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Research Article

Error estimates for approximating fixed points and best proximity points for noncyclic and cyclic contraction mappings

A. Safari-Hafshejani

Abstract

In this article, we find a priori and a posteriori error estimates of the fixed point for the Picard iteration associated with a noncyclic contraction map, which is defined on a uniformly convex Banach space with a modulus of convexity of power type. As a result, we obtain priori and posteriori error estimates of Zlatanov for approximating the best proximity points of cyclic contraction maps on this type of space.

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Keywords: Fixed point; Noncyclic contraction map; Uniformly convex Banach space; Modulus of convexity; Priori and posteriori errors estimates.

1 Introduction

A basic result in fixed point theory is the Banach contraction principle. Fixed point theory is an important tool to solve the equation $Tx = x$ for mappings T is defined on subsets of metric or normed spaces. One of the advantages of Banach's fixed point theorem is the estimation of the error of successive iterations and the rate of convergence. There are equations $Tx = x$ for which the exact solution is not easy to find or even is not possible to find. The error estimate is very useful in these cases. An extensive study about approximations of fixed points for self-maps can be found in [2]. In 2016, Zlatanov [17] obtained error estimates for approximating the best proximity

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Akram Safari-Hafshejani
Department of Pure Mathematics, Payame Noor University (PNU), P. O. Box: 19395-3697, Tehran, Iran. e-mail: asafari@pnu.ac.ir

points for cyclic contraction maps as generalization of the Banach contraction principle. More cases can be found in [10, 11, 12, 16] and references therein.

One other kind of a generalization of the Banach contraction principle is the notation of noncyclical maps; that is, $T : A \cup B \to A \cup B$ such that $T(A) \subseteq A$ and $T(B) \subseteq B$. Also, a sufficient condition for the existence and the uniqueness of fixed points in uniformly convex Banach spaces are given in [15].

In this article, we obtain "a priori error estimates" and "a posteriori error estimates" for approximating the fixed point of noncyclic contractions. As a result, we obtain "a priori error estimates" and "a posteriori error estimates" of Zlatanov for approximating the best proximity point of cyclic contractions.

2 Preliminaries

In this section, we recall some definitions and facts, which will be used hereafter. Let A and B be nonempty subsets of a metric space (X, d). The map $T : A \cup B \to A \cup B$ is called a noncyclic map if $T(A) \subseteq A$ and $T(B) \subseteq B$. The noncyclic map $T : A \cup B \to A \cup B$ is called a noncyclic contraction map if there holds the inequality $d(Tx, Ty) \leq kd(x, y) + (1 - k)d(A, B)$ for some $k \in (0, 1)$ and all $x \in A$ and $y \in B$, where $d(A, B) := \inf\{d(x, y) : x \in A, y \in B\}$. We say that $(\xi, \eta) \in A \times B$ is an optimal pair of fixed points of the noncyclic mapping T provided that

$$T\xi = \xi, \quad T\eta = \eta \quad \text{and} \quad d(\xi, \eta) = d(A, B),$$

The definition for noncyclic contraction was introduced in [8].

The map $T : A \cup B \to A \cup B$ is called a cyclic map if $T(A) \subseteq B$ and $T(B) \subseteq A$. The cyclic map $T : A \cup B \to A \cup B$ is called a cyclic contraction map if there holds the inequality $d(Tx, Ty) \leq kd(x, y) + (1 - k)d(A, B)$ for some $k \in (0, 1)$ and all $x \in A$ and $y \in B$. A point $\xi \in A \cup B$ is called a best proximity point for T if $d(\xi, T\xi) = d(A, B)$; see [4, 6, 7] and references therein. If sets A and B have a nonempty intersection, then every best proximity point of T is a fixed point of T.

