkin method and wavelets precondition for first kind Fredholm integral equations [14], spectral collocation method for Fredholm integral equations on the half-line [16], hybrid Legendre Block-Pulse functions for the system of nonlinear Fredholm–Hammerstein integral equations [17], single-term Walsh series for nonlinear Volterra-Hammerstein integral equations [18], piecewise constant functions method for nonlinear Fredholm-Volterra integral equations [19], and so on. The existence and uniqueness of such equations were discussed in [8, 11].

In this work, we consider the Fredholm and Volterra integral equations of Hammerstein type

\[ u(x) = \nu(x) + \int_a^b \kappa(x, t)\psi(t, u(t))dt, \quad x \in [a, b], \]

\[ u(x) = \nu(x) + \int_a^x \kappa(x, t)\psi(t, u(t))dt, \quad x \in [a, b], \]

where \( \nu \in L^2[a, b] \) and \( \kappa \in L^2[a, b]^2 \) are known functions, \( \psi \) is a given nonlinear function defined on \( [a, b] \), and \( u \) is unknown to be determined.

2 Method of solution

In this section, first, we describe the Newton–Cotes quadrature rule.

Let the interval \([a, b]\) be partitioned into \( n \) sub-intervals, by \( n \) equally spaced points. That is,

\[ x_0 = a, \quad x_i = x_0 + ih, \quad \text{for } i = 0, 1, \ldots, n, \quad (1) \]
where the step size $h$ is defined by $h = \frac{x_n - x_0}{n} = \frac{b-a}{n}$.

The Newton–Cotes quadrature rule for a function $f$ defined on $[a,b]$ with known values at equally spaced points $x_i$, $i = 0, 1, \ldots, n$, is as follows:

$$\int_a^b f(x) \, dx \approx \sum_{i=0}^n \omega_i f(x_i), \quad (2)$$

for the set of weights $\{\omega_i\}_{i=0}^n$. Now consider the Lagrange basis polynomials $l_i(t) = \prod_{j=0, j \neq i}^{n} \left( \frac{t - t_j}{t_i - t_j} \right)$ and let $\rho_n(t)$ be the interpolation polynomial in the Lagrange form for the given data points $(t_0, f(t_0)), (t_1, f(t_1)), \ldots, (t_n, f(t_n))$. Then

$$\int_a^b f(t) \, dt \approx \int_a^b \rho_n(t) \, dt = \int_a^b \left( \sum_{i=0}^n f(t_i) l_i(t) \right) \, dt = \sum_{i=0}^n f(t_i) \int_a^b l_i(t) \, dt = \sum_{i=0}^n \omega_i f(t_i), \quad (3)$$

where $\omega_i = \int_a^b l_i(t) \, dt$.

Note that we take the Lagrange interpolation points $t_i$ to be the same as the points $x_i$ for the Newton–Cotes quadrature rule. Now consider the following Fredholm and Volterra integral equations of the second kind

$$u(x) = \nu(x) + \int_a^b \kappa(x, t) \psi(t, u(t)) \, dt, \quad x \in [a, b], \quad (4)$$

$$u(x) = \nu(x) + \int_x^b \kappa(x, t) \psi(t, u(t)) \, dt, \quad x \in [a, b]. \quad (5)$$

These can be written as

$$u(x) = \nu(x) + \int_a^b \kappa(x, t) \Psi(t) \, dt, \quad (6)$$

$$u(x) = \nu(x) + \int_x^b \kappa(x, t) \Psi(t) \, dt, \quad (7)$$

where $\Psi(t) = \psi(t, u(t))$. By considering $u_n(x)$ as an approximation for $u(x)$, we can turn equations (6) and (7) into the following equations:
\[ u_n(x) = \nu(x) + \int_a^b \kappa(x, t)\Psi_n(t)\,dt, \quad (8) \]

\[ u_n(x) = \nu(x) + \int_a^x \kappa(x, t)\Psi_n(t)\,dt, \quad (9) \]

where \( \Psi_n(t) = \psi(t, u_n(t)) \), which immediately implies

\[ \Psi_n(t) = \psi(t, \nu(t) + \int_a^b \kappa(t, x)\Psi_n(x)\,dx), \quad (10) \]

\[ \Psi_n(t) = \psi(t, \nu(t) + \int_a^t \kappa(t, x)\Psi_n(x)\,dx). \quad (11) \]

