Ordering results of extreme order statistics from multiple-outlier scale models with dependence

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Abstract. Many authors have studied ordering results between extreme order statistics from multiple-outlier models when the observations are independent. However, the independence assumption is not very attractive in many situations due to the complexity of the problems. This paper focuses on stochastic comparisons of extreme order statistics stemming from multiple-outlier scale models with dependence. Archimedean copula is used to model dependence structure among nonnegative random variables. Sufficient conditions are obtained to compare the largest order statistics in the sense of the usual stochastic, reversed hazard rate, likelihood ratio, dispersive, star, and Lorenz orders. The smallest order statistics are also compared with respect to the usual stochastic, hazard rate, star, and Lorenz orders. Here, the sufficient conditions are based on the weak-super majorization, weak-sub majorization, and *p*-larger orders between the model parameters. To illustrate the theoretical establishments, some examples are provided. Furthermore, some counterexamples are provided to establish that ignorance of sufficient conditions may not lead to the established ordering results between the order statistics.

Keywords: Archimedean copula; Dispersive order; Hazard rate order; Majorization; Multiple-outlier model; Reversed hazard rate order.

1 Introduction

Order statistics play a vital role in many fields such as statistical inference, economics, reliability theory and operations research. Consider a random sample $X_1, \ldots, X_n$ from a population. Then, the ith order statistic is denoted by $X_{i:n}$, where $i = 1, \ldots, n$. In reliability theory, the ith order statistic represents the lifetime of an $(n-i+1)$-out-of-n system, which functions if at least

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$n - i + 1$ of n components work. In particular, the order statistics $X_{1:n}$ and $X_{n:n}$ represent the lifetimes of series and parallel systems, respectively. Due to the correspondence between the order statistics and the systems' reliability, a lot of effort has been put to study ordering results between order statistics in terms of many well-known stochastic orders. In this paper, we deal with the comparison of extreme order statistics arising from dependent multiple-outlier scale models in the sense of the usual stochastic, reversed hazard rate, hazard rate, star, and Lorenz orders.

Due to the robustness of different estimators of model parameters, multiple-outlier models have been widely used by many researchers. Now, we present some developments on stochastic comparisons between order statistics arising from multiple-outlier models.

Kochar and Xu (2011) considered multiple-outlier exponential models. They showed that more heterogeneity among the scale parameters of the model results more skewed order statistics. Zhao and Balakrishnan (2012) took similar model and obtained ordering results between the largest order statistics with respect to the likelihood ratio, reversed hazard rate, hazard rate, and usual stochastic orderings. Zhao and Balakrishnan (2015) discussed stochastic comparisons of the largest order statistics from multiple-outlier gamma models in terms of various stochastic orderings such as the likelihood ratio, hazard rate, star, and dispersive orders. Kochar and Torrado (2015) established likelihood ratio ordering between the largest order statistics arising from independent multiple-outlier scale models. Sufficient conditions for the comparison of the lifetimes of series systems with respect to dispersive order have been obtained by Fang et al. (2016). They considered that the components of the series systems follow multiple-outlier Weibull models. Amini Seresht et al. (2016) studied multiple-outlier proportional hazard rate models and developed ordering results with respect to the star, Lorenz, and dispersive orders. Furthermore, they proved that more heterogeneity among the multiple-outlier components led to a more skewed lifetime of a *k*-out-of-*n* system consisting of these components. Balakrishnan and Torrado (2016) obtained conditions under which the likelihood ratio order holds between largest order statistics under the set-up of multiple-outlier exponential model. Torrado (2017) developed the comparison result similar to Balakrishnan and Torrado (2016) for the multiple-outlier scale models when the random variables are independent. Wang and Cheng (2017) studied an open problem on mean residual life ordering between two parallel systems under multiple-outlier exponential models, which was proposed by Balakrishnan and Zhao (2013).

It is noted that almost all concerned research in this area has been developed under the assumption of statistically independent component lifetimes. However, there are some practical situations, where the condition of statistically mutual independence among the component lifetimes is evidently unsuitable. For an example, let us consider a mechanical system. The components of the system are suffering a common stress. Then, it is of huge interest to include statistical dependence among component lifetimes into the study of stochastic comparison of the lifetimes of the series and parallel systems. Furthermore, note that due to the complexity of working with the dependent random variables, marginal effort was put to the study of dependent multiple-outlier models by the researchers (see Navarro et al. 2018). These are the main motivations to investigate ordering properties of the extreme order statistics arising from multiple-outlier dependent scale components. The dependency structure among the random variables is modeled by the concept of Archimedean copulas. We recall that a nonnegative random variable X with distribution function F_X is said to follow the scale model if there exists

$\lambda > 0$ such that $F_X(x) = F(\lambda x)$, where F is the baseline distribution function and λ is the scale parameter.

In this paper, we will develop different ordering results between the largest as well as the smallest order statistics stemming from multiple-outlier dependent scale models with respect to several stochastic orderings such as the usual stochastic, hazard rate, reversed hazard rate, star, and Lorenz orders. Let $\{X_1, \ldots, X_{n_1^*}, X_{n_1^*+1}, \ldots, X_{n^*}\}$ be a set of dependent and heterogeneous random observations. The observations are sharing a common Archimedean copula with generator ψ_1 and are taken from the multiple-outlier scale model, where for $i = 1, \ldots, n_1^*$, $X_i \sim F_1(\lambda_1 x)$ and for $j = n_1^* + 1, \ldots, n^*$, $X_j \sim F_2(\lambda_2 x)$, where $\lambda_1, \lambda_2 > 0$. Note that $F_1(\cdot)$ and $F_2(\cdot)$ are two different baseline distribution functions. Also, let $\{Y_1, \ldots, Y_{n_1^*}, Y_{n_1^*+1}, \ldots, Y_{n^*}\}$ be another set of dependent and heterogeneous random observations sharing a common Archimedean copula with generator ψ_2, drawn from the multiple-outlier scale model, where for $i = 1, \ldots, n_1^*$, $Y_i \sim F_1(\mu_1 x)$ and for $j = n_1^* + 1, \ldots, n^*$, $Y_j \sim F_2(\mu_2 x)$, where $\mu_1, \mu_2 > 0$.

Denote by r_1, $\tilde{r}_1$ and r_2, $\tilde{r}_2$ the hazard rate and reversed hazard rate functions for F_1 and F_2, respectively. Furthermore, $X_{n:n}(n_1, n_2)$, $Y_{n^*:n^*}(n_1^*, n_2^*)$ and $X_{1:n}(n_1, n_2)$, $Y_{1:n^*}(n_1^*, n_2^*)$ denote the largest and the smallest order statistics, respectively, arising from $\{X_1, \ldots, X_{n_1}, X_{n_1+1}, \ldots, X_n\}$ and $\{Y_1, \ldots, Y_{n_1^*}, Y_{n_1^*+1}, \ldots, Y_{n^*}\}$, where $1 \le n_1 \le n_1^* \le n_2^* \le n_2$, $n = n_1 + n_2$ and $n^* = n_1^* + n_2^*$. We aim to establish sufficient conditions, under which the following implications hold:

$$(\underbrace{\lambda_1, \ldots, \lambda_1}_{n_1^*}, \underbrace{\lambda_2, \ldots, \lambda_2}_{n_2^*}) \succeq^w (\underbrace{\mu_1, \ldots, \mu_1}_{n_1^*}, \underbrace{\mu_2, \ldots, \mu_2}_{n_2^*}) \quad \Rightarrow \quad Y_{n^*:n^*}(n_1^*, n_2^*) \le_{st} [\le_{rh}] X_{n:n}(n_1, n_2),$$

$$(\underbrace{\lambda_1, \ldots, \lambda_1}_{n_1^*}, \underbrace{\lambda_2, \ldots, \lambda_2}_{n_2^*}) \succeq_w (\underbrace{\mu_1, \ldots, \mu_1}_{n_1^*}, \underbrace{\mu_2, \ldots, \mu_2}_{n_2^*}) \quad \Rightarrow \quad X_{1:n}(n_1, n_2) \le_{st} Y_{1:n^*}(n_1^*, n_2^*)$$

and

$$(\underbrace{u_1, \ldots, u_1}_{n_1^*}, \underbrace{u_2, \ldots, u_2}_{n_2^*}) \succeq_w (\underbrace{v_1, \ldots, v_1}_{n_1^*}, \underbrace{v_2, \ldots, v_2}_{n_2^*}) \Rightarrow X_{1:n}(n_1, n_2) \le_{hr} Y_{1:n^*}(n_1^*, n_2^*),$$

where $u_i = \log \lambda_i$ and $v_i = \log \mu_i$, $i = 1, 2$.