Definition 1. [9] The modulus of convexity of a Banach space X is the function $\delta_X : [0, 2] \to [0, 1]$ defined by

$$\delta_X(\epsilon) = \inf\left\{1 - \|\frac{x + y}{2}\| : \ \|x\| \leq 1, \ \|y\| \leq 1, \ \|x - y\| \geq \epsilon\right\}.$$

The norm is called uniformly convex if $\delta_X(\epsilon) > 0$ for all $\epsilon > 0$. The space $(X, \|.\|)$ is called a uniformly convex space.

As a result of [15, Lemma 2.2 and Theorem 2.7], we have the next theorem.

Theorem 1. [15] Let A and B be nonempty, closed, and convex subsets of a uniformly convex Banach space $(X, \| \cdot \|)$ and let $T : A \cup B \to A \cup B$

be a noncyclic contraction map. Then T has a unique optimal pair of fixed points (ξ, η) such that for every $x_0 \in A$ and $y_0 \in B$ the sequences $\{T^n x_0\}$ and $\{T^n y_0\}$ converge to ξ and η, respectively.

Definition 2. [9] A Banach space X is said to be uniformly convex if there exists a strictly increasing function $\delta : [0,2] \to [0,1]$ such that the following implication holds for all $x, y, p \in X$, $R > 0$ and $r \in [0, 2R]$:

$$\left.\begin{array}{l} \|x-p\| \leq R \\ \|y-p\| \leq R \\ \|x-y\| \geq r \end{array}\right\} \quad \Rightarrow \quad \|\frac{x+y}{2} - p\| \leq \big(1 - \delta(\frac{r}{R})\big)R. \tag{1}$$

If $(X, \|\cdot\|)$ is a uniformly convex Banach space, then $\delta_X(\epsilon)$ is strictly increasing function. Therefore if $(X, \|\cdot\|)$ is a uniformly convex Banach space, then there exists the inverse function δ^{-1} of the modulus of convexity. If there exist constants $C > 0$ and $q > 0$ such that the inequality $\delta_X(\epsilon) \geq C\epsilon^q$ holds for every $\epsilon \in (0,2]$, then we say that the modulus of convexity is of power type q. It is well known that the modulus of convexity with respect to the canonical norm $\|\cdot\|_p$ in l_p or L_p is of power type, and there holds the inequalities $\delta_X(\epsilon) \geq \frac{\epsilon^p}{p2^p}$ for $p \geq 2$ and $\delta_X(\epsilon) \geq \frac{(p-1)\epsilon^2}{8}$ for $p \in (1,2)$; see [13]. An extensive study of the geometry of Banach spaces can be found in [1, 3, 5].

3 Main results

In this section, we begin with the following lemma as a result of [15, Lemma 2.2], which will be used later.

Lemma 1. Let A and B be nonempty subsets of a metric space (X, d) and let $T : A \cup B \to A \cup B$ be a noncyclic contraction map. Then, for every $x \in A$ and $y \in B$, there holds the inequality

$$d(T^n x, T^n y) - d(A,B) \leq k^n (d(x,y) - d(A,B)). \tag{2}$$

In the following result, we obtain our main result in this section.

Theorem 2. Suppose that A and B are nonempty, closed, and convex subsets of a uniformly convex Banach space $(X, \|\cdot\|)$ such that $d := d(A,B) > 0$, and that $T : A \cup B \to A \cup B$ is a noncyclic contraction map. Let $\delta_X(\epsilon) \geq C\epsilon^q$ for some $C > 0$, $q \geq 2$ and every $\epsilon \in (0,2]$. Then

(i) T has a unique optimal pair of fixed points $(\xi, \eta) \in A \times B$;

(ii) for every $x_0 \in A$ and $y_0 \in B$ the sequences $\{T^n x_0\}$ and $\{T^n y_0\}$ converge to ξ and η, respectively;

(iii) a priori error estimate holds

$$\|\xi - T^m x_0\| \leq \frac{M_{x_0,y_0}}{1-\sqrt[q]{k}} \sqrt[q]{\frac{M_{x_0,y_0}-d}{Cd}} (\sqrt[q]{k})^m;$$

(iv) a posteriori error estimate holds

$$\|T^n x_0 - \xi\| \leq \frac{M_{x_n,y_n}}{1-\sqrt[q]{k}} \sqrt[q]{\frac{M_{x_n,y_n}-d}{Cd}};$$

where for every $x \in A$ and $y \in B$, $M_{x,y} := \max\{\|x-y\|, \|Tx-y\|\}$.