Using quadrature formula (2) to evaluate the integral part of (10), we obtain

\[ \int_a^b \kappa(t, x)\Psi_n(x)\,dx = \sum_{i=0}^n \omega_i \kappa(t, x_i) \Psi_n(x_i) = \sum_{i=0}^n \omega_i \kappa(t, x_i) \Psi_{n,i}, \quad (12) \]

where \( \Psi_{n,i} = \Psi_n(x_i) \) and \( \omega_i = \int_a^b l_i(t)\,dt \). Similarly, using the Lagrange interpolation for the integrand of (11) gives

\[ \int_a^t \kappa(t, x)\Psi_n(x)\,dx = \int_a^t \sum_{i=0}^n \kappa(t, x_i)\Psi_n(x_i) l_i(x)\,dx = \sum_{i=0}^n \kappa(t, x_i)\Psi_n(x_i) \int_a^t l_i(x)\,dx = \sum_{i=0}^n \omega_i(t) \kappa(t, x_i) \Psi_n(x_i) = \sum_{i=0}^n \omega_i(t) \kappa(t, x_i) \Psi_{n,i}, \quad (13) \]

where \( \omega_i(t) = \int_a^t l_i(x)\,dx \) and \( \Psi_{n,i} = \Psi_n(x_i) \). Substituting (12) into (10) and (13) into (11), we obtain

\[ \Psi_n(t) = \psi(t, \nu(t) + \sum_{i=0}^n \omega_i \kappa(t, x_i) \Psi_{n,i}) \quad (14) \]

and

\[ \Psi_n(t) = \psi(t, \nu(t) + \sum_{i=0}^n \omega_i(t) \kappa(t, x_i) \Psi_{n,i}), \quad (15) \]
respectively. Evaluating (14) and (15) at the points \( t = x_j, j = 0, 1, \ldots, n \) (the points for Newton–Cotes quadrature rule), respectively, gives

\[
\Psi_{n,j} = \psi \left( x_j, \nu(x_j) + \sum_{i=0}^{n} \omega_i \kappa(x_j, x_i) \Psi_{n,i} \right), \quad j = 0, 1, \ldots, n, (16)
\]

and

\[
\Psi_{n,j} = \psi \left( x_j, \nu(x_j) + \sum_{i=0}^{n} \omega_{i,j} \kappa(x_j, x_i) \Psi_{n,i} \right), \quad j = 0, 1, \ldots, n, (17)
\]

where \( \omega_{i,j} = \omega_j(x_j) = \int_{a}^{x_j} l_i(x)dx \). Nonlinear systems of algebraic equations (16) and (17) can be solved by numerical methods such as Newton’s method. By solving the above systems, the values \( \Psi_{n,i}, i = 0, 1, \ldots, n \), will be known. Finally, by substituting (12) into (8) and (13) into (9), we find the numerical solutions of the integral equations (4) and (5) by

\[
u_n(x) = \nu(x) + \sum_{i=0}^{n} \omega_i \kappa(x, t_i) \Psi_{n,i}, (18)\]

and

\[
u_n(x) = \nu(x) + \sum_{i=0}^{n} \omega_i(x) \kappa(x, t_i) \Psi_{n,i}, (19)\]

respectively, where \( t_i = x_i, i = 0, 1, \ldots, n \).

### 3 Convergence of the method

In this section, we analyze the convergence of the prescribed method in Section 2, which enables us to control the estimated errors. First, we provide an interpolation polynomial error bound, which is given in the following theorem.