The remainder of the paper is rolled out as follows. Some basic definitions and important lemmas are provided in Section 2. Section 3 consists of two subsections. In Subsection 3.1, we obtain sufficient conditions, under which two largest order statistics are comparable according to the usual stochastic order, reversed hazard rate order, likelihood ratio, dispersive order, star order, and Lorenz order, whereas in Subsection 3.2, we study the usual stochastic order, hazard rate order, star order, and Lorenz order between two smallest order statistics. We also present some examples to illustrate the established results. Finally, we conclude the paper in Section 4.

Throughout the paper, we only concern about nonnegative random variables. Increasing and decreasing mean nondecreasing and nonincreasing, respectively. Also, the prime "*′*" stands for the first order derivative.

2 Basic notions

In this section, we recall some basic definitions and well-known concepts of stochastic orders and majorization. Let $\mathbf{x} = (x_1, \ldots, x_n)$ and $\mathbf{y} = (y_1, \ldots, y_n)$ be two n-dimensional vectors such that

$\mathbf{x}$, $\mathbf{y} \in \mathbb{A}$, where $\mathbb{A} \subset \mathbb{R}^n$ and $\mathbb{R}^n$ be an n-dimensional Euclidean space. Also, consider the order coordinates of the vectors $\mathbf{x}$ and $\mathbf{y}$ as $x_{1:n} \leq \cdots \leq x_{n:n}$ and $y_{1:n} \leq \cdots \leq y_{n:n}$, respectively.

Definition 1. *A vector $\mathbf{x}$ is said to be*

- *majorized by another vector $\mathbf{y}$, (denoted by $\mathbf{x} \preceq^m \mathbf{y}$), if for each $l = 1, \ldots, n-1$, we have $\sum_{i=1}^{l} x_{i:n} \geq \sum_{i=1}^{l} y_{i:n}$ and $\sum_{i=1}^{n} x_{i:n} = \sum_{i=1}^{n} y_{i:n}$;*

- *weakly submajorized by another vector $\mathbf{y}$, denoted by $\mathbf{x} \preceq_w \mathbf{y}$, if for each $l = 1, \ldots, n$, we have $\sum_{i=l}^{n} x_{i:n} \leq \sum_{i=l}^{n} y_{i:n}$;*

- *weakly supermajorized by another vector $\mathbf{y}$, denoted by $\mathbf{x} \preceq^w \mathbf{y}$, if for each $l = 1, \ldots, n$, we have $\sum_{i=1}^{l} x_{i:n} \geq \sum_{i=1}^{l} y_{i:n}$.*

Note that $\mathbf{x} \preceq^m \mathbf{y}$ implies both $\mathbf{x} \preceq_w \mathbf{y}$ and $\mathbf{x} \preceq^w \mathbf{y}$. For a brief introduction of majorization orders and their applications, we refer to Marshall et al. (2011). Now, we present notions of stochastic orderings. Let X_1 and X_2 be two nonnegative random variables with probability density functions f_{X_1} and f_{X_2}, cumulative density functions F_{X_1} and F_{X_2}, survival functions $\bar{F}_{X_1} = 1 - F_{X_1}$ and $\bar{F}_{X_2} = 1 - F_{X_2}$, hazard rate functions $r_{X_1} = f_{X_1}/\bar{F}_{X_1}$ and $r_{X_2} = f_{X_2}/\bar{F}_{X_2}$, and reversed hazard rate functions $\tilde{r}_{X_1} = f_{X_1}/F_{X_1}$ and $\tilde{r}_{X_2} = f_{X_1}/F_{X_2}$, respectively.

Definition 2. *A random variable X_1 is said to be smaller than X_2 in the*

- *likelihood ratio order (denoted by $X_1 \leq_{lr} X_2$) if $f_{X_2}(x)/f_{X_1}(x)$ is increasing in $x > 0$,*

- *hazard rate order (denoted by $X_1 \leq_{hr} X_2$) if $r_{X_1}(x) \geq r_{X_2}(x)$, for all $x > 0$;*

- *reversed hazard rate order (denoted by $X_1 \leq_{rh} X_2$) if $\tilde{r}_{X_1}(x) \leq \tilde{r}_{X_2}(x)$, for all $x > 0$;*

- *usual stochastic order (denoted by $X_1 \leq_{st} X_2$) if $\bar{F}_{X_1}(x) \leq \bar{F}_{X_2}(x)$, for all x;*

- *star order (denoted by $X_1 \leq_* X_2$ or $F_{X_1}(x) \leq_* F_{X_2}(x)$) if $F_{X_2}^{-1} F_{X_1}(x)$ is star shaped in the sense that $\frac{F_{X_2}^{-1} F_{X_1}(x)}{x}$ is increasing in x on the support of X_1;*

- *Lorenz order (denoted by $X_1 \leq_{Lorenz} X_2$) if $\frac{1}{E(X_1)} \int_0^{F_{X_1}^{-1}(u)} x \, dF_{X_1}(x) \geq \frac{1}{E(X_2)} \int_0^{F_{X_2}^{-1}(u)} x \, dF_{X_2}(x)$, for all $u \in (0, 1]$;*

- *dispersive order, (denoted by $X_1 \leq_{disp} X_2$) if $F_{X_1}^{-1}(\beta) - F_{X_1}^{-1}(\alpha) \leq F_{X_2}^{-1}(\beta) - F_{X_2}^{-1}(\alpha)$ for all $0 < \alpha \leq \beta < 1$.*

Note that both the hazard rate and reversed hazard rate orderings imply the usual stochastic ordering. Also, star order implies Lorenz order (see Marshall and Olkin 2007). One may refer to Shaked and Shanthikumar (2007) for a detailed discussion on various stochastic orderings. The next definition is for the Schur-convex and Schur-concave functions.

Definition 3. *A function $\Psi : \mathbb{R}^n \to \mathbb{R}$ is said to be Schur-convex (Schur-concave) in $\mathbb{R}^n$ if*

$$\mathbf{x} \overset{m}{\succeq} \mathbf{y} \Rightarrow \Psi(\mathbf{x}) \geq (\leq) \Psi(\mathbf{y}), \;\; \text{for all} \;\; \mathbf{x}, \mathbf{y} \in \mathbb{R}^n.$$

Throughout the article, we will use the notations. (i) $\mathscr{D}_+ = \{(x_1, \ldots, x_n) : x_1 \geq x_2 \geq \cdots \geq x_n > 0\}$ and (ii) $\mathscr{E}_+ = \{(x_1, \ldots, x_n) : 0 < x_1 \leq x_2 \leq \cdots \leq x_n\}$. Set $h'(z) = \frac{dh(z)}{dz}$. The following consecutive lemmas due to Kundu et al. (2016) are useful to prove the results in the subsequent sections. The partial derivative of h with respect to its kth argument is denoted by $h_{(k)}(\mathbf{z}) = \partial h(\mathbf{z})/\partial z_k$, for $k = 1, \ldots, n$, where $\mathbf{z} = (z_1, \ldots, z_n)$.

Lemma 1. *Let $h : \mathscr{D}_+ \to \mathbb{R}$ be a function, continuously differentiable on the interior of $\mathscr{D}_+$. Then, for $\mathbf{x}$, $\mathbf{y} \in \mathscr{D}_+$,*

$$\mathbf{x} \succeq^m \mathbf{y} \text{ implies } h(\mathbf{x}) \geq (\leq) h(\mathbf{y}),$$

if and only if $h_{(k)}(\mathbf{z})$ is decreasing (increasing) in $k = 1, \ldots, n$.