Proof. The proof of (i) and (ii) follows from Theorem 1. (iii) For every $n \in \mathbb{N}$ let $x_n = T^n x_0$ and let $y_n = T^n y_0$. From Lemma 1, we have the inequalities

$$\|x_n - y_n\| \leq k^n(\|x_0 - y_0\| - d) + d \leq k^n(M_{x_0,y_0} - d) + d,$$

$$\|x_{n+1} - y_n\| \leq k^n(\|Tx_0 - y_0\| - d) + d \leq k^n(M_{x_0,y_0} - d) + d,$$

and

$$\|x_n - x_{n+1}\| \leq 2\big(k^n(M_{x_0,y_0} - d) + d\big).$$

Now, from (1) with $x = x_n$, $y = x_{n+1}$, $z = y_n$, $r = \|x_n - x_{n+1}\|$, $R = k^n(M_{x_0,y_0} - d) + d$, and using the convexity of the set A, we get the chain of inequalities

$$\begin{aligned} d &\leq \|\frac{x_n + x_{n+1}}{2} - y_n\| \\ &\leq \left(1 - \delta\big(\frac{\|x_n - x_{n+1}\|}{d + k^n(M_{x_0,y_0} - d)}\big)\right)\left(d + k^n(M_{x_0,y_0} - d)\right). \end{aligned} \tag{3}$$

Using (3), we obtain the inequality

$$\delta\big(\frac{\|x_n - x_{n+1}\|}{d + k^n(M_{x_0,y_0} - d)}\big) \leq \frac{k^n(M_{x_0,y_0} - d)}{d + k^n(M_{x_0,y_0} - d)}. \tag{4}$$

From the uniform convexity of X, it follows that δ is strictly increasing, and therefore there exists its inverse function δ^{-1}, which is strictly increasing. From (4), we get

$$\|x_n - x_{n+1}\| \leq \left(d + k^n(M_{x_0,y_0} - d)\right)\delta^{-1}\left(\frac{k^n(M_{x_0,y_0} - d)}{d + k^n(M_{x_0,y_0} - d)}\right). \tag{5}$$

It follows from the inequality $\delta_X(t) \geq Ct^q$ that $\delta_X^{-1}(t) \leq \big(\frac{t}{C}\big)^{\frac{1}{q}}$. Using (5), we obtain

$$\begin{aligned} \|x_n - x_{n+1}\| &\le M_{x_0,y_0} \sqrt[q]{\frac{k^n(M_{x_0,y_0}-d)}{C\big(d+k^n(M_{x_0,y_0}-d)\big)}} \\ &\le M_{x_0,y_0} \sqrt[q]{\frac{M_{x_0,y_0}-d}{Cd}} (\sqrt[q]{k})^n. \end{aligned} \tag{6}$$

So, from (6), we obtain

$$\|x_n - x_{n+1}\| \le M_{x_0,y_0} \sqrt[q]{\frac{M_{x_0,y_0}-d}{Cd}} (\sqrt[q]{k})^n. \tag{7}$$

From (i) and (ii), there exists a unique fixed point $\xi \in A$ such that for every $x_0 \in A$, the sequence $\{T^n x_0\}$ converges to ξ. After substitution in (7), we get the inequality

$$\sum_{n=1}^{\infty} \|x_n - x_{n+1}\| \le M_{x_0,y_0} \sqrt[q]{\frac{M_{x_0,y_0}-d}{Cd}} \frac{\sqrt[q]{k}}{1-\sqrt[q]{k}}.$$

