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Investigation of a variable-power FGM-type copula

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Abstract. Copula theory is essentially based on the use of multivariate functions, commonly called copulas. These functions provide a versatile toolset for capturing a broad spectrum of dependence structures, highlighting their indispensability in a multitude of applied domains. However, in light of the evolving complexities inherent in real-world data, there is a growing demand for pioneering copula constructions. In this paper, our main goal is to increase the field of variable-power copulas by introducing an innovative Farlie–Gumbel–Morgenstern (FGM)-type power copula. It is distinguished by a unique one-parameter formulation that recovers the independence copula. In the main part, we establish its mathematical validity, which is based on differentiation techniques, appropriate factorizations, and two complementary logarithmic inequalities. Then we provide a comprehensive exploration of its modeling properties, with a focus on its negative dependence through the beta medial correlation, rho of Spearman and tau of Kendall. The corresponding copula data generation is examined with different values of the parameter. A new bivariate normal distribution is also derived, and its shapes are discussed. Finally, the minimum and maximum of two random variables connected through the proposed copula are examined from a distributional viewpoint. Our findings contribute to the advancement of copula theory, thereby enhancing its practical utility across a wide range of disciplines.

Keywords: Bivariate distributions; Bivariate plots; Copula approach; Correlation measures; Data generation; Dependence model; Inequalities.

1 Introduction

Copula theory, first introduced by Sklar (1959), has transformed into an indispensable tool for analyzing multivariate data and characterizing the interrelationships among random variables. This theory is based on copulas, which are pivotal multivariate functions enabling the separation of marginal distributions from the underlying dependence structure. This separation empowers

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practitioners to independently model and estimate joint distributions, making copulas highly adaptable and applicable in various domains such as finance (as discussed in Joe 2015), insurance (as studied in Embrechts et al. 2001), and environmental sciences (as detailed in Becherini et al. 2013), among many others. For a comprehensive examination of the theoretical and practical aspects of copulas, we recommend consulting Nelsen (2006) and Durante and Sempi (2016). Recent advancements are documented in Susam (2020a), Susam (2020b), Chesneau (2022), Chesneau (2023), Chesneau (2021b), Michimae and Emura (2022), Shih et al. (2002), El Ktaibi et al. (2022), and Yeh et al. (2023). These references collectively provide a solid foundation for better understanding and applying copula theory.

In the current context, there is a growing demand for the development of new copula constructions, driven by several compelling factors. First, existing copula families often have limitations in capturing the complexities of dependence structures marked by asymmetric behavior of tails or outliers, which are prevalent in various domains. Second, real-world data often exhibit complex, nonlinear, or nonmonotonic dependence patterns that traditional copula models struggle to accurately represent. Finally, the rise of high-dimensional datasets requires copula models capable of efficiently adapting to large dependence structures.

The development of innovative copula families to overcome these limitations is thus of paramount importance. A recent contribution to the subject is the variable-power copula family, briefly presented below. In the bivariate setting, a variable-power copula is characterized by the following general expression:

$$G(x,y) = x^{P(x,y)} y^{Q(x,y)}, \quad (x,y) \in [0,1]^2. \tag{1}$$

Here, $P(x,y)$ and $Q(x,y)$ are bivariate functions that adhere to specific conditions while satisfying the following requirements: $G(0,0) = 0$, $G(0,1) = 0$, $G(1,0) = 0$, and $G(1,1) = 1$. It is evident that when $P(x,y) = Q(x,y) = 1$, the result is the independence copula, that is, $G(x,y) = xy = \Pi(x,y)$. The investigation of scenarios, where $P(x,y) = 1$ and $Q(x,y)$ is a function exclusively dependent on x, was explored in Chesneau (2023). This exploration led to the discovery of several new bivariate copulas with varying power, providing new insights into dependence patterns that extend beyond the conventional framework.

On the other hand, the Farlie-Gumbel-Morgenstern (FGM) copula can be considered as a cornerstone of copula theory. Indeed, it has received considerable attention in recent research due to its versatility in modeling multivariate dependence structure. This widespread interest has stimulated innovative developments in our understanding and use of copulas. From a mathematical point of view, the FGM copula is defined by the following expression:

$$G^{FGM}(x,y) = xy[1 + a(1-x)(1-y)], \quad (x,y) \in [0,1]^2,$$

with $a \in [-1,1]$. The primary advantage of the FGM copula lies in its simplicity, ease of manipulation, and its capacity to capture both negative and positive dependences. However, it does have limitations, notably its restricted ability to represent diverse copula density shapes and its moderate degree of dependence. Recent research efforts have been devoted to improving its flexibility by introducing modifications and extensions, thereby providing more robust tools for risk assessment, financial modeling, and various other applications. Notably, several articles, such as Morgenstern (1956), Celebioglu (1997), Bairamov et al. (2001), Rodríguez-Lallena and

Úbeda-Flores (2004), Domma and Giordano (2012), Mirhosseini et al. (2015), Bairamov and Kotz (2002), Amini et al. (2011), and Chesneau (2021a), have contributed by introducing various modified versions of the FGM copula. These advancements are continually expanding the horizons of copula theory and its practical applications across diverse fields.

In this article, we introduce an innovative copula expression that merges the fundamental principles of variable-power and FGM copulas. Specifically, by selecting $P(x,y) = Q(x,y) = G^{FGM}(x,y)/(xy)$ in (1), our proposed copula exhibits the following distinctive expression:

$$G(x,y) = x^{G^{FGM}(x,y)/(xy)} y^{G^{FGM}(x,y)/(xy)}, \quad (x,y) \in [0,1]^2,$$

which can also be represented in a more concise manner as follows:

$$G(x,y) = (xy)^{1+a(1-x)(1-y)}, \quad (x,y) \in [0,1]^2.$$

It is worth noting that, at this stage, the parameter a remains undetermined, with the exception of the trivial case where $a = 0$ giving the independence copula. The proposed copula stands out primarily due to its novel expression, which, to the best of our knowledge, has not been examined previously. In the main result, we determine a range of values for a making it valid in the mathematical sense. The proof is based on differentiation, appropriate factorizing, and two complementary logarithmic inequalities. Then we investigate its properties through a comprehensive analysis that encompasses analytical, graphical, and numerical methods. A focus is put on diverse kinds of negative dependence. In addition, the data generation based on the proposed copula is described, and a new bivariate normal distribution is derived. Finally, a distributional perspective is used to analyze the minimum and maximum of two random variables connected by the suggested copula.