Lemma 2. *Let $h : \mathscr{E}_+ \to \mathbb{R}$ be a function, continuously differentiable on the interior of $\mathscr{E}_+$. Then, for $\mathbf{x}$, $\mathbf{y} \in \mathscr{E}_+$,*

$$\mathbf{x} \succeq^m \mathbf{y} \text{ implies } h(\mathbf{x}) \geq (\leq) h(\mathbf{y}),$$

if and only if $h_{(k)}(\mathbf{z})$ is increasing (decreasing) in $k = 1, \ldots, n$.

The following lemma due to Saunders and Moran (1978) is useful to establish star order between the order statistics.

Lemma 3. *Let $\{F_\lambda | \lambda \in \mathbb{R}\}$ be a class of distribution functions, such that F_λ is supported on some interval $(a, b) \subseteq (0, \infty)$ and has density f_λ that does not vanish on any subinterval of (a, b). Then,*

$$F_\lambda \leq_* F_{\lambda^*}, \quad \lambda \leq \lambda^*$$

if and only if

$$\frac{F_\lambda'(x)}{x f_\lambda(x)} \text{ is decreasing in } x,$$

where F_λ' is the derivative of F_λ with respect to λ.

To model the dependency structure among the random variables, the concept of copulas plays a vital role. One of the important characteristics of the copula is that it involves the information of the dependencies between the random variables apart from the behavior of the marginal distributions. Archimedean copulas are important class of copulas. These are used widely because of its simplicity. Let F and $\bar{F}$ be the joint distribution function and the joint survival function of the random vector $\mathbf{X} = (X_1, \ldots, X_n)$. Suppose there exist functions $C(\mathbf{z}) : [0, 1]^n \to [0, 1]$ and $\hat{C}(\mathbf{z}) : [0, 1]^n \to [0, 1]$ such that for all x_i, $i \in \mathscr{I}_n$, where $\mathscr{I}_n$ is the index set

$$F(x_1, \ldots, x_n) = C(F_1(x_1), \ldots, F_n(x_n))$$

and

$$\bar{F}(x_1, \ldots, x_n) = \hat{C}(\bar{F}_1(x_1), \ldots, \bar{F}_n(x_n))$$

hold. Then, $C(\mathbf{z})$ and $\hat{C}(\mathbf{z})$ are said to be the copula and survival copula of $\mathbf{X}$, respectively. Here, $F_1, \ldots, F_n$ and $\bar{F}_1, \ldots, \bar{F}_n$ are the univariate marginal distribution functions and survival functions of the random variables $X_1, \ldots, X_n$, respectively.

Now, let $\psi : [0, \infty) \to [0, 1]$ be a nonincreasing and continuous function, satisfying $\psi(0) = 1$ and $\psi(\infty) = 0$. Also, let $\psi = \phi^{-1} = \sup\{x \in \mathcal{R} : \phi(x) > v\}$ be the right continuous inverse. Furthermore, suppose ψ satisfies the conditions (i) $(-1)^i \psi^i(x) \geq 0$, $i = 0,\ 1, \ldots, d-2$ and (ii) $(-1)^{d-2} \psi^{d-2}$ is nonincreasing and convex. That implies the generator ψ is d-monotone. Then, a copula C_ψ is said to be an Archimedean copula if it can be written as the following form

$$C_\psi(v_1, \ldots, v_n) = \psi(\phi(v_1) + \cdots + \phi(v_n)), \quad \text{for all } v_i \in [0,1],\ i \in \mathscr{I}_n.$$

For further discussion on Archimedean copulas, one may refer to McNeil and Nešlehová (2009); Nelsen (2006). A few well known Archimedan copulas including different properties of the generators are enlisted in Table 1. Next lemma is taken from Li and Fang (2015), which has been used to prove the results in Theorems 1, 7 and 9.

Lemma 4. *For two n-dimensional Archimedean copulas C_{ψ_1} and C_{ψ_2}, if $\phi_2 \circ \psi_1$ is super-additive, then $C_{\psi_1}(\mathbf{z}) \leq C_{\psi_2}(\mathbf{z})$, for all $\mathbf{z} \in [0,1]^n$. A function f is said to be super-additive, if $f(x) + f(y) \leq f(x+y)$, for all x and y in the domain of f.*

3 Main results

This section is completely devoted to establishing sufficient conditions, under which the extreme order statistics arising from multiple outlier dependent scale models are comparable in different stochastic senses. The usual stochastic, hazard rate, reversed hazard rate, star, and Lorenz orders are used in what follows. Throughout this section, we denote two-dimensional vectors by bold symbols. For example, $\lambda = (\lambda_1, \lambda_2)$ and $\mu = (\mu_1, \mu_2)$.

3.1 Orderings between the largest order statistics

This subsection addresses ordering results between the largest order statistics arising from multiple-outlier models. The following three consecutive theorems present different conditions, for which the usual stochastic order between the largest order statistics holds.

The distribution functions of $X_{n^*:n^*}(n_1^*, n_2^*)$ and $Y_{n^*:n^*}(n_1^*, n_2^*)$ are, respectively, given by

$$F_{X_{n^*:n^*}(n_1^*, n_2^*)}(x) = \psi_1 \left[\sum_{i=1}^{n^*} \phi_1(F_i(x\lambda_i)) \right] = \psi_1 \left[n_1^* \phi_1 \left(F_1\left(x\lambda_1 \right) \right) + n_2^* \phi_1 \left(F_2\left(x\lambda_2 \right) \right) \right] \tag{1}$$

and

$$F_{X_{n^*:n^*}(n_1^*, n_2^*)}(x) = \psi_2 \left[\sum_{i=1}^{n^*} \phi_2(F_i(x\mu_i)) \right] = \psi_2 \left[n_1^* \phi_2 \left(F_1\left(x\mu_1 \right) \right) + n_2^* \phi_2 \left(F_2\left(x\mu_2 \right) \right) \right]. \tag{2}$$

Li et al. (2016) established comparison results between two largest order statistics in the sense of the usual stochastic order when the underlying random variables follow the dependent scale models. The authors have used same baseline distribution.

In this section, we consider two sets of dependent observations where the random variables follow multiple-outliers dependent scale models sharing Archimedean copula with same/different generators. Here, the first n_1 observations of $\{X_1, \ldots, X_{n_1}, X_{n_1+1}, \ldots, X_n\}$ have baseline distribution