Consequently, the series $\sum_{n=1}^{\infty} \|x_n - x_{n+1}\|$ is absolutely convergent. Thus, for any $m \ge 1$, there holds $\xi = x_m - \sum_{n=m}^{\infty}(x_n - x_{n+1})$, and we get the inequality

$$\|\xi - x_m\| \le \sum_{n=m}^{\infty} \|x_n - x_{n+1}\| \le M_{x_0,y_0} \sqrt[q]{\frac{M_{x_0,y_0}-d}{Cd}} \frac{(\sqrt[q]{k})^m}{1-\sqrt[q]{k}}.$$

Hence,

$$\|\xi - T^m x_0\| \le \frac{M_{x_0,y_0}}{1-\sqrt[q]{k}} \sqrt[q]{\frac{M_{x_0,y_0}-d}{Cd}} (\sqrt[q]{k})^m.$$

(iv) In a similar way (7), we have

$$\|x_{n+i} - x_{n+i+1}\| \le M_{x_n,y_n} \sqrt[q]{\frac{M_{x_n,y_n}-d}{Cd}} (\sqrt[q]{k})^i.$$

So,

$$\begin{aligned} \|x_n - x_{n+m}\| &\le \sum_{i=0}^{m-1} \|x_{n+i} - x_{n+i+1}\| \\ &\le M_{x_n,y_n} \sqrt[q]{\frac{M_{x_n,y_n}-d}{Cd}} \sum_{i=0}^{m-1} (\sqrt[q]{k})^i. \end{aligned}$$

Hence,

$$\|x_n - x_{n+m}\| \leq \frac{M_{x_n,y_n}}{1-\sqrt[q]{k}} \sqrt[q]{\frac{M_{x_n,y_n}-d}{Cd}}(1-(\sqrt[q]{k})^m). \tag{8}$$

After letting $m \to \infty$ in (8), we obtain the inequality

$$\|T^n x_0 - \xi\| \leq \frac{M_{x_n,y_n}}{1-\sqrt[q]{k}} \sqrt[q]{\frac{M_{x_n,y_n}-d}{Cd}}.$$

□

In the sequence, we obtain the main result of [17] as a special case of Theorem 2.

Corollary 1. [17, Theorem 3.2] Suppose that A and B are nonempty, closed and convex subsets of a uniformly convex Banach space $(X, \|.\|)$ such that $d := d(A,B) > 0$, and that $T : A \cup B \to A \cup B$ is a cyclic contraction map. Let $\delta_X(\epsilon) \geq C\epsilon^q$ for some $C > 0$, $q \geq 2$, and every $\epsilon \in (0,2]$. Then

(i) there exists a unique best proximity point ξ of T in A, $T\xi$ is a unique best proximity point of T in B and $\xi = T^2\xi$;

(ii) for every $x_0 \in A$, the sequence $\{T^{2n}x_0\}$ converges to ξ and $\{T^{2n+1}x_0\}$ converges to $T\xi$.

(iii) a priori error estimate holds

$$\|\xi - T^{2n}x_0\| \leq \frac{\|x_0 - Tx_0\|}{1-\sqrt[q]{k^2}} \sqrt[q]{\frac{\|x_0 - Tx_0\| - d}{Cd}}(\sqrt[q]{k})^{2n};$$

(iv) a posteriori error estimate holds

$$\|T^{2n}x_0 - \xi\| \leq \frac{\|T^{2n-1}x_0 - T^{2n}x_0\|}{1-\sqrt[q]{k^2}} \sqrt[q]{\frac{\|T^{2n-1}x_0 - T^{2n}x_0\| - d}{Cd}} \sqrt[q]{k}.$$

Proof. The proof of (i) and (ii) follows from [17, Theorem 2.1].