The article is structured as follows: In Section 2, we introduce the variable-power FGM-type copula. Section 3 delves into an in-depth exploration of its key characteristics. Section 4 contains complementary studies to highlight some computational or distributional aspects. Finally, in Section 5, we provide our concluding remarks.

2 A variable-power FGM-type copula

2.1 Notion of copula

Before introducing our novel variable-power copula, it is essential to provide the notion of copula in the standard bivariate absolutely continuous (SBAC) context.

Lemma 1. *Nelsen (2006) In the SBAC context, we characterize a copula as a bivariate function, denoted as $B(x,y)$, defined over $[0,1]^2$, which is twice continuously differentiable on $(0,1)^2$ and satisfies the following conditions:*

C1: $B(x,0) = 0$, $B(0,y) = 0$, $B(x,1) = x$, *and* $B(1,y) = y$,

C2: $\partial_{x,y}B(x,y) = \dfrac{\partial^2}{\partial x \partial y}B(x,y) \geq 0.$

In fact, the original definition of a bivariate copula considers the 2-increasing condition, which is equivalent to C2 in the SBAC setting. From a mathematical point of view, proving C2 is often more difficult than C1, as it may require complex differentiations, factorizations, and inequalities. We refer to Nelsen (2006) for all the technical details behind the notion of copula in the SBAC context. We will adopt this notion in this article.

2.2 Novel copula

The upcoming result emphasizes the variable-power FGM-type copula under consideration.

Proposition 1. *The bivariate function defined by*

$$G(x,y) = (xy)^{1+a(1-x)(1-y)}, \quad (x,y) \in [0,1]^2, \tag{2}$$

with $a \in [0,1]$*, is a valid copula.*

Proof. Let us examine C1 and C2 outlined in Lemma 1 with the proposed bivariate function, assuming that $a \in [0,1]$. Beginning with C1, for any $x \in [0,1]$, since $1 + a(1-x) \geq 0$, we can observe that

$$G(x,0) = (x \times 0)^{1+a(1-x)(1-0)} = 0.$$

Likewise, for any $y \in [0,1]$, we find that $G(0,y) = 0$. Furthermore, for any $x \in [0,1]$, we have

$$G(x,1) = (x \times 1)^{1+a(1-x)(1-1)} = x.$$

Similarly, for any $y \in [0,1]$, we can determine that $G(1,y) = y$. Thus, C1 is satisfied.

Now, let us delve into C2. By employing conventional differentiation principles and conducting a comprehensive factorization, we arrive at the following result:

$$\begin{aligned}
\partial_{x,y} G(x,y) =& (xy)^{a(1-x)(1-y)} \times \{ax(y-1) + a(x-1)y + axy\log(xy) \\
& + [a(1-x)(1-y) + ax(y-1)\log(xy) + 1][a(1-x)(1-y) + a(x-1)y\log(xy) + 1]\} \\
=& (xy)^{a(1-x)(1-y)}\{ax(y-1) + a(x-1)y + axy\log(xy) + 2a(1-x)(1-y) + 1 \\
& + a(x-1)y\log(xy) + a(y-1)x\log(xy) \\
& + a^2(1-x)^2(1-y)^2 - a^2(1-x)^2(1-y)y\log(xy) - a^2(1-x)(1-y)^2 x\log(xy) \\
& + a^2(1-x)(1-y)xy[\log(xy)]^2\} \\
=& (xy)^{a(1-x)(1-y)}[R(x,y) + S(x,y)],
\end{aligned}$$

where

$$\begin{aligned}
R(x,y) =& ax(y-1) + a(x-1)y + axy\log(xy) + 2a(1-x)(1-y) + 1 \\
& + a(1-x)y[-\log(xy)] + a(1-y)x[-\log(xy)]
\end{aligned}$$

and

$$\begin{aligned}
S(x,y) =& a^2(1-x)^2(1-y)^2 + a^2(1-x)^2(1-y)y[-\log(xy)] \\
& + a^2(1-x)(1-y)^2 x[-\log(xy)] + a^2(1-x)(1-y)xy[\log(xy)]^2.
\end{aligned}$$

Since $(x,y) \in (0,1)^2$ and $a \in [0,1]$, it is obvious that $(xy)^{a(1-x)(1-y)} \geq 0$. Let us now prove that we have $R(x,y) \geq 0$ and $S(x,y) \geq 0$.

With regard to $S(x,y)$, since $(x,y) \in (0,1)^2$, we have $x \geq 0$, $y \geq 0$, $1-x \geq 0$, $1-y \geq 0$, $-\log(xy) \geq 0$, and, obviously, $a^2 \geq 0$ $(1-x)^2 \geq 0$, $(1-y)^2 \geq 0$, and $[\log(xy)]^2 \geq 0$. So, as a direct sum of nonnegative main terms, we have $S(x,y) \geq 0$.

Concerning $R(x,y)$, the steps are more technical. The following logarithmic inequality is well known: $\log(1+u) \leq u$ for $u > -1$. As a result, since $(x,y) \in (0,1)^2$, we have $\log(xy) = \log[1+(xy-1)] \leq xy-1$, so $-\log(xy) \geq 1-xy$. On the other hand, the following complementary logarithmic inequality is well known too: $\log(1+u) \geq u/(1+u)$ for $u > -1$. As a result, since $(x,y) \in (0,1)^2$, we have $\log(xy) = \log[1+(xy-1)] \geq (xy-1)/[1+(xy-1)] = (xy-1)/(xy)$. Therefore, since $a \geq 0$, we have

$$\begin{aligned} R(x,y) &\geq ax(y-1) + a(x-1)y + a(xy-1) + 2a(1-x)(1-y) + 1 \\ &\quad + a(1-x)y(1-xy) + a(1-y)x(1-xy) \\ &= 2ax^2y^2 - ax^2y - axy^2 + 3axy - 2ax - 2ay + a + 1 \\ &= axy[xy + (1-x)(1-y) + 1 - a] + a(1-x)(1-y) + (1-ax)(1-ay). \end{aligned}$$

Since $a \in [0,1]$ specifically and $(x,y) \in (0,1)^2$, it is clear that $axy \geq 0$, $xy + (1-x)(1-y) \geq 0$, $1-a \geq 0$, $a(1-x)(1-y) \geq 0$ and $(1-ax)(1-ay) \geq (1-a)^2 \geq 0$. Hence, we have $R(x,y) \geq 0$.