**Theorem 1.** [20] Suppose that \( f \in C^{n+1}[a, b] \), and let \( p_n \) be a polynomial of degree \( \leq n \) that interpolates the function \( f \) at \( n + 1 \) distinct points \( x_0, x_1, \ldots, x_n \in [a, b] \). Then for each \( x \in [a, b] \), there exists a point \( \zeta_x \in [a, b] \) such that

\[
f(x) - p_n(x) = \frac{\pi(x)}{(n + 1)!} f^{(n+1)}(\zeta_x), (20)
\]

where \( \pi(x) = \Pi_{i=0}^{n} (x - x_i) \).

**Theorem 2.** In the case of equally spaced interpolation points \( x_0 = a \) and \( x_i = x_0 + ih \), for \( i = 0, 1, \ldots n \), where \( h = \frac{b-a}{n} \), we have
390 Shahsavaran

\[ |\pi(x)| \leq \frac{n!}{4} h^{n+1}. \tag{21} \]

**Proof.** Suppose \( x \in [x_i, x_{i+1}] \). Then for the first \( i \) terms of \( \pi(x) \), that is \( \prod_{j=0}^{i-1} (x - x_j) = (x - x_0)(x - x_1) \ldots (x - x_{i-1}) \), and due to equally spaced points \( x_i \), we have

\[ |x - x_i - j| \leq x_{i+1} - x_{i-j} = (j + 1)h, \quad j = 1, 2, \ldots, i. \tag{22} \]

Thus

\[ |\prod_{j=0}^{i-1} (x - x_j)| \leq (i + 1)! h^i. \tag{23} \]

For the next two terms of \( \pi(x) \), that is \( (x - x_i)(x - x_{i+1}) \), and using the simple identity \( \alpha \beta \leq \left( \frac{\alpha + \beta}{2} \right)^2 \), we can write

\[
| (x - x_i)(x - x_{i+1}) | = (x - x_i)(x_{i+1} - x) \quad (x_i \leq x \leq x_{i+1}) \\
\leq \left( \frac{x_{i+1} - x_i}{2} \right)^2 \\
= \frac{h^2}{4}. \tag{24} 
\]

For the \( n - i - 1 \) remaining terms of \( \pi(x) \), that is \( \prod_{j=i+2}^{n} (x - x_j) = (x - x_{i+2})(x - x_{i+3}) \ldots (x - x_n) \), we may proceed as follows:

\[
|x - x_{i+j}| = x_{i+j} - x \quad (x_i \leq x \leq x_{i+1}) \\
\leq x_{i+j} - x_0 \\
= (i + j)h, \quad j = 2, 3, \ldots, n - i. \tag{25} 
\]

Thus

\[ |\prod_{j=i+2}^{n} (x - x_j)| \leq \frac{n!}{(i + 1)!} h^{n-i}. \tag{26} \]

Therefore combining (23), (24), and (26) leads to

\[ |\pi(x)| = |\prod_{i=0}^{n} (x - x_i)| \\
\leq \frac{n!}{4} h^{n+1}. \tag{27} \]

**Theorem 3.** Suppose that \( \kappa \in C^{n+1} [a, b]^2 \) and that \( \psi \) in (4) is a function in \( C^{n+1} [a, b] \) with \( n \geq 0 \). If \( u(x) \), the exact solution, and \( u_n(x) \), the approximate solution defined by (18), are both in \( C^{n+1} [a, b] \), then \( u_n(x) \) is uniformly convergent to \( u(x) \).