Because T is a cyclic contraction map, it is clear that T^2 is a noncyclic contraction map and

$$d(T^2x, T^2y) \leq k^2 d(x,y) + (1-k^2)d(A,B).$$

(iii) As T is a cyclic contraction map, we have

$$\|T^2x_0 - Tx_0\| \leq k\|Tx_0 - x_0\| + (1-k)d(A,B) \leq \|Tx_0 - x_0\|.$$

So,

$$\max\left\{\|x_0 - Tx_0\|, \|T^2x_0 - Tx_0\|\right\} = \|x_0 - Tx_0\|.$$

Hence,

$$M_{x_0,Tx_0} = \|x_0 - Tx_0\|.$$

Applying Theorem 2(iii) for noncyclic contraction T^2, we obtain

$$\begin{aligned}\|\xi - T^{2m}x_0\| &\leq \frac{M_{x_0,Tx_0}}{1-\sqrt[q]{k^2}}\sqrt[q]{\frac{M_{x_0,Tx_0}-d}{Cd}}(\sqrt[q]{k^2})^m\\ &= \frac{\|x_0 - Tx_0\|}{1-\sqrt[q]{k^2}}\sqrt[q]{\frac{\|x_0 - Tx_0\|-d}{Cd}}(\sqrt[q]{k})^{2m}.\end{aligned}$$

(iv) Since T is a cyclic contraction map, we get

$$\begin{aligned}\|T^{2n+2}x_0 - T^{2n+1}x_0\| &\leq k\|T^{2n+1}x_0 - T^{2n}x_0\| + (1-k)d(A,B)\\ &\leq \|T^{2n+1}x_0 - T^{2n}x_0\|,\end{aligned}$$

for every $n \in \mathbb{N}$. So,

$$\max\left\{\|T^{2n}x_0 - T^{2n+1}x_0\|, \|T^{2n+2}x_0 - T^{2n+1}x_0\|\right\} = \|T^{2n}x_0 - T^{2n+1}x_0\|.$$

Hence, we have relations

$$M_{T^{2n}x_0,T^{2n+1}x_0} = \|T^{2n}x_0 - T^{2n+1}x_0\|, \tag{9}$$

$$M_{T^{2n}x_0,T^{2n+1}x_0} \leq \|T^{2n-1}x_0 - T^{2n}x_0\|, \tag{10}$$

$$M_{T^{2n}x_0,T^{2n+1}x_0} - d \leq k(\|T^{2n-1}x_0 - T^{2n}x_0\| - d). \tag{11}$$

Applying Theorem 2(iv) for noncyclic contraction T^2, (9), (10), and (11), we obtain

$$\begin{aligned}\|T^{2n}x_0 - \xi\| &\leq \frac{M_{T^{2n}x_0,T^{2n+1}x_0}}{1-\sqrt[q]{k^2}}\sqrt[q]{\frac{M_{T^{2n}x_0,T^{2n+1}x_0}-d}{Cd}}\\ &\leq \frac{\|T^{2n}x_0 - T^{2n+1}x_0\|}{1-\sqrt[q]{k^2}}\sqrt[q]{k}\sqrt[q]{\frac{\|T^{2n-1}x_0 - T^{2n}x_0\|-d}{Cd}}\\ &\leq \frac{\|T^{2n-1}x_0 - T^{2n}x_0\|}{1-\sqrt[q]{k^2}}\sqrt[q]{\frac{\|T^{2n-1}x_0 - T^{2n}x_0\|-d}{Cd}}\sqrt[q]{k}.\end{aligned}$$