The above results imply that $\partial_{x,y} G(x,y) \geq 0$. Thus, C2 is satisfied. We conclude that $G(x,y)$ is a valid copula. □

We refer to the copula mentioned in (2) as the variable-power FGM (VPFGM) copula. To the best of our understanding, this is a new addition to the variable-power copula family, as discussed in Chesneau's earlier work (see Chesneau 2022, 2023).

Some related comments are formulated below. First, the independence copula is recovered by taking $a = 0$, and we can write the VPFGM copula as

$$G(x,y) = [\Pi(x,y)]^{1+a(1-x)(1-y)}, \quad (x,y) \in [0,1]^2.$$

We thus see how the independence structure is modified with a variable-power term.

We can also express it in an exponential-logarithmic form as follows:

$$G(x,y) = \exp\{[1 + a(1-x)(1-y)]\log(xy)\}$$

or, eventually,

$$G(x,y) = \exp\{[1 + a(1-x)(1-y)]\log(x) + [1 + a(1-x)(1-y)]\log(y)\}.$$

By analyzing these expressions, the VPFGM copula is distinguished from the extreme value copula family because no function of $\log(x)/\log(xy)$ or $\log(y)/\log(xy)$ can be exhibited in the power-variable term. Furthermore, we can demonstrate that it lacks associativity, which excludes its classification in the Archimedean copula family. Another comment is that we can write it as

$$G(x,y) = xy\exp[a(1-x)(1-y)\phi(xy)], \quad (x,y) \in [0,1]^2,$$

with $\phi(t) = \log(t)$. Under this form, we remark a strong connection with the Celebioglu–Cuadras copula, which is defined with $\phi(t) = 1$ (see Celebioglu 1995). This concordance in form is intriguing and perhaps reveals an unidentified copula family.

With the use of the software R and the package `plotly` (see R Core Team 2016), Figure 1 illustrates Proposition 1 by plotting the VPFGM copula for selected values of $a \in [0,1]$.

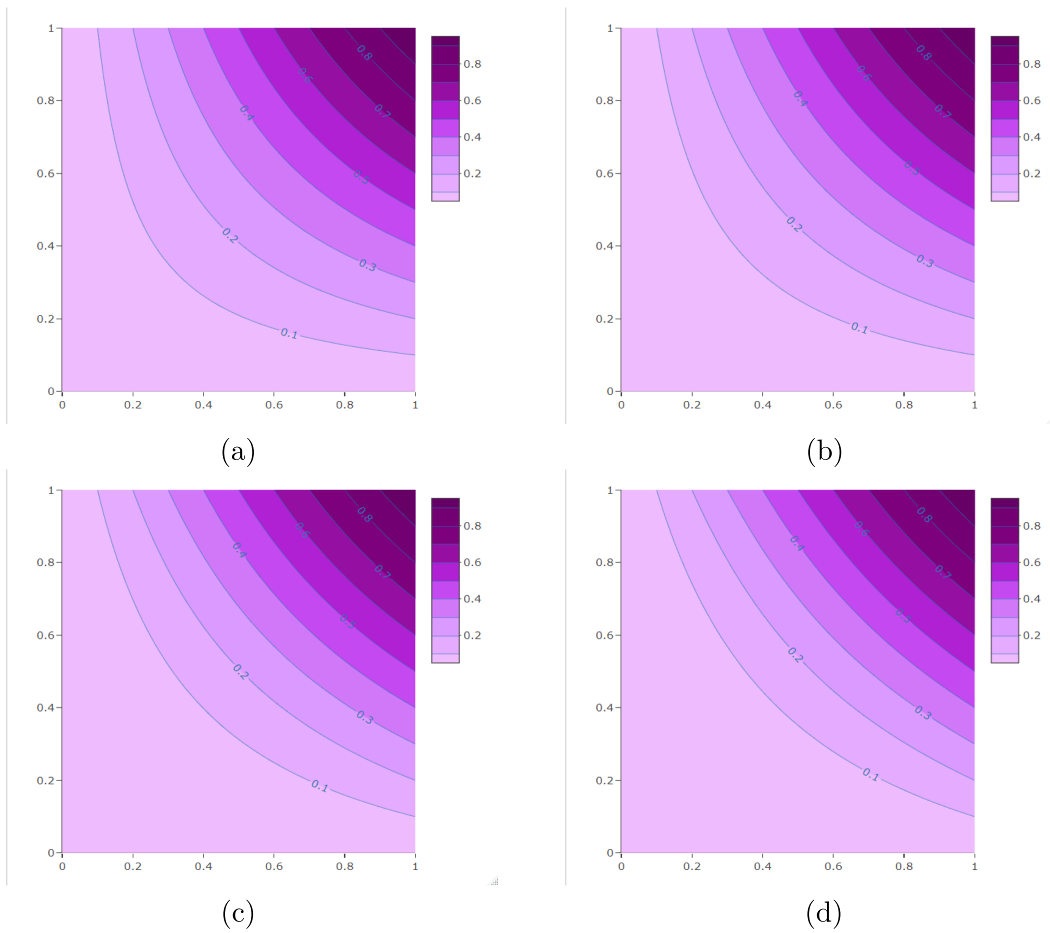


Figure 1: Plots of the intensity zones of the VPFGM copula for (a) $a = 0.05$, (b) $a = 0.3$, (c) $a = 0.7$, and (d) $a = 1$.

We can discern the characteristic circular intensity zones indicative of a valid copula. The remainder of the study considers the associated copula density and explores several of its pivotal characteristics.

3 Copula density and characteristics

3.1 Copula density

The copula density is a crucial component in achieving a comprehensive understanding of a copula, particularly with regard to its various shapes and their implications. In our context, the VPFGM copula density is given as

$$\begin{aligned} g(x,y) =& \partial_{x,y} G(x,y) = (xy)^{a(1-x)(1-y)} \times \left\{ ax(y-1) + a(x-1)y + axy\log(xy) \right. \\ & \left. + [a(1-x)(1-y) + ax(y-1)\log(xy) + 1][a(1-x)(1-y) + a(x-1)y\log(xy) + 1] \right\}, \\ & (x,y) \in (0,1)^2. \end{aligned} \tag{3}$$

This function is relatively complex, primarily due to the presence of the logarithmic term $\log(xy)$. However, with the assistance of mathematical software, such as R, its implementation becomes straightforward. Figure 2 showcases the VPFGM copula density for a range of selected values of $a \in [0,1]$.

A wide panel of different colored zones is observed, highlighting the versatility of the VPFGM copula density. The most pronounced and intense zones are particularly noticeable at the coordinates $(0,1)$ and $(1,0)$. This observation becomes increasingly evident as the values of a increase.