**Proof.** From (6) and (18), for every \( x \in [a, b] \), we have
\[ u(x) - u_n(x) = \int_a^b \kappa(x,t)\Psi(t)dt - \sum_{i=0}^n \omega_i \kappa(x,t_i)\Psi_{n,i} \quad (\omega_i = \int_a^b l_i(t)dt) \]
\[ = \int_a^b \left( \kappa(x,t)\Psi(t) - \sum_{i=0}^n \kappa(x,t_i)\Psi_{n,i}l_i(t) \right) dt, \]  
(28)  
but \( \sum_{i=0}^n \kappa(x,t_i)\Psi_{n,i}l_i(t) \) interpolates \( \kappa(x,t)\Psi(t) \) at the points \( t_i, \ i = 0,1,\ldots,n, \) \( (t_i = x_i, \) the points for Newton–Cotes quadrature rule). If we set \( F(x,t) = \kappa(x,t)\Psi(t), \) from (28) and Theorem 1, then
\[ u(x) - u_n(x) = \int_a^b \left( \frac{\pi(t)}{(n+1)!} \frac{\partial^{n+1}F}{\partial t^{n+1}}(x,\zeta_t) \right) dt, \ \zeta_t \in [a,b], \] (29)
where \( \pi(t) = \Pi_{i=0}^n (t - t_i). \) Since \( \psi, u \in C^{n+1}[a,b], \) there is some \( M_1 > 0 \) with \( |\psi^{(n+1)}(t)| \leq M_1 \) for all \( t \in [a,b], \) where \( \Psi(t) = \psi(t,u(t)). \) Also \( \kappa \in C^{n+1}[a,b]^2; \) thus there is some \( M_2 > 0 \) with \( |\frac{\partial^{n+1}F}{\partial t^{n+1}}(x,t)| \leq M_2 \) for all \( x,t \in [a,b]. \) Then \( F(x,t) = \kappa(x,t)\Psi(t), \) necessitates that \( |\frac{\partial^{n+1}F}{\partial t^{n+1}}(x,t)| \leq M \) for a real number \( M \) and for all \( x,t \in [a,b]. \) Therefore from (27) and (29), we obtain
\[ |u(x) - u_n(x)| \leq \frac{1}{(n+1)!} \int_a^b \left| \frac{\pi(t)}{h} \frac{\partial^{n+1}F}{\partial t^{n+1}}(x,\zeta_t) \right| dt \]
\[ \leq \frac{(b-a)M}{4(n+1)} h^{n+1} \quad (h = \frac{b-a}{n}) \]
\[ \leq \frac{M}{4} \left( \frac{b-a}{n} \right)^{n+2}, \] (30)
where we used \( (n+1)n^{n+1} \geq nn^{n+1} = n^{n+2} \) to get the last inequality. We note that
\[ \left( \frac{b-a}{n} \right)^{n+2} \leq \left( \frac{b-a}{n} \right)^2 \left( 1 + \frac{b-a}{n} \right)^n. \]
Thus from (30), we obtain
\[ 0 \leq |u(x) - u_n(x)| \leq \frac{M}{4} \left( \frac{b-a}{n} \right)^2 \left( 1 + \frac{b-a}{n} \right)^n. \] (31)
It is clear to see that
\[ \lim_{n \to \infty} \left( \frac{b-a}{n} \right)^2 = 0, \] (32)
\[ \lim_{n \to \infty} \left( 1 + \frac{b-a}{n} \right)^n = e^{b-a}. \] (33)
Considering the limit of the both sides of (31) as \( n \) approaches infinity and using (32)–(33), yield
\[ \lim_{n \to \infty} u_n(x) = u(x), \quad \text{for every } x \in [a, b]; \]

that is, \( u_n(x) \) is uniformly convergent to \( u(x) \).

As a result of Theorem 3 and from (30), it is also clearly seen that

\[ |u(x) - u_n(x)| = O(h^{n+2}), \quad \text{where } h = \frac{b - a}{n}. \tag{34} \]

4 Illustrative examples

In this section, we apply the method proposed in Section 2 to some test examples. All numerical calculations are performed by Maple 13.

4.1 Fredholm integral equation

Example 1. Consider

\[ u(x) = \frac{7}{8} x + \int_0^1 \frac{1}{2} x t u^2(t) dt, \quad 0 \leq x \leq 1, \]

with exact solution \( u(x) = x \).