Table 1: List of several Archimedean copulas and generators

Archimedean copula	$\psi(x) \equiv \psi(x;\sigma)$	$\sigma \in$	ψ is log-convex	ψ is log-concave	$\frac{1-\psi}{\psi'}$ is decreasing	$[\frac{1-\psi}{\psi'}][\frac{1-\psi}{\psi'}]'$ is increasing	$[\frac{1-\psi}{\psi'}]'/[\frac{\psi}{\psi'}]$ is increasing	$\psi_1(x)$	$\psi_2(x)$	$\phi_2 \circ \psi_1$ is super-additive
Independence copula	e^{-x}	-	Yes	Yes	Yes	Yes	Yes	e^{-x}	e^{-x}	Yes
Ali–Mikhail–Haq copula	$\frac{1-\sigma}{e^x-\sigma}$	$[-1,1)$	Yes $\sigma\in(0,1)$	Yes $\sigma\in[-1,0)$	Yes $\sigma\in[-1,1)$	Yes $\sigma\in[-1,1)$	-	$\frac{1-\sigma_1}{e^x-\sigma_1}$	$\frac{1-\sigma_2}{e^x-\sigma_2}$	Yes $\lessgtr\ \vee\ \vee$ $0\quad\sigma_1\quad\sigma_2\quad1$
Gumbel–Barnett copula	$e^{\frac{1}{\sigma}(1-e^x)}$	$(0,1]$	No	Yes $\sigma\in(0,1]$	Yes $\sigma\in(0,1]$	-	$\sigma=0.99$	$e^{\frac{1}{\sigma_1}(1-e^x)}$	$e^{\frac{1}{\sigma_2}(1-e^x)}$	Yes $\vee\ \lessgtr\ \lessgtr$ $0\quad\sigma_2\quad\sigma_1\quad1$
Gumbel–Hougaard copula	$e^{1-(1+x)^{\sigma}}$	$[1,\infty)$	No	Yes $\sigma\in[1,\infty)$	Yes $\sigma\in[1,\infty)$	-	-	$e^{1-(1+x)^{\sigma_1}}$	$e^{1-(1+x)^{\sigma_2}}$	Yes $\lessgtr$ $\sigma_1\quad\sigma_2$
Gumbel copula	$e^{-x^{\frac{1}{\sigma}}}$	$[1,\infty)$	Yes $\sigma\in[1,\infty)$	No	Yes $\sigma\in[1,\infty)$	-	-	$e^{-x^{\frac{1}{\sigma_1}}}$	$e^{-x^{\frac{1}{\sigma_2}}}$	Yes $\lessgtr$ $\sigma_1\quad\sigma_2$
Clayton copula	$(1+\sigma t)^{-\frac{1}{\sigma}}$	$[1,\infty)$	Yes $\sigma\in[1,\infty)$	No	Yes $\sigma\in[1,\infty)$	-	-	$(1+\sigma_1 t)^{-\frac{1}{\sigma_1}}$	$(1+\sigma_2 t)^{-\frac{1}{\sigma_2}}$	Yes $\lessgtr$ $\sigma_1\quad\sigma_2$
-	$(1-t)^{\sigma}$	$[1,\infty)$	No	Yes $\sigma\in[1,\infty)$	Yes $\sigma\in[1,\infty)$	Yes $\sigma\in[1,\infty)$	-			

function F_1 and remaining observations have baseline distribution function F_2. Furthermore, assume that first n_1^* observations of $\{Y_1,\ldots,Y_{n_1^*},Y_{n_1^*+1},\ldots,Y_{n^*}\}$ have a baseline distribution function F_1, and remaining $(n^*-n_1^*)$ observations have baseline distribution function F_2. It is also assumed that the number of observations n $(=n_1+n_2)$ and n^* $(=n_1^*+n_2^*)$ may be different. Now, we investigate whether the usual stochastic order between $Y_{n^*:n^*}(n_1^*,n_2^*)$ and $X_{n:n}(n_1,n_2)$ holds under this set-up. To obtain this result, we first prove following two theorems. In the following theorem, it is assumed that the dependence structures of two sets of samples having multiple-outliers are different. Also, first n_1^* observations of $\{X_1,\ldots,X_{n_1^*},X_{n_1^*+1},\ldots,X_{n^*}\}$ have baseline distribution function F_1 and remaining observations have baseline distribution function F_2. Before presenting the first result, we state the following assumption.

Assumption 1. *Let $X_1,\ldots,X_{n^*}$ be n^* dependent nonnegative random variables sharing Archimedean (survival) copula with generator ψ_1, with $X_i \sim F_1(x\lambda_1)$, for $i=1,\ldots,n_1^*$ and $X_j \sim F_2(x\lambda_2)$, for $j=n_1^*+1,\ldots,n^*$. Also, let $Y_1,\ldots,Y_{n^*}$ be n^* dependent nonnegative random variables sharing Archimedean copula with generator ψ_2, with $Y_i \sim F_1(x\mu_1)$, for $i=1,\ldots,n_1^*$ and $Y_j \sim F_2(x\mu_2)$, for $j=n_1^*+1,\ldots,n^*$. Here, $n_1^*+n_2^*=n^*$, $\psi_1=\phi_1^{-1}$ and $\psi_2=\phi_2^{-1}$.*

Theorem 1. *Under the set-up as in Assumption 1, let $\tilde{r}_1(x)\geq(\leq)\tilde{r}_2(x)$ and let $n_1^*\geq(\leq)n_2^*$. Also, let λ, $\mu\in\mathscr{E}_+$ $(\mathscr{D}_+)$, $\phi_2\circ\psi_1$ be super-additive and let ψ_1 or ψ_2 be log-convex. Then,*

(i) $(\underbrace{\lambda_1,\ldots,\lambda_1}_{n_1^*},\underbrace{\lambda_2,\ldots,\lambda_2}_{n_2^*})\succeq^w(\underbrace{\mu_1,\ldots,\mu_1}_{n_1^*},\underbrace{\mu_2,\ldots,\mu_2}_{n_2^*})\Rightarrow Y_{n^*:n^*}(n_1^*,n_2^*)\leq_{st}X_{n^*:n^*}(n_1^*,n_2^*)$, *provided* $\tilde{r}_1(x)$
    or $\tilde{r}_2(x)$ is decreasing;

(ii) $(\underbrace{\lambda_1,\ldots,\lambda_1}_{n_1^*},\underbrace{\lambda_2,\ldots,\lambda_2}_{n_2^*})\succeq^p(\underbrace{\mu_1,\ldots,\mu_1}_{n_1^*},\underbrace{\mu_2,\ldots,\mu_2}_{n_2^*})\Rightarrow Y_{n^*:n^*}(n_1^*,n_2^*)\leq_{st}X_{n^*:n^*}(n_1^*,n_2^*)$, *provided* $x\tilde{r}_1(x)$
    or $x\tilde{r}_2(x)$ is decreasing.

Proof. Here, we prove the first part. Second part can be proved by similar argument. Denote $A(\lambda,\psi_1,x)=F_{X_{n^*:n^*}(n_1^*,n_2^*)}(x)$ and $B(\mu,\psi_2,x)=F_{Y_{n^*:n^*}(n_1^*,n_2^*)}(x)$ (given in (1) and (2)). Using the fact that $\phi_2\circ\psi_1$ is super-additive, one can easily obtain $A(\mu,\psi_1,x)\leq B(\mu,\psi_2,x)$. Therefore, to prove the desired result, we have to show that $A(\lambda,\psi_1,x)\leq A(\mu,\psi_1,x)$. This is equivalent to establish that the function $A(\lambda,\psi_1,x)$ is increasing and Schur-concave with respect to λ (see Marshall et al. 2011, Theorem A.8). Furthermore, on differentiating $A(\lambda,\psi_1,\mathbf{x})$ with respect to λ_i partially, we get

$$\frac{\partial A(\lambda,\psi_1,x)}{\partial\lambda_i}=xn_i^*\tilde{r}_i(x\lambda_i)\frac{\psi_1\left[\phi_1\left[F_i\left(x\lambda_i\right)\right]\right]}{\psi_1'\left[\phi_1\left[F_i\left(x\lambda_i\right)\right]\right]}\psi_1'\left[\sum_{m=1}^{n^*}\phi_1(F_m(x\lambda_m))\right],\tag{3}$$

where $i=1,2$. From (3), it is not difficult to check that $\frac{\partial A(\lambda,\psi_1,x)}{\partial\lambda_i}\geq0$ for $i=1,2$. Thus, $A(\lambda,\psi_1,x)$ is increasing in λ_i, for $i=1,2$. To establish Schur-concavity of $A(\lambda,\psi_1,x)$, in view of Lemma 2 (Lemma 1), we only need to show that for $1\leq i\leq j\leq n^*$, the following inequality holds:

$$\frac{\partial A(\lambda,\psi_1,x)}{\partial\lambda_i}-\frac{\partial A(\lambda,\psi_1,x)}{\partial\lambda_j}\geq(\leq)0,\text{ for }\lambda\in\mathscr{E}_+\ (\mathscr{D}_+).\tag{4}$$

Next, consider three cases.
Case I: For $1\leq i\leq j\leq n_1^*$, $\lambda_i=\lambda_j=\lambda_1$. In this case, $\frac{\partial A(\lambda,\psi_1,x)}{\partial\lambda_i}-\frac{\partial A(\lambda,\psi_1,x)}{\partial\lambda_j}=0$.