For a deeper analysis, we can also visualize the shapes of the VPFGM copula density. Utilizing the R package `plot3D`, Figure 3 showcases some of them for selected values of $a \in [0,1]$.

By examining the relationship between Figures 2 and 3, we gain deeper insights into the behavior of the copula density, particularly its characteristics around the critical corner points.

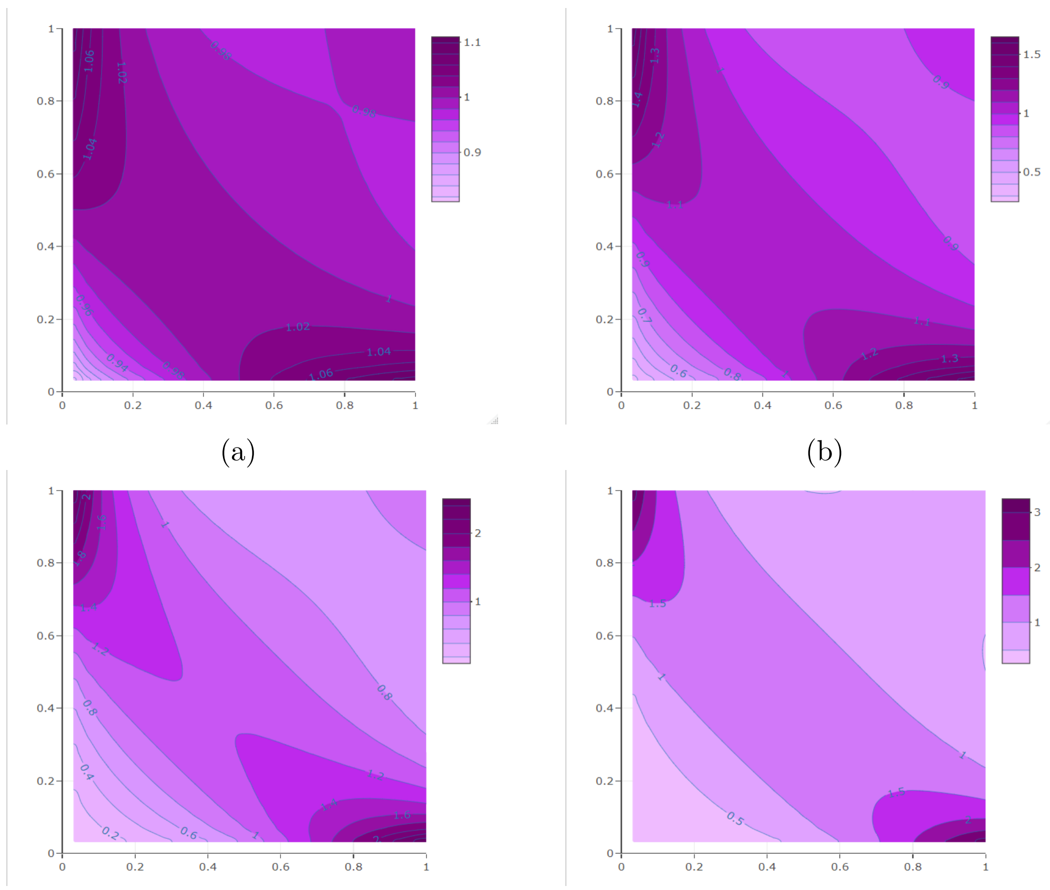
To begin a discussion involving the copula density, let us recall that Proposition 1 holds for $a \in [0,1]$. However, it is not claimed that $[0,1]$ is the optimal range of values of a. Based on Lemma 1, we can remark that C1 holds for $a > -1$. Furthermore, we have the following value for the copula density, among the rare comprehensive ones:

$$g\left(\frac{1}{2}, 1\right) = 1 - \frac{a}{2}.$$

Since a valid copula density must be nonnegative by C2, the inequality $a \leq 2$ is a necessary condition.

On the other hand, numerical tests exclude the negative values for a; in this case, we can always find $(x_\iota, y_\iota) \in (0,1)^2$ such that $g(x_\iota, y_\iota) < 0$, with x_ι approaching 1 and y_ι approaching 0; C2 is not satisfied. For example, by taking $a = -0.2$, $x_\iota = 0.999$ and $y_\iota = 0.001$, we have $g(x_\iota, y_\iota) \approx -0.178378 < 0$.

In addition, as a result of a more deep computational study, the situation is not clarified for $a \in (1,2]$; C2 is satisfied in many pointwise cases. Thus, we conjecture that the optimal range of values for a is an interval I_{op} such that $[0,1] \subseteq I_{op} \subseteq [0,2]$, but more investigations are needed for its identification.



(a) (b) (c) (d)

Figure 2: Plots of the intensity zones of the VPFGM copula density for (a) $a = 0.05$, (b) $a = 0.3$, (c) $a = 0.7$, and (d) $a = 1$.