For this example, by solving the nonlinear system (16) for \( n = 4 \), we have

\[ \Psi_{4,0} = 0, \quad \Psi_{4,1} = 0.0625, \quad \Psi_{4,2} = 0.25, \quad \Psi_{4,3} = 0.5625, \quad \Psi_{4,4} = 1. \]

Also the nodes \( x_i \) and the weights \( \omega_i \) of the Newton–Cotes quadrature rule for the same \( n \) are

\[ t_0 = 0, \quad t_1 = \frac{1}{4}, \quad t_2 = \frac{1}{2}, \quad t_3 = \frac{3}{4}, \quad t_4 = 1, \]

\[ \omega_0 = \frac{7}{90}, \quad \omega_1 = \frac{16}{45}, \quad \omega_2 = \frac{2}{15}, \quad \omega_3 = \frac{16}{45}, \quad \omega_4 = \frac{7}{90}. \]

Substituting the values of \( x_i, \omega_i \) and \( \Psi_{4,i} \) for \( i = 0, \ldots, 4 \) into (18), we have
Application of Newton–Cotes quadrature rule for nonlinear Hammerstein... 393

\[ u_4(x) = \nu(x) + \sum_{i=0}^{4} \omega_i \kappa(x, t_i) \Psi_{4,i} \]

\[ = \frac{7}{8} x + \frac{7}{90} \left( \frac{1}{2} x \times 0 \right) \times 0 \]
\[ + \frac{16}{45} \left( \frac{1}{2} x \times \frac{1}{4} \right) \times 0.0625 \]
\[ + \frac{2}{15} \left( \frac{1}{2} x \times \frac{1}{2} \right) \times 0.25 \]
\[ + \frac{16}{45} \left( \frac{1}{2} x \times \frac{3}{4} \right) \times 0.5625 \]
\[ + \frac{7}{90} \left( \frac{1}{2} x \times 1 \right) \times 1 \]
\[ = x, \]

which is the exact solution of Example 1

**Example 2.** [9] Consider

\[ u(x) = e^{x} + 1 - \int_{0}^{1} (x + t)e^{u(t)}dt, \quad 0 \leq x \leq 1, \]

with the exact solution \( u(x) = x \).

The absolute error \(|u_n(x) - u(x)|\) of Example 2 for \( n = 2, 4, 6 \) is shown in Table 1.

<table>
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<th>( x )</th>
<th>( n = 2 )</th>
<th>( n = 4 )</th>
<th>( n = 6 )</th>
<th>method of [9] ((N = 7))</th>
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</tr>
</tbody>
</table>

**Example 3.** [1] Consider

\[ u(x) = e^{x+1} - \int_{0}^{1} e^{x-2t} (u(t))^3 dt, \quad 0 \leq x \leq 1, \]

with exact solution \( u(x) = e^x \).

The absolute error \(|u_n(x) - u(x)|\) of Example 3 for \( n = 2, 4, 6 \) is shown in Table 2.
Example 4. [12] In this example, the proposed method in Section 2 is used to
solve an integral equation reformulation of the nonlinear two-point boundary
value problem

\[ u''(t) - \exp(u(t)) = 0, \quad t \in (0, 1), \quad u(0) = u(1) = 0. \]

This problem has the unique solution

\[ u(t) = -\ln(2) + 2\ln\left(\frac{c}{\cos\left(\frac{c(t-\frac{1}{2})}{2}\right)}\right), \]

where \( c \) is the only solution of \( \frac{c}{\cos(\frac{c}{2})} = \sqrt{2} \), and may be reformulated as the
integral equation

\[ u(x) = \int_0^1 \kappa(x, t) \exp(u(t)) dt, \quad x \in [0, 1], \]

where

\[ \kappa(x, t) = \begin{cases} -t(1-x), & t \leq x, \\ -x(1-t), & t > x. \end{cases} \]

The uniform norm \( ||u_n - u|| = \sup\{||u_n(t) - u(t)||, t \in [0, 1]\} \) of Example
4 for \( n = 5, 9 \) is shown in Table 3.

| \( n \) | \( ||u_n - u|| \) (presented method) | \( ||u_n - u|| \) (method of [12]) |
|---|---|---|
| 5 | 5.94e-3 | 5.19e-4 |
| 9 | 2.19e-3 | 1.28e-4 |
4.2 Volterra integral equation

Example 5. [5] Consider
\[ u(x) = \frac{3}{2} - \frac{1}{2} e^{-2x} - \int_0^x (u^2(t) + u(t)) dt, \quad 0 \leq x \leq 1, \]
with the exact solution \( u(x) = e^{-x} \).