Case II: For $n_1^* + 1 \leq i \leq j \leq n^*$, $\lambda_i = \lambda_j = \lambda_2$. Here, $\frac{\partial A(\lambda, \psi_1, x)}{\partial \lambda_i} - \frac{\partial A(\lambda, \psi_1, x)}{\partial \lambda_j} = 0$.

Case III: For $1 \leq i \leq n_1^*$ and $n_1^* + 1 \leq j \leq n^*$, $\lambda_i = \lambda_1$ and $\lambda_j = \lambda_2$. For this case, consider $\lambda_1 \leq (\geq)\lambda_2$, which implies $\phi_1(F_1(x\lambda_1)) \geq (\leq)\phi_1(F_1(x\lambda_2))$. Furthermore, under the given assumption, we get $\phi_1(F_1(x\lambda_2)) \geq (\leq)\phi_1(F_2(x\lambda_2))$. Hence, $\phi_1(F_1(x\lambda_1)) \geq (\leq)\phi_1(F_2(x\lambda_2))$. Again, ψ_1 is log-convex. Therefore, we have

$$-\frac{\psi_1(w)}{\psi_1'(w)}\bigg|_{w=\phi_1[F_1(x\lambda_1)]} \geq (\leq) -\frac{\psi_1(w)}{\psi_1'(w)}\bigg|_{w=\phi_1[F_2(x\lambda_2)]}. \tag{5}$$

Moreover, $\tilde{r}_1(w)$ is decreasing in $w > 0$; hence

$$\tilde{r}_1(x\lambda_1) \geq (\leq)\tilde{r}_1(x\lambda_2). \tag{6}$$

Also, $\tilde{r}_1(x) \geq (\leq)\tilde{r}_2(x)$ gives

$$\tilde{r}_1(x\lambda_2) \geq (\leq)\tilde{r}_2(x\lambda_2). \tag{7}$$

Equations (6), (7) and $n_1^* \geq (\leq)n_2^*$, together imply

$$n_1^*\tilde{r}_1(x\lambda_1) \geq (\leq)n_2^*\tilde{r}_2(x\lambda_2). \tag{8}$$

Finally, combining (5) and (8), we obtain (4).   □

The following corollary, which is a direct consequence of Theorem 1 presents some special cases.

Corollary 1. *In addition to Assumption 1, let $\psi_1 = \psi_2 = \psi$, let $n_1^* \geq (\leq)n_2^*$, and let ψ be log-convex. Furthermore, let λ, $\mu \in \mathcal{E}_+ (\mathcal{D}_+)$. Then,*

(i) $(\underbrace{\lambda_1,\ldots,\lambda_1}_{n_1^*},\underbrace{\lambda_2,\ldots,\lambda_2}_{n_2^*}) \succeq^w (\underbrace{\mu_1,\ldots,\mu_1}_{n_1^*},\underbrace{\mu_2,\ldots,\mu_2}_{n_2^*}) \Rightarrow Y_{n^*:n^*}(n_1^*,n_2^*) \leq_{st} X_{n^*:n^*}(n_1^*,n_2^*)$, provided $\tilde{r}_1(x)$
    or $\tilde{r}_2(x)$ is decreasing and $\tilde{r}_1(x) \geq (\leq)\tilde{r}_2(x)$;

(ii) for $\tilde{r}_1 = \tilde{r}_2 = \tilde{r}$, we have

$$(\underbrace{\lambda_1,\ldots,\lambda_1}_{n_1^*},\underbrace{\lambda_2,\ldots,\lambda_2}_{n_2^*}) \succeq^w (\underbrace{\mu_1,\ldots,\mu_1}_{n_1^*},\underbrace{\mu_2,\ldots,\mu_2}_{n_2^*}) \Rightarrow Y_{n^*:n^*}(n_1^*,n_2^*) \leq_{st} X_{n^*:n^*}(n_1^*,n_2^*),$$

provided $\tilde{r}(x)$ is decreasing.

The next theorem states that the ordering result holds between the largest order statistics $X_{n:n}(n_1,n_2)$ and $X_{n^*:n^*}(n_1^*,n_2^*)$ according to the usual stochastic ordering. Here, the samples are collected from multiple-outlier dependent scale models. Also, it is assumed that the samples are sharing Archimedean copula with a common generator.

Assumption 2. *Let $X_1,\ldots,X_{n^*}$ be n^* dependent nonnegative random variables sharing Archimedean (survival) copula with generator ψ_1, such that $X_i \sim F_1(x\lambda_1)$, for $i = 1,\ldots,n_1^*$ and $X_j \sim F_2(x\lambda_2)$, for $j = n_1^*+1,\ldots,n^*$. We assume that there exist two natural numbers n_1 and n_2 such that $1 \leq n_1 \leq n_1^* \leq n_2^* \leq n_2$. Also, $n = n_1 + n_2$, $n^* = n_1^* + n_2^*$ and $\psi_1 = \phi_1^{-1}$.*

Theorem 2. *Let Assumption 2 hold with $F_1 \geq F_2$. Then, for $\lambda \in \mathscr{D}_+$, we have*

$$(n_1, n_2) \succeq_w (n_1^*, n_2^*) \Rightarrow X_{n^*:n^*}(n_1^*, n_2^*) \leq_{st} X_{n:n}(n_1, n_2).$$

Proof. The distribution functions of $X_{n:n}(n_1, n_2)$ and $X_{n^*:n^*}(n_1^*, n_2^*)$ can be, respectively, written as

$$F_{X_{n:n}(n_1,n_2)}(x) = \psi_1 \left(\sum_{i=1}^{n} \phi_1 \left(F_i (x\lambda_i) \right) \right) = \psi_1 \left[n_1 \phi_1 \left(F_1 (x\lambda_1) \right) + n_2 \phi_1 \left(F_2 (x\lambda_2) \right) \right]$$

and

$$F_{X_{n^*:n^*}(n_1^*,n_2^*)}(x) = \psi_1 \left(\sum_{i=1}^{n^*} \phi_1 \left(F_i (x\lambda_i) \right) \right) = \psi_1 \left[n_1^* \phi_1 \left(F_1 (x\lambda_1) \right) + n_2^* \phi_1 \left(F_2 (x\lambda_2) \right) \right].$$

To obtain the desired result, one needs to show $F_{X_{n:n}(n_1,n_2)}(x) \leq F_{X_{n^*:n^*}(n_1^*,n_2^*)}(x)$, which is equivalent to establish

$$\psi_1 \left(\sum_{i=1}^{n} \phi_1 \left(F_i (x\lambda_i) \right) \right) \leq \psi_1 \left(\sum_{i=1}^{n^*} \phi_1 \left(F_i (x\lambda_i) \right) \right)$$

$$\Rightarrow n_1 \phi_1 \left(F_1 (x\lambda_1) \right) + n_2 \phi_1 \left(F_2 (x\lambda_2) \right) \geq n_1^* \phi_1 \left(F_1 (x\lambda_1) \right) + n_2^* \phi_1 \left(F_2 (x\lambda_2) \right)$$

$$\Rightarrow (n_1^* - n_1) \phi_1 \left(F_1 (x\lambda_1) \right) \leq (n_2 - n_2^*) \phi_1 \left(F_2 (x\lambda_2) \right). \tag{9}$$

Now, $(n_1, n_2) \succeq_w (n_1^*, n_2^*) \Rightarrow (n_1 + n_2) \geq (n_1^* + n_2^*) \Rightarrow (n_2 - n_2^*) \geq (n_1^* - n_1) \geq 0$. Also, $\lambda_1 \geq \lambda_2 \Rightarrow \phi_1 \left(F_2 (x\lambda_2) \right) \geq \phi_1 \left(F_1 (x\lambda_1) \right) \geq 0$. Combining above two inequalities, we have the inequality given in (9). $\square$

Next, we observe that two largest order statistics $X_{n:n}(n_1, n_2)$ and $Y_{n^*:n^*}(n_1^*, n_2^*)$ are comparable with respect to the usual stochastic order under the presence of Archimedean copula. It is worth mentioning that the order statistics are constructed from two multiple-outlier dependent samples having sample sizes n and n^*. The pairs of the sizes of both the outliers (n_1, n_2) and (n_1^*, n_2^*) are assumed to be connected according to the weakly submajorization order and p-larger order. The following assumption is useful for the next theorem.