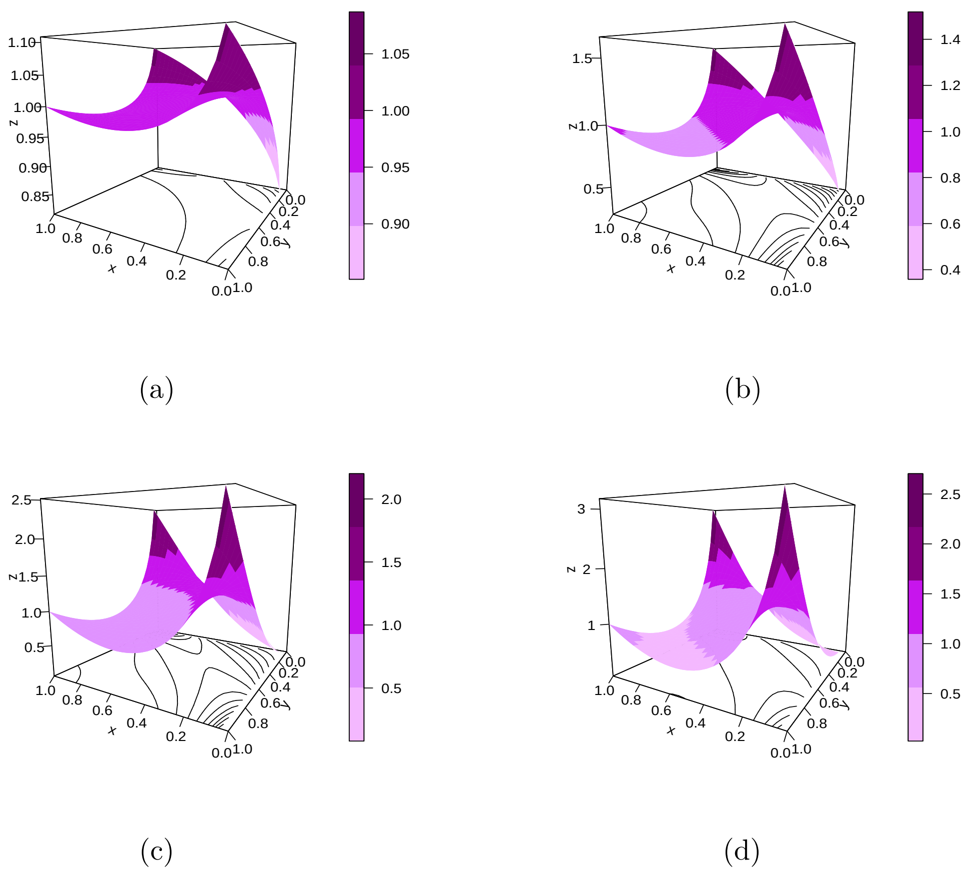


Figure 3: Plots of the shapes of the VPFGM copula density for (a) $a = 0.05$, (b) $a = 0.3$, (c) $a = 0.7$, and (d) $a = 1$.

3.2 Characteristics

We now explore the main properties of the VPFGM copula. Clearly, it is diagonally symmetric because, for any $(x,y) \in [0,1]^2$, we have $G(x,y) = G(y,x)$. Based on the well-known copula theory, the Fréchet-Hoeffding bounds hold (see Nelsen 2006). They imply the following inequalities: $\max(x+y-1,0) \leq G(x,y) \leq \min(x,y)$, that is,

$$\max(x+y-1,0) \leq (xy)^{1+a(1-x)(1-y)} \leq \min(x,y).$$

We can easily establish the upper bound by direct proof, while the lower bound seems less immediately verifiable.

Another direct and important property of a copula is the Lipschitz condition of constant 1, that is, for any $(x_1,x_2,y_1,y_2) \in [0,1]^4$, we have

$$|G(x_2,y_2) - G(x_1,y_1)| \leq |x_2 - x_1| + |y_2 - y_1|,$$

which, in expanded form, corresponds to

$$|(x_2y_2)^{1+a(1-x_2)(1-y_2)} - (x_1y_1)^{1+a(1-x_1)(1-y_1)}| \leq |x_2 - x_1| + |y_2 - y_1|.$$

On the other hand, the function $\psi(x,y) = \partial_y G(x,y) = \partial G(x,y)/(\partial y)$ is nondecreasing and satisfies $0 \leq \psi(x,y) \leq 1$. Since

$$\psi(x,y) = x(xy)^{a(1-x)(1-y)}\left[1 + a(1-x)(1-y) - a(1-x)y\log(xy)\right], \tag{4}$$

the following inequalities hold:

$$0 \leq x(xy)^{a(1-x)(1-y)}\left[1 + a(1-x)(1-y) - a(1-x)y\log(xy)\right] \leq 1.$$

Let us mention that $\psi(x,y)$ will play an important role in some coming parts, especially in the definition of integral-type correlation measures and the data generation process related to the proposed copula.

More technical properties are described below.

The VPFGM copula satisfies a notable weighted geometric result. Indeed, by setting $G(x,y;a) = G(x,y)$, for any $(x,y) \in [0,1]^2$ and $(a_1,a_2,b) \in [0,1]^3$, we have

$$\begin{aligned}[G(x,y;a_1)]^b[G(x,y;a_2)]^{1-b} &= \left[(xy)^{1+a_1(1-x)(1-y)}\right]^b \left[(xy)^{1+a_2(1-x)(1-y)}\right]^{1-b} \\ &= (xy)^{1+[ba_1+(1-b)a_2](1-x)(1-y)} = G[x,y;ba_1+(1-b)a_2].\end{aligned}$$

The tail dependence coefficients for the lower left (LL), lower right (LR), upper left (UL), and upper right (UR) directions can be calculated using the formulas provided in Nelsen (2006) and Jaworski (2023). They can be obtained through conventional limit methods in the following manner:

$$\lambda_{LL} = \lim_{x\to 0}\frac{G(x,x)}{x} = \lim_{x\to 0} x^{1+2a(1-x)^2} = \lim_{x\to 0} x^{2a+1} = 0,$$

$$\lambda_{LR} = \lim_{x\to 0}\frac{x - G(1-x,x)}{x} = \lim_{x\to 0}\frac{x - [(1-x)x]^{1+ax(1-x)}}{x} = \lim_{x\to 0} x[1 - a\log(x)] = 0.$$

Since $G(x,y)$ is diagonally symmetric, we have

$$\lambda_{UL} = \lim_{x\to 0} \frac{x - G(x,1-x)}{x} = \lambda_{RL} = 0$$

and

$$\lambda_{UR} = \lim_{x\to 1} \frac{1-2x+G(x,x)}{1-x} = \lim_{x\to 1} \frac{1-2x+x^{2+2a(1-x)^2}}{1-x} = \lim_{x\to 1}[(1-x)-2a(1-x)^2] = 0,$$

respectively. As a result, the VPFGM copula is completely free of tail dependence, which is relatively rare for a variable-power copula (see Chesneau 2022, 2023).

For any $(x,y) \in [0,1]$ and $a \in [0,1]$, since $1+a(1-x)(1-y) \geq 1$, we have

$$G(x,y) = (xy)^{1+a(1-x)(1-y)} \leq xy,$$

which implies that the VPFGM copula is negatively quadrant dependent. Also, owing to this inequality, it is obvious that

$$G(x,y) \leq xy \leq xy[1+a(1-x)(1-y)] = G^{FGM}(x,y),$$

giving an immediate concordance ordering between the VPFGM and FGM copulas.

By using the following well-known power (Bernoulli) inequality: $(1+u)^r \geq 1+ru$ for $u \geq -1$ and $r \in \mathbb{R}/(0,1)$, then we have

$$\begin{aligned} G(x,y) &= (xy)^{1+a(1-x)(1-y)} \geq 1+[1+a(1-x)(1-y)](xy-1) \\ &= G^{FGM}(x,y) - a(1-x)(1-y). \end{aligned}$$

We can also remark that

$$\partial_x \left(\frac{G(x,y)}{x} \right) = a\frac{1}{x}y(1-y)(xy)^{a(1-x)(1-y)}[1-x-x\log(xy)] \geq 0,$$

implying that the VPFGM copula is left tail increasing (with respect to x and y since it is diagonally symmetric). For more details on the notion of tail monotonicity, we may refer to Nelsen (2006) and Izadkhah et al. (2015).