For this example, we define \( e_n(x) = |u_n(x) - u(x)| \). The maximum norm \( ||e_n||_{\infty} = \max\{|e_n(x_i)|, x_i = .1 * i, i = 0, 1, \ldots, 10\} \) for \( n = 1, 3, 5, 7, 9 \) is presented in Table 4.

| \( n \) | \( ||e_n||_{\infty} \) (presented method) | \( ||e_n||_{\infty} \) (method of [5]) |
|-------|------------------------------------------|------------------------------------------|
| 1     | 1.174e-1                                 | 6.282e-2                                 |
| 3     | 5.235e-4                                 | 1.001e-3                                 |
| 5     | 3.829e-6                                 | 8.294e-6                                 |
| 7     | 2.020e-8                                 | 3.913e-8                                 |
| 9     | 1.000e-10                                | 1.163e-10                                |

Example 6. [18] Consider
\[ u(x) = 1 + \sin^2 x - 3 \int_0^x \sin(x - t)(u(t))^2 dt, \quad 0 \leq x \leq 1, \]
with exact solution \( u(x) = \cos x \).

The absolute error \( |u_n(x) - u(x)| \) of Example 6 for \( n = 3, 5, 7, 9 \) is shown in Table 5.

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<th>( n = 3 )</th>
<th>( n = 5 )</th>
<th>( n = 7 )</th>
<th>( n = 9 )</th>
<th>method of [18] (( m = 60 ))</th>
</tr>
</thead>
<tbody>
<tr>
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<td>4.6e-3</td>
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<tr>
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<td>2.0e-5</td>
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<td>1.0</td>
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<td>2.9e-7</td>
<td>2.1e-9</td>
<td>1.0e-5</td>
</tr>
</tbody>
</table>

Example 7 (Constructed by author). Consider
\[ u(x) = e^{-x} - e^x(x + 1) + \int_{-1}^x e^{x+t} u(t) dt, \quad -1 \leq x \leq 1. \]
To calculate the error in the interval $[-1, 1]$, we define the error function $e_n(x)$ as

$$e_n(x) = u_n(x) - \nu(x) - \int_{-1}^{x} \kappa(x, t)u_n(t)dt.$$ 

Actually, on the right-hand side of the above equation, we put the approximated solution $u_n(x)$ instead of the exact solution $u(x)$ for (4). Now the absolute error $|e_n(x)|$ of Example 7 for $n = 2, 3, 5, 7, 9$ and some $x \in [-1, 1]$ is shown in Table 6.