□

Let A and B be nonempty, closed, and convex subsets of a uniformly convex Banach space $(X, \|\cdot\|)$ with a modulus of convexity of power type. Theorem 2 shows that if noncyclic contraction T has a fixed point $\xi \in A$ such that $\{T^n x_0\}$ converges to ξ for some $x_0 \in A$ and (2) holds for every $x \in A$

and $y \in B$, then priori and posteriori errors estimates hold in relations (iii) and (iv) of Theorem 2, respectively. Also, Zlatanov [17] showed that if the cyclic contraction T has the best proximity point $\xi \in A$ such that $\{T^{2n}x_0\}$ converges to ξ for some $x_0 \in A$ and

$$d(T^n x, T^{n+1}x) - d(A,B) \leq k^n(d(x,Tx) - d(A,B)) \tag{12}$$

for every $x \in A \cup B$, then priori and posteriori errors estimates hold in relations (iii) and (iv) of Corollary 1, respectively. In fact, these results can be generalized to contractions that satisfy these conditions. For instance, consider the generalized cyclic quasi-contraction $T : A \cup B \to A \cup B$ introduced in [14]. The author proved that if A and B are nonempty, closed, and convex subsets of a uniformly convex Banach space and $T : A \cup B \to A \cup B$ is a generalized cyclic quasi-contraction, that is, for which there exists $k \in [0,1)$ such that

$$\begin{aligned}\|Tx - Ty\| \leq & k \max\left\{\|x-y\|, \|x-Tx\|, \|y-Ty\|, \frac{\|x-Ty\| + \|Tx-y\|}{2}\right\} \\ & + (1-c)d(A,B),\end{aligned}$$

for all $x \in A$ and $y \in B$; then for every $x_0 \in A$ the sequence $\{T^{2n}x_0\}$ converges to some best proximity point $\xi \in A$ and (12) holds. So priori and posteriori errors estimates for each best proximity point of a generalized cyclic quasi-contraction hold in relations (iii) and (iv) of Corollary 1, respectively. Ilchev [11] used exactly this point to get the main results for the Kannan cyclic contractive maps[12].

4 A numerical example

We know that the space $(\mathbb{R}^p, \|\cdot\|_p)$ is uniformly convex with modulus of convexity of power type, provided that $p > 1$. The following example illustrates Theorem 2.

Example 1. Consider the space $\mathbb{R}^2$ endowed with the norms $\|(x,y)\|_2 = \sqrt[2]{|x|^2 + |y|^2}$. Let

$$A = \{(x,y) \in \mathbb{R}^2 : y - x + 1 \leq 0, y + x - 1 \geq 0\}$$

and

$$B = \{(x,y) \in \mathbb{R}^2 : y - x - 1 \geq 0, y + x + 1 \leq 0\}.$$

It is easy to calculate $d(A,B) = 2$. Suppose that $\lambda \in (0,1)$. Let us define a map $T : \mathbb{R}^2_2 \to \mathbb{R}^2_2$ by

$$T(x,y)=\begin{cases}(1-\lambda+\lambda x,\lambda y) & \text{if } (x,y)\in A,\\ (-1+\lambda+\lambda x,\lambda y) & \text{if } (x,y)\in B.\end{cases}$$

We will show that the map $T : A \cup B \to A \cup B$ is a noncyclic contraction with $k = \lambda$. Consider $(x, y) \in A$, and let $(x', y') := T(x, y)$. Then

$$y' - x' + 1 = \lambda y - 1 + \lambda - \lambda x + 1 = \lambda(y - x + 1) \leq 0$$

and

$$y' + x' - 1 = \lambda y + 1 - \lambda + \lambda x - 1 = \lambda(y + x - 1) \geq 0.$$

Therefore, $T(A) \subseteq A$. The inclusion $T(B) \subseteq B$ is proved in a similar fashion. It is easy to observe that $(1, 0)$ is a fixed point of T in A, that $(-1, 0)$ is a fixed point of T in B, and that $\|(1, 0) - (-1, 0)\|_2 = 2$. Let $u_1 = (x, y)$ and let $u_2 = (x', y')$. Then

$$\begin{aligned}\|T(x,y)-T(x',y')\|_2 &= \|(2(1-\lambda)+\lambda(x-x'),\lambda(y-y'))\|_2\\ &=\sqrt{|2(1-\lambda)+\lambda(x-x')|^2+\lambda|y-y'|^2}\\ &=\|2(1-\lambda)e_1+\lambda(u_1-u_2)\|_2\\ &\leq \lambda\|u_1-u_2\|_2+(1-\lambda)d(A,B).\end{aligned}$$