As commented in Izadkhah et al. (2015), the FGM copula benefits from interesting likelihood ratio dependence properties. Such properties are not excluded for the VPFGM copula, but its complex copula density is an obstacle to a deep study on this aspect.

Regarding the information provided in Nelsen (2006), the expression for the beta medial correlation coefficient of the VPFGM copula is as follows:

$$\beta = 4G(0.5,0.5) - 1 = 4 \times 2^{-2-2a(1/2^2)} - 1 = 2^{-a/2} - 1.$$

It is clear that $\beta \leq 0$, implying the negative dependence feature of the VPFGM copula.

Also, based on the theory in Nelsen (2006), the basic definition of the rho of Spearman related to the VPFGM copula is

$$\begin{aligned} \rho_S &= 12 \int_0^1 \int_0^1 [G(x,y) - xy]\, dxdy \\ &= 12 \int_0^1 \int_0^1 xy \left[(xy)^{a(1-x)(1-y)} - 1 \right] dxdy. \end{aligned}$$

Since no direct primitive of the integrand exists and no integral technique gives a satisfying result, a numerical evaluation of ρ_S is necessary. Table 1 gives the values of ρ_S for some values of $a \in [0,1]$.

Table 1: Some values of the rho of Spearman of the VPFGM copula for $a = 0, 0.1, 0.2, \dots, 1$.

a	0.0	0.1	0.2	0.3	0.4	0.5	0.6	0.7	0.8	0.9	1.0
ρ_S	0	-0.0534	-0.1028	-0.1487	-0.1917	-0.232	-0.27	-0.3059	-0.3399	-0.3722	-0.4029

From this table, we can observe that the VPFGM copula exhibits varying degrees of negative dependence, as indicated by the values of ρ_S into the interval of $[-0.4029, 0]$. Notably, this range demonstrates a larger negative dependence compared to the FGM copula, which attains values within the range of $[-0.3333, 0]$. The FGM copula permits positive dependence, whereas the VPFGM copula, for $a \in [0,1]$, lacks this property. The main distinction between the VPFGM and FGM copulas lies in their inherent characteristics and the manner in which they shape the behavior of the associated copula densities.

To complete the above study, one can also investigate the tau of Kendall of the VPFGM copula defined as

$$\tau_K = 1 - 4\int_0^1 \int_0^1 \psi(x,y)\psi(y,x)dxdy,$$

where $\psi(x,y)$ is given in (4). A numerical exploration gives the results in Table 2.

Table 2: Some values of the tau of Kendall of the VPFGM copula for $a = 0, 0.1, 0.2, \dots, 1$.

a	0.0	0.1	0.2	0.3	0.4	0.5	0.6	0.7	0.8	0.9	1.0
τ_K	0	-0.0357	-0.0691	-0.1005	-0.1302	-0.1585	-0.1854	-0.2112	-0.2359	-0.2596	-0.2825

The same conclusion drawn for the rho of Spearman holds: if we focus on the negative values, the tau of Kendall of the VPFGM copula attains the range of values $[-0.2825, 0]$ against $[-0.2222, 0]$ for the FGM copula, with still the originality arguments in terms of density copula shapes.

4 Some computational and distributional aspects

This section completes the previous study by investigating the data generation, bivariate distributions, and minimum and maximum random variables, all connected with the VPFGM copula through the subject dependence structure.

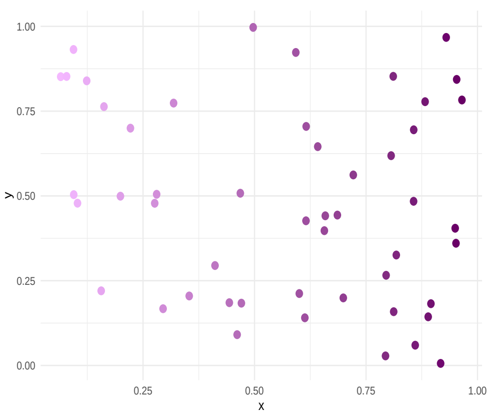
4.1 Data generation

By considering the VPFGM copula as a bivariate cumulative distribution function, we may produce random pairs of values (data) from a random vector, say (X,Y). Thus, by introducing the probability operator $\mathbb{P}$, we suppose that

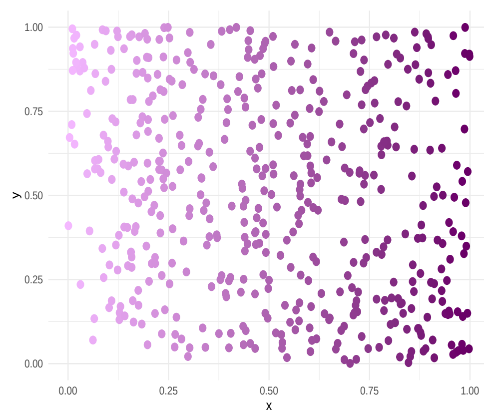
$$\mathbb{P}(X \leq x, Y \leq y) = G(x,y), \quad (x,y) \in [0,1]^2.$$

Based on the data generation result in Nelsen (2006), the process to generate a single pair of values of (X,Y), say $(x_\star, y_\star)$, is described below. First, we generate a pair of independent values $(x_\star, z)$, each from random values of the uniform distribution on the interval $[0,1]$. Second, we determine $y_\star$ as the solution of the following nonlinear equation: $G_{cond}(x_\star, y_\star) = z$, where $G_{cond}(x,y) = \psi(y,x)$ as defined in (4) (with the exchange of x and y). Then the desired pair of values is $(x_\star, y_\star)$.

We can repeat this process m times to generate m such pairs of values. As an illustrative application, we generate several pairs of values for different values of m and a in Figures 4 and 5.