Assumption 3. *Let $X_1, \ldots, X_n$ be n nonnegative dependent random variables sharing Archimedean (survival) copula with generator ψ_1, such that $X_i \sim F_1(x\lambda_1)$, for $i = 1, \ldots, n_1$ and $X_j \sim F_2(x\lambda_2)$, for $j = n_1 + 1, \ldots, n$. Also, let $Y_1, \ldots, Y_{n^*}$ be n^* dependent nonnegative random variables sharing Archimedean copula with generator ψ_2, such that $Y_i \sim F_1(x\mu_1)$, for $i = 1, \ldots, n_1^*$ and $Y_j \sim F_2(x\mu_2)$, for $j = n_1^* + 1, \ldots, n^*$. Here, $1 \leq n_1 \leq n_1^* \leq n_2^* \leq n_2$, $n = n_1 + n_2$ and $n^* = n_1^* + n_2^*$.*

The following result states as remark immediately follows from Theorems 1 and 2.

Remark 1. *Assume that Assumption 3 holds with $\tilde{r}_1(x) \leq \tilde{r}_2(x)$. Let $(n_1, n_2) \succeq_w (n_1^*, n_2^*)$, let $\lambda, \mu \in \mathscr{D}_+$, let $\phi_2 \circ \psi_1$ be super-additive, and let ψ_1 or ψ_2 be log-convex.*

(i) $(\underbrace{\lambda_1, \ldots, \lambda_1}_{n_1^*}, \underbrace{\lambda_2, \ldots, \lambda_2}_{n_2^*}) \succeq^w (\underbrace{\mu_1, \ldots, \mu_1}_{n_1^*}, \underbrace{\mu_2, \ldots, \mu_2}_{n_2^*}) \Rightarrow Y_{n^*:n^*}(n_1^*, n_2^*) \leq_{st} X_{n:n}(n_1, n_2),$ *provided $\tilde{r}_1(x)$ or $\tilde{r}_2(x)$ is decreasing;*

(ii) $(\underbrace{\lambda_1,\ldots,\lambda_1}_{n_1^*},\underbrace{\lambda_2,\ldots,\lambda_2}_{n_2^*}) \succeq^p (\underbrace{\mu_1,\ldots,\mu_1}_{n_1^*},\underbrace{\mu_2,\ldots,\mu_2}_{n_2^*}) \Rightarrow Y_{n^*:n^*}(n_1^*,n_2^*) \leq_{st} X_{n:n}(n_1,n_2), \text{ provided } x\tilde{r}_1(x)$
 or $x\tilde{r}_2(x)$ is decreasing.

The condition stated in Remark 1 "$\phi_2 \circ \psi_1$ is super-additive" demonstrates that the copula having generator ψ_2 is more positively dependent than that having generator ψ_1. Therefore, from Remark 1(*i*), one can conclude that in the presence of the Archimedean copula for more positively dependent components having multiple-outlier dependent scale models, less heterogeneous scale parameter vector (with respect to the weakly supermajorized order) leads to less reliable parallel system in the sense of the stochastically smaller lifetime.

Remark 2. *From Table 1, one can see that in literature, there are several Archimedean copulas whose generators are satisfied all the conditions of Theorems 1 and 2 and Remark 1. For example, we can consider Independence copula, Clayton copula, Ali-Mikhail-Haq copula, and Gumbel copula.*

We now present a numerical example, which provides an illustration of the result in Remark 1.

Example 1. Set $\lambda = (\lambda_1,\lambda_2) = (5,2)$, $\mu = (\mu_1,\mu_2) = (6,3)$, $(n_1,n_2) = (1,11)$, $(n_1^*,n_2^*) = (5,6)$, $\psi_1(x) = e^{-x^{\frac{1}{9}}}$, $\psi_2(x) = e^{-x^{\frac{1}{10}}}$, $x > 0$. Consider the baseline distribution functions as $F_2(x) = 1 - e^{1-(1+x^2)^{\frac{1}{5}}}$ and $F_1(x) = 1 - e^{-x}$, $x > 0$. Here, both the reversed hazard rate functions $\tilde{r}_1$ and $\tilde{r}_2$ are decreasing and satisfy $\tilde{r}_1(x) \leq \tilde{r}_2(x)$, for $x > 0$. Furthermore, ψ_1 and ψ_2 are log-convex and $\phi_2 \circ \psi_1$ is super-additive (see Table 1). Thus, all the conditions of Remark 1 are satisfied. Now, we plot the graphs of $F_{X_{12:12}(1,11)}(x)$ and $F_{Y_{11:11}(5,6)}(x)$ in Figure 1, which shows that $Y_{11:11}(5,6) \leq_{st} X_{12:12}(1,11)$ holds.
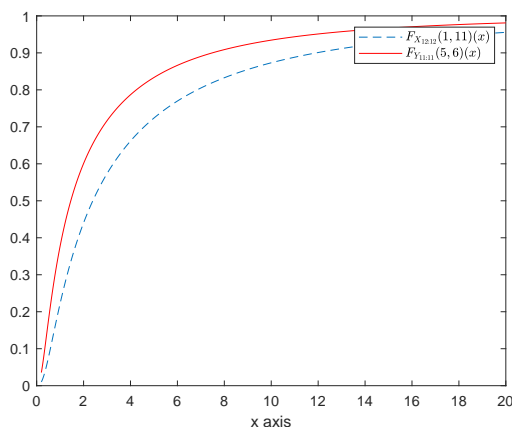

Figure 1: Plots of the distribution functions $F_{X_{12:12}(1,11)}(x)$ and $F_{Y_{11:11}(5,6)}(x)$ as in Example 1.

Next, we present a counterexample to illustrate that the result does not hold if $\tilde{r}_1(x) \geq \tilde{r}_2(x)$ and $\lambda \in \mathscr{E}_+$ in Remark 1.

Counterexample 1. Consider $\lambda = (\lambda_1, \lambda_2) = (2,6)$, $\mu = (\mu_1, \mu_2) = (8,2)$, $(n_1, n_2) = (1,8)$, $(n_1^*, n_2^*) = (3,4)$, $\psi_1(x) = e^{-x^{\frac{1}{3}}}$, $\psi_2(x) = e^{-x^{\frac{1}{10}}}$, $x > 0$. Baseline distribution functions are taken as $F_1(x) = 1 - e^{-x}$ and $F_2(x) = 1 - (1 + 2x)^{-0.5}$, $x > 0$. It can be seen that all the conditions of Remark 1 are satisfied except $\lambda \in \mathscr{D}_+$ and $\tilde{r}_1(x) \leq \tilde{r}_2(x)$. Now, we plot the graph of $F_{X_{9:9}(1,8)}(x) - F_{Y_{7:7}(3,4)}(x)$ in Figure 2, which reveals that $Y_{7:7}(3,4) \not\leq_{st} X_{9:9}(1,8)$.
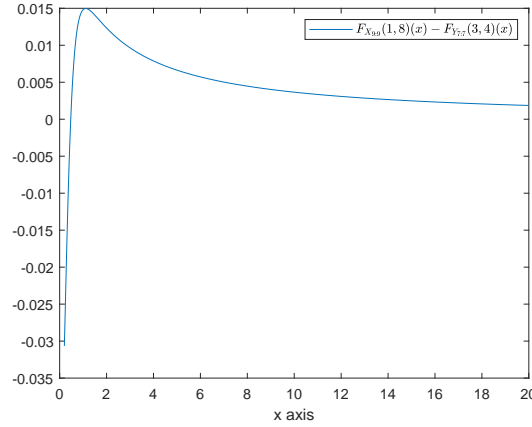
Figure 2: Plot of $F_{X_{9:9}(1,8)}(x) - F_{Y_{7:7}(3,4)}(x)$ as in Counterexample 1

In the preceding results, we have derived sufficient conditions, under which the largest order statistics from multiple-outlier dependent scale models obey the usual stochastic order. However, naturally, it is of interest to extend the ordering results to some other stronger concepts of the stochastic orders. In this part of the subsection, we establish sufficient conditions, under which the reversed hazard rate order holds between the largest order statistics. The following theorem shows that the largest order statistics $X_{n^*:n^*}(n_1^*, n_2^*)$ and $Y_{n^*:n^*}(n_1^*, n_2^*)$ have the reversed hazard rate ordering when the scale parameters are associated with the weakly supermajorization order. The samples are heterogeneous and follow multiple-outlier dependent scale models.