Thus we can apply Theorem 2 to get error estimates of the successive iterations $\{x_n\}$, where $x_{n+1} = Tx_n$. We will consider a numerical example with $\lambda = \frac{1}{16}$. From [13], we get $C = \frac{1}{8}$ and $q = 2$.

Applying Theorem 2(iv), we obtain

$$\|x_n - \xi\| \leq M_n,$$

for $n \geq 0$, where

$$M_n := \frac{8}{3} M_{x_n,y_n}\sqrt{M_{x_n,y_n} - 2}.$$

In the following table, we obtain the number n of iterations, needed by a posteriori estimate less than 0.005 with initial points $x_0 = (1000, 8)$ and $y_0 = (-500.5, -4)$, which is at least 8.

Applying Theorem 2(iii), we get

$$\|\xi - x_n\| \leq \frac{8}{3} M_{x_0,y_0}\sqrt{M_{x_0,y_0} - 2}(\frac{1}{4})^n,$$

The number n of iterations, needed by a priori error estimate less than 0.005 with an initial points $x_0 = (1000, 8)$ and $y_0 = (-500.5, -4)$, is at least 13.

Similarly, it is shown that the number n of iterations, needed by a posteriori estimate less than 0.005 for $\lambda = \frac{1}{4}$ with initial points $x_0 = (1000, 8)$ and

$\lambda = \frac{1}{16}$	x_n y_n	M_{x_n,y_n}	M_n
$n = 0$	$(1000, 8)$ $(-500.5, -4)$	1500.547983	154900.90193
$n = 1$	$(63.4375, 0.5)$ $(-32.21875, -0.25)$	95.65919017	2468.71315
$n = 2$	$(4.90234375, 3.125 \times 10^{-2})$ $(-2.951171875, -1.5625 \times 10^{-2})$	7.8536555	50.67037
$n = 3$	$(1.243896484, 1.953125 \times 10^{-3})$ $(-1.121948242, -9.765625 \times 10^{-4})$	2.3658465	3.81596
$n = 4$	$(1.01524353, 1.220703125 \times 10^{-4})$ $(-1.007621765, -6.103515625 \times 10^{-5})$	2.0228653	0.81568
$n = 5$	$(1.000952721, 7.629394531 \times 10^{-6})$ $(-1.00047636, -3.81469726 \times 10^{-6})$	2.0014290	0.20176
$n = 6$	$(1.000059545, 4.768371582 \times 10^{-7})$ $(-1.000029773, -2.384185791 \times 10^{-7})$	2.0000893	0.05040
$n = 7$	$(1.0000037215, 2.98023223 \times 10^{-8})$ $(-1.0000018607, -1.490116119 \times 10^{-8})$	2.0000055	0.01260
$n = 8$	$(1.0000002325, 1.8626451 \times 10^{-9})$ $(-1.0000001162, -9.31322574 \times 10^{-10})$	2.0000003	0.00315

$y_0 = (-500.5, -4)$, is at least 16. Also, the number n of iterations, needed by a priori error estimate less than 0.005, is at least 26.

5 Conclusion

In this article, we found a priori and a posteriori errors estimates for approximating fixed points for noncyclic contraction maps, which is defined on a uniformly convex Banach space with a modulus of convexity of power type. As seen in Example 1, a priori error estimate gives a larger number of iterations that are needed than a posteriori estimate. Therefore, it can be concluded that formula (iv) of Theorem 2 provides a better upper bound for error estimates.

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