(a)



(b)

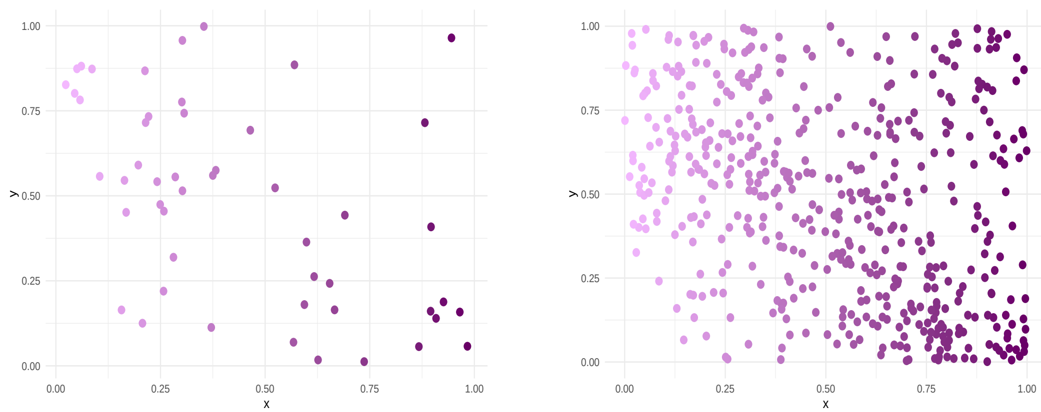
Figure 4: Examples of generated pairs of values (data) from the VPFGM copula for $a = 0.3$ and (a) $m = 50$ and (b) $m = 500$.

Various point structures, including clusters and "empty areas", are visible in these figures, and they are all caused by the multiple dependence features of the VPFGM copula.

On the other hand, from a statistical point of view, these generated pairs of values can be used to test the effectiveness of parametric estimation methods for a. This can therefore be considered as a first step towards the statistical use of the VPFGM copula.

4.2 Bivariate distributions

The primary aim of the VPFGM copula is to create bivariate distributions. The fundamental approach is summarized below. Suppose we have two cumulative distribution functions, denoted



(a) (b)

Figure 5: Examples of generated pairs of values (data) from the VPFGM copula for $a = 0.9$ and (a) $m = 50$ and (b) $m = 500$.

as $U(x)$ and $V(y)$, representing univariate continuous distributions. We establish a novel bivariate cumulative distribution function through the following expression:

$$W(x,y) = G[U(x),V(y)] = [U(x)V(y)]^{1+a[1-U(x)][1-V(y)]}, \quad (x,y) \in \mathbb{R}^2,$$

and the corresponding probability density function is given as

$$w(x,y) = u(x)v(y)g[U(x),V(y)], \quad (x,y) \in \mathbb{R}^2,$$

where $u(x)$ and $v(y)$ are the probability density functions associated with $U(x)$ and $V(y)$, respectively, and $g(x,y)$ is the VPFGM copula density presented in (3). A novel bivariate distribution emerges from these functions. When it comes to selecting appropriate functions for $U(x)$ and $V(y)$ within the context of lifetime analysis, the comprehensive review by Taketomi et al. (2022) can be consulted. For valuable choices of $U(x)$ and $V(y)$ defined on $\mathbb{R}$, we may think to the logistic distribution (see Ali et al. 1978), or the normal distribution.

Just to give an example of a bivariate distribution defined on $\mathbb{R}^2$, let us consider $U(x)$ and $V(y)$ as both the cumulative distribution functions of the standard normal distribution, and call the related VPFGM bivariate distribution the VPFGM normal (VPFGMN) distribution. Figure 6 presents the intensity zones of the corresponding probability density function $w(x,y)$ for selected values of $a \in [0,1]$.

We immediately see how the circle zone of the independence bivariate standard normal distribution is deformed as the values of the parameter a increase and tends to be more dispersed at the top-right corner. This figure is the first representation of a bivariate distribution generated by the VPFGM copula.

4.3 Minimum and maximum random variables

Studying the minimum and maximum of two dependent random variables within the support $[0,1]$ is crucial to understanding extreme values in proportional type data. This analysis provides

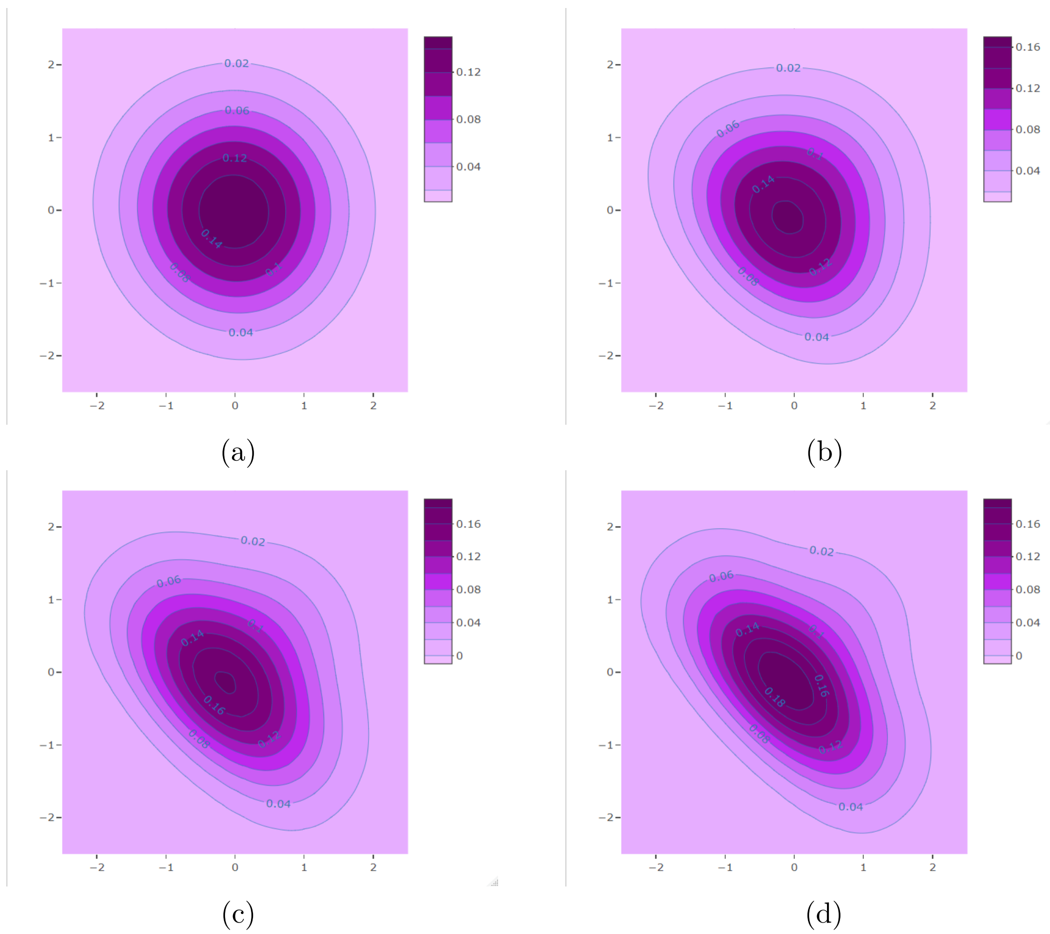


Figure 6: Plots of the intensity zones of the VPFGMN probability density function for (a) $a = 0.05$, (b) $a = 0.3$, (c) $a = 0.7$, and (d) $a = 1$.

insight into the potential range of outcomes and helps identify critical scenarios. By exploring the interdependence of these variables, we can better understand the variability inherent in proportions, providing valuable information for decision-making and risk assessment in diverse fields such as finance, biology, social sciences or sport sciences. For the general context of the bivariate distributions within the support $[0,1]^2$, we may refer to Arnold and Ng (2011), Nadarajah et al. (2017), and Martínez-Flórez et al. (2022).