Theorem 3. *Let Assumption 1 hold with $r_1 = r_2 = r$, $n_1^* \geq (\leq) n_2^*$ and $\psi_1 = \psi_2 = \psi$. Also, suppose that ψ is log-concave, that $\frac{1-\psi}{\psi'}$ is decreasing, and that $\frac{1-\psi}{\psi'}[\frac{1-\psi}{\psi'}]'$ is increasing. Then,*

$$\underbrace{(\lambda_1, \ldots, \lambda_1}_{n_1^*}, \underbrace{\lambda_2, \ldots, \lambda_2)}_{n_2^*} \succeq^w \underbrace{(\mu_1, \ldots, \mu_1}_{n_1^*}, \underbrace{\mu_2, \ldots, \mu_2)}_{n_2^*} \Rightarrow Y_{n^*:n^*}(n_1^*, n_2^*) \leq_{rh} X_{n^*:n^*}(n_1^*, n_2^*),$$

provided λ, $\mu \in \mathscr{E}_+ (\mathscr{D}_+)$, $r(x)$ is decreasing and $xr(x)$ is decreasing and convex.

Proof. Under the given assumption, $r_1 = r_2 = r$ implies $F_1 = F_2 = F$. The reversed hazard rate

function of $X_{n^*:n^*}(n_1^*, n_2^*)$ is

$$
\begin{aligned}
\tilde{r}_{X_{n^*:n^*}(n_1^*,n_2^*)}(x) &= \frac{\psi'\left[\sum_{i=1}^{n^*}\phi\left(F\left(x\lambda_i\right)\right)\right]}{\psi\left[\sum_{i=1}^{n^*}\phi\left(F\left(x\lambda_i\right)\right)\right]}\left[\sum_{i=1}^{n^*}\frac{n_1^*\lambda_i f(x\lambda_i)}{\psi'[\phi\left(F\left(x\lambda_i\right)\right)]}\right] \\
&= \frac{\psi'\left[\sum_{i=1}^{n^*}\phi\left(F\left(x\lambda_i\right)\right)\right]}{\psi\left[\sum_{i=1}^{n^*}\phi\left(F\left(x\lambda_i\right)\right)\right]}\left[\sum_{i=1}^{n^*}\frac{n_i^*\lambda_i r(x\lambda_i)[1-\psi[\phi\left(F\left(x\lambda_i\right)\right)]]}{\psi'[\phi\left(F\left(x\lambda_i\right)\right)]}\right],
\end{aligned}
$$

where f is the probability density function corresponding to F. Denote $z = n_1^*\phi\left(F\left(x\lambda_1\right)\right) + n_2^*\phi\left(F\left(x\lambda_2\right)\right)$. The partial derivative of $\tilde{r}_{X_{n^*:n^*}(n_1^*,n_2^*)}(x)$ with respect to λ_i is obtained as

$$
\begin{aligned}
\frac{\partial[\tilde{r}_{X_{n^*:n^*}(n_1^*,n_2^*)}(x)]}{\partial\lambda_i} &= n_i^* x r(x\lambda_i)\frac{d}{dz}\left[\frac{\psi'(z)}{\psi(z)}\right]\left[\frac{1-\psi[\phi\left[F\left(x\lambda_i\right)\right]]}{\psi'[\phi\left(F\left(x\lambda_i\right)\right)]}\right]\left[\sum_{i=1}^{n^*}\frac{\lambda_i f(x\lambda_i)}{\psi'[\phi\left(F\left(x\lambda_i\right)\right)]}\right] \\
&+ n_i^* x\lambda_i[r(x\lambda_i)]^2\frac{\psi'(z)}{\psi(z)}\left[\frac{1-\psi(v)}{\psi'(v)}\frac{d}{dv}\left[\frac{1-\psi(v)}{\psi'(v)}\right]\right]_{v=\phi(F(x\lambda_i))} \\
&+ n_i^*\frac{d}{dw}[wr(w)]_{w=x\lambda_i}\left[\frac{1-\psi[\phi\left[F\left(x\lambda_i\right)\right]]}{\psi'[\phi\left[F\left(x\lambda_i\right)\right]]}\right]\frac{\psi'(z)}{\psi(z)},
\end{aligned} \tag{10}
$$

for $i = 1, 2$. Utilizing (Marshall et al., 2011, Theorem $A.8$), to obtain the desired result, we need to prove that $\tilde{r}_{X_{n^*:n^*}(n_1^*,n_2^*)}(x)$ is decreasing and Schur-convex with respect to λ. Using the given assumptions and (10), the decreasing property of $\tilde{r}_{X_{n^*:n^*}(n_1^*,n_2^*)}(x)$ with respect to λ is obvious. Furthermore, according to Lemma 2 (Lemma 1), to show the Schur-convexity of $\tilde{r}_{X_{n^*:n^*}(n_1^*,n_2^*)}(x)$, we have to establish for $1 \le i \le j \le n^*$,

$$
\left[\frac{\partial[\tilde{r}_{X_{n^*:n^*}}(n_1^*,n_2^*)(x)]}{\partial\lambda_i} - \frac{\partial[\tilde{r}_{X_{n^*:n^*}}(n_1^*,n_2^*)(x)]}{\partial\lambda_j}\right] \le (\ge)0, \text{ for } \lambda \in \mathscr{E}_+ \ (\mathscr{D}_+). \tag{11}
$$

Now, consider the following three cases.

Case I: For $1 \le i \le j \le n_1^*$, $\lambda_i = \lambda_j = \lambda_1$. Here, $\frac{\partial[\tilde{r}_{X_{n^*:n^*}(n_1^*,n_2^*)}(x)]}{\partial\lambda_i} - \frac{\partial[\tilde{r}_{X_{n^*:n^*}(n_1^*,n_2^*)}(x)]}{\partial\lambda_j} = 0$.

Case II: For $n_1^* + 1 \le i \le j \le n^*$, $\lambda_i = \lambda_j = \lambda_2$. Hence, $\frac{\partial[\tilde{r}_{X_{n^*:n^*}(n_1^*,n_2^*)}(x)]}{\partial\lambda_i} - \frac{\partial[\tilde{r}_{X_{n^*:n^*}(n_1^*,n_2^*)}(x)]}{\partial\lambda_j} = 0$.

Case III: For $1 \le i \le n_1^*$ and $n_1^* + 1 \le j \le n^*$, $\lambda_i = \lambda_1$ and $\lambda_j = \lambda_2$. Consider $\lambda_1 \le \lambda_2$, which gives $\phi(F(x\lambda_1)) \ge \phi(F(x\lambda_2))$. Here, we only discuss the proof when $\lambda_1 \le \lambda_2$. The other case when $\lambda_1 \ge \lambda_2$ can be proved in the similar way. The concavity property of $\ln\psi$ provides $\frac{d}{dz}\left[\frac{\psi'(z)}{\psi(z)}\right] \le 0$. Again, using decreasing property of $\frac{1-\psi}{\psi'}$, we have

$$
\left.\frac{1-\psi(w)}{\psi'(w)}\right|_{w=\phi[F(x\lambda_1)]} \le \left.\frac{1-\psi(w)}{\psi'(w)}\right|_{w=\phi[F(x\lambda_2)]} \le 0. \tag{12}
$$

Furthermore, it has been assumed that $r(x)$ is decreasing and that $xr(x)$ is decreasing and convex. Therefore, using $n_1^* \ge n_2^*$, we have

$$
r(x\lambda_1) \ge r(x\lambda_2), \tag{13}
$$

$$
n_1^* x\lambda_1 r(x\lambda_1) \ge n_2^* x\lambda_2 r(x\lambda_2), \qquad \text{and} \tag{14}
$$

$$n_1^* \frac{d}{dw}[wr(w)]_{w=x\lambda_1} \le n_2^* \frac{d}{dw}[wr(w)]_{w=x\lambda_2} \le 0. \tag{15}$$

Moreover, $\frac{1-\psi(w)}{\psi'(w)} \frac{d}{dw}[\frac{1-\psi(w)}{\psi'(w)}]$ is increasing. Therefore, we obtain the following inequality:

$$\left[\frac{1-\psi(w)}{\psi'(w)} \frac{d}{dw}\left[\frac{1-\psi(w)}{\psi'(w)}\right]\right]_{w=\phi[F(x\lambda_1)]} \ge \left[\frac{1-\psi(w)}{\psi'(w)} \frac{d}{dw}\left[\frac{1-\psi(w)}{\psi'(w)}\right]\right]_{w=\phi[F(x\lambda_2)]} \ge 0. \tag{16}$$

Now, combining (12)–(16) and the given assumptions, we obtain that the inequality given by (11) holds. $\qquad\square$

In the next theorem, we show that the largest order statistics $X_{n:n}(n_1, n_2)$ and $X_{n^*:n^*}(n_1^*, n_2^*)$ are comparable according to the reversed hazard rate order.

Theorem 4. *Let Assumption 2 hold with $\psi_1 = \psi$ and $r_1 = r_2 = r$. Then, for $\lambda \in \mathscr{D}_+$, we have*

$$(n_1, n_2) \succeq_w (n_1^*, n_2^*) \Rightarrow X_{n^*:n^*}(n_1^*, n_2^*) \le_{rh} X_{n:n}(n_1, n_2),$$

provided $\ln \psi$ is concave and $\frac{1-\psi}{\psi'}$ and $xr(x)$ are decreasing.

Proof. The stated result will be proved, if we show that $\tilde{r}_{X_{n:n}(n_1,n_2)}(x) \ge \tilde{r}_{X_{n^*:n^*}(n_1^*,n_2^*)}(x)$. Equivalently,

$$\frac{\psi'\left[\sum_{i=1}^{n} \phi\left(F\left(x\lambda_i\right)\right)\right]}{\psi\left[\sum_{i=1}^{n} \phi\left(F\left(x\lambda_i\right)\right)\right]} \times \frac{\psi\left[\sum_{i=1}^{n^*} \phi\left(F\left(x\lambda_i\right)\right)\right]}{\psi'\left[\sum_{i=1}^{n^*} \phi\left(F\left(x\lambda_i\right)\right)\right]} \ge \frac{\sum_{i=1}^{n^*} \frac{\lambda_i r(x\lambda_i)[1-\psi[\phi\left(F\left(x\lambda_i\right)\right)]]}{\psi'[\phi\left(F\left(x\lambda_i\right)\right)]}}{\sum_{i=1}^{n} \frac{\lambda_i r(x\lambda_i)[1-\psi[\phi\left(F\left(x\lambda_i\right)\right)]]}{\psi'[\phi\left(F\left(x\lambda_i\right)\right)]}}. \tag{17}$$

The preceding inequality holds if the following two inequalities are satisfied:

$$\frac{\psi'\left[\sum_{i=1}^{n^*} \phi\left(F\left(x\lambda_i\right)\right)\right]}{\psi\left[\sum_{i=1}^{n^*} \phi\left(F\left(x\lambda_i\right)\right)\right]} \ge \frac{\psi'\left[\sum_{i=1}^{n} \phi\left(F\left(x\lambda_i\right)\right)\right]}{\psi\left[\sum_{i=1}^{n} \phi\left(F\left(x\lambda_i\right)\right)\right]} \Leftrightarrow (n_1^* - n_1)\phi(F(x\lambda_1)) \le (n_2 - n_2^*)\phi(F(x\lambda_2)) \tag{18}$$

and

$$\sum_{i=1}^{n^*} \frac{\lambda_i r(x\lambda_i)[1-\psi[\phi\left(F\left(x\lambda_i\right)\right)]]}{\psi'[\phi\left(F\left(x\lambda_i\right)\right)]} \ge \sum_{i=1}^{n} \frac{\lambda_i r(x\lambda_i)[1-\psi[\phi\left(F\left(x\lambda_i\right)\right)]]}{\psi'[\phi\left(F\left(x\lambda_i\right)\right)]}$$

$$\Leftrightarrow (n_1^* - n_1)\frac{\lambda_1 r(x\lambda_1)[1-\psi[\phi\left(F\left(x\lambda_1\right)\right)]]}{\psi'[\phi\left(F\left(x\lambda_1\right)\right)]} \ge (n_2 - n_2^*)\frac{\lambda_2 r(x\lambda_2)[1-\psi[\phi\left(F\left(x\lambda_2\right)\right)]]}{\psi'[\phi\left(F\left(x\lambda_2\right)\right)]}. \tag{19}$$

Furthermore,

$$(n_1, n_2) \succeq_w (n_1^*, n_2^*) \Rightarrow (n_1 + n_2) \ge (n_1^* + n_2^*) \Rightarrow (n_2 - n_2^*) \ge (n_1^* - n_1) \ge 0. \tag{20}$$

Also,

$$\lambda_1 \ge \lambda_2 \Rightarrow \phi\left(F\left(x\lambda_2\right)\right) \ge \phi\left(F\left(x\lambda_1\right)\right) \ge 0.$$

Moreover, $\frac{1-\psi}{\psi'}$ is decreasing. Thus,

$$\frac{1-\psi(w)}{\psi'(w)}\Big|_{w=\phi[F(x\lambda_2)]} \le \frac{1-\psi(w)}{\psi'(w)}\Big|_{w=\phi[F(x\lambda_1)]} \le 0. \tag{21}$$

Using the decreasing property of $xr(x)$, we have

$$x\lambda_1 r(x\lambda_1) \leq x\lambda_2 r(x\lambda_2). \tag{22}$$

Combining (20), (21), and (22), the inequality (19) can be obtained. Using (20) and the assumption that ψ is log-concave, we get the inequality (18). $\square$

Now, we are ready to state a result which shows that the largest order statistics $X_{n:n}(n_1, n_2)$ and $Y_{n^*:n^*}(n_1^*, n_2^*)$ can be compared with respect to the reversed hazard rate order. Here, n and n^* may be different.

Theorem 5. *Let the set-up in Assumption 3 hold with $\psi_1 = \psi_2 = \psi$ and $r_1 = r_2 = r$. Also, assume λ, $\mu \in \mathcal{D}_+$ and $(n_1, n_2) \succeq_w (n_1^*, n_2^*)$. Then,*

$$(\underbrace{\lambda_1, \ldots, \lambda_1}_{n_1^*}, \underbrace{\lambda_2, \ldots, \lambda_2}_{n_2^*}) \succeq^w (\underbrace{\mu_1, \ldots, \mu_1}_{n_1^*}, \underbrace{\mu_2, \ldots, \mu_2}_{n_2^*}) \Rightarrow Y_{n^*:n^*}(n_1^*, n_2^*) \leq_{rh} X_{n:n}(n_1, n_2),$$

provided ψ is log-concave, $\frac{1-\psi}{\psi'}$ is decreasing, $\frac{1-\psi}{\psi'}[\frac{1-\psi}{\psi'}]'$ is increasing, $xr(x)$ is decreasing and convex, and $r(x)$ is decreasing.

Proof. According to Theorem 3, we have

$$(\underbrace{\lambda_1, \ldots