<table>
<thead>
<tr>
<th>$x$</th>
<th>$e_2(x)$</th>
<th>$e_3(x)$</th>
<th>$e_5(x)$</th>
<th>$e_7(x)$</th>
<th>$e_9(x)$</th>
</tr>
</thead>
<tbody>
<tr>
<td>-1.0</td>
<td>0.0</td>
<td>0.0</td>
<td>0.0</td>
<td>0.0</td>
<td>0.0</td>
</tr>
<tr>
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<td>2.0e-3</td>
<td>3.4e-4</td>
<td>2.3e-4</td>
<td>5.7e-6</td>
<td>1.0e-9</td>
</tr>
<tr>
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<td>1.0e-2</td>
<td>1.2e-3</td>
<td>4.5e-4</td>
<td>6.9e-6</td>
<td>1.0e-9</td>
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<tr>
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<td>2.8e-2</td>
<td>2.2e-3</td>
<td>4.4e-4</td>
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<td>1.2e-9</td>
</tr>
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<td>2.5e-3</td>
<td>4.2e-4</td>
<td>1.2e-5</td>
<td>9.0e-10</td>
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<td>1.6e-3</td>
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<td>3.0e-9</td>
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<td>5.9e-5</td>
<td>9.4e-4</td>
<td>3.2e-5</td>
<td>6.1e-9</td>
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<td>2.0e-2</td>
<td>3.7e-4</td>
<td>9.0e-4</td>
<td>1.8e-5</td>
<td>9.2e-9</td>
</tr>
<tr>
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<td>1.0e-1</td>
<td>1.5e-3</td>
<td>7.1e-4</td>
<td>6.9e-6</td>
<td>2.1e-8</td>
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<td>6.9e-8</td>
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<td>1.3e-3</td>
<td>1.4e-3</td>
<td>5.1e-5</td>
<td>2.2e-7</td>
</tr>
</tbody>
</table>

Table 6: Absolute error of Example 7

Example 8. Consider

$$u(x) = -\frac{x^5}{4} - \frac{2x^4}{3} - \frac{5x^3}{6} - x^2 + 1 + \int_0^x (xt + 1)(u(t))^2dt, \quad 0 \leq x \leq 1.$$ 

Similar to Example 7, to calculate the error in the interval $[0, 1]$, we define the error function $e_n(x)$ as

$$e_n(x) = u_n(x) - \nu(x) - \int_0^x \kappa(x, t)(u_n(t))^2dt.$$ 

Now the absolute error $|e_n(x)|$ of Example 8 for $n = 2, 3, 5, 7$ and some $x \in [0, 1]$ is shown in Table 7.

5 Conclusion

In this work, the Newton–Cotes quadrature rule together with the Lagrange interpolation were used to transform Fredholm and Volterra integral equations to a system of algebraic equations. As shown in Section 4, via some
Application of Newton–Cotes quadrature rule for nonlinear Hammerstein...

Table 7: Absolute error of Example 8

<table>
<thead>
<tr>
<th>x</th>
<th>$e_2(x)$</th>
<th>$e_3(x)$</th>
<th>$e_5(x)$</th>
<th>$e_7(x)$</th>
</tr>
</thead>
<tbody>
<tr>
<td>0.0</td>
<td>0.0</td>
<td>0.0</td>
<td>0.0</td>
<td>0.0</td>
</tr>
<tr>
<td>0.1</td>
<td>4.7e-4</td>
<td>9.5e-3</td>
<td>0.0</td>
<td>0.0</td>
</tr>
<tr>
<td>0.2</td>
<td>1.1e-3</td>
<td>2.7e-2</td>
<td>1.0e-10</td>
<td>1.0e-10</td>
</tr>
<tr>
<td>0.3</td>
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<td>4.0e-2</td>
<td>1.0e-10</td>
<td>3.0e-10</td>
</tr>
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<td>1.0e-10</td>
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<tr>
<td>0.5</td>
<td>4.4e-4</td>
<td>3.1e-2</td>
<td>1.0e-10</td>
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<tr>
<td>0.6</td>
<td>9.4e-4</td>
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<td>2.0e-9</td>
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<tr>
<td>0.7</td>
<td>3.4e-3</td>
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<td>8.0e-9</td>
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<td>1.0e-8</td>
</tr>
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<td>0.9</td>
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<td>5.1e-2</td>
<td>1.0e-9</td>
<td>4.0e-8</td>
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<tr>
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<td>1.6e-2</td>
<td>5.7e-2</td>
<td>3.0e-9</td>
<td>7.7e-8</td>
</tr>
</tbody>
</table>

test examples, as $n$ increases, the error decreases. The calculated errors in the test examples were compatible with the presented error bound in (30). It was stated that a high accuracy is achieved even by using a small number of $n$. Also the method can be extended to solve a system of such equations.

References