In the result below, we examine the probability functions associated with the minimum and maximum of two dependent random variables whose joint distribution is characterized by the VPFGM copula, which mainly governs the dependence between these random variables (the marginal distributions being in fact the uniform distribution on the interval $[0,1]$).

Proposition 2. *Let us consider a random vector (X,Y) that has the VPFGM copula as the cumulative distribution function. Then we define the minimum and maximum of X and Y by $M_{\min} = \min(X,Y)$ and $M_{\max} = \max(X,Y)$, respectively. Then the results below hold.*

- *The cumulative distribution function of $M_{\max}$ is given by*

$$F_{\max}(t) = t^{2+2a(1-t)^2}, \quad t \in [0,1],$$

(with the standard complementary values for $t \notin [0,1]$).

Furthermore, the corresponding probability density function is indicated as

$$f_{\max}(t) = 2t^{1+2a(1-t)^2}[a(1-t)^2 - 2at(1-t)\log(t) + 1], \quad t \in (0,1],$$

(with the standard complementary value for $t \notin (0,1]$).

- *The cumulative distribution function of $M_{\min}$ is given by*

$$F_{\min}(t) = 2t - t^{2+2a(1-t)^2}, \quad t \in [0,1].$$

Furthermore, the corresponding probability density function is indicated as

$$f_{\min}(t) = 2\left\{1 - t^{1+2a(1-t)^2}[a(1-t)^2 - 2at(1-t)\log(t) + 1]\right\}, \quad t \in (0,1].$$

Proof. Let us prove each item with the use of the probability operator $\mathbb{P}$.

- For any $t \in [0,1]$, we have

$$F_{\max}(t) = \mathbb{P}(M_{\max} \leq t) = \mathbb{P}(X \leq t, Y \leq t) = G(t,t) = t^{2+2a(1-t)^2}.$$

The corresponding probability density function is obtained upon differentiation with respect to t; we have

$$f_{\max}(t) = F'_{\max}(t) = 2t^{1+2a(1-t)^2}[a(1-t)^2 - 2at(1-t)\log(t) + 1].$$

- Concerning $M_{\min}$, for any $t \in [0,1]$, since $\mathbb{P}(X > t) = \mathbb{P}(Y > t) = 1 - t$, by using the previous result, we have

$$\begin{aligned} F_{\min}(t) &= \mathbb{P}(M_{\min} \le t) = 1 - \mathbb{P}(M_{\min} > t) = 1 - \mathbb{P}(X > t, Y > t) \\ &= 1 - \mathbb{P}(X > t) - \mathbb{P}(Y > t) + \mathbb{P}(M_{\max} > t) \\ &= 1 - \mathbb{P}(X > t) - \mathbb{P}(Y > t) + 1 - F_{\max}(t) \\ &= 2t - F_{\max}(t) = 2t - t^{2+2a(1-t)^2}. \end{aligned}$$

  The corresponding probability density function is obtained upon differentiation with respect to t; we have

$$f_{\min}(t) = F'_{\min}(t) = 2\left\{1 - t^{1+2a(1-t)^2}[a(1-t)^2 - 2at(1-t)\log(t) + 1]\right\}.$$

The desired functions are obtained. □

Beyond the minimum and maximum paradigms, an interest in the two exhibited distributions in Proposition 2 is that they are new in the literature and can be considered one-parameter statistical models in the field of proportional or rate data analysis. Thus, in some senses, the nature of the VPFGM copula is detoured for such univariate perspectives. Further details and applications of the univariate distributions with support $[0,1]$ can be found in Mazucheli et al. (2019), Korkmaz (2020), and Korkmaz and Chesneau (2021).

5 Conclusion

In this article, we introduced and comprehensively analyzed a novel copula that amalgamates the structural elements of variable-power and FGM-type copulas, giving rise to an innovative dependence model. Our examination demonstrated that it has a diagonal symmetry and manifests negative correlation characteristics, including negatively quadrant dependence and negative-valued beta medial correlation, among others.

Moreover, through a numerical investigation, it was determined that the rho of Spearman is confined to the interval $[-0.4029, 0]$, while the tau of Kendall is found within the range of $[-0.2825, 0]$. In addition, an adapted data generation process is described, a new bivariate normal distribution is derived, and a distributional analysis is conducted on the minimum and maximum of two random variables that are related by the suggested copula. These findings represent a slight enhancement over the corresponding negative range associated with the FGM copula. Consequently, our work makes a contribution to the advancement of variable-power-type copulas, establishing a theoretical foundation for applied research using them.

Some potential avenues of work are described below.

- The study of the following more general and possibly diagonally asymmetric version of the proposed copula:

$$G_{\dagger}(x,y) = (xy)^{1+a\theta(x)\xi(y)}, \quad (x,y) \in [0,1]^2,$$

where $\theta(x)$ and $\xi(y)$ are functions that satisfy $\theta(1)=0$ and $\xi(1)=0$, and other conditions that need to be determined to make it valid in the mathematical sense. Like this, we transpose the idea in Rodríguez-Lallena and Úbeda-Flores (2004) for the FGM copula to the VPFGM copula.

- The development of a multivariate extension of the proposed copula in the following form:

$$G(x_1,x_2,\ldots,x_n)=\left(\prod_{i=1}^{n}x_i\right)^{1+a\prod_{i=1}^{n}(1-x_i)}, \quad (x_1,x_2,\ldots,x_n)\in[0,1]^n,$$

where n denotes a positive integer. In this setting, the possible range of values on a making $G(x_1,x_2,\ldots,x_n)$ a valid copula is a challenge that we postpone for the future.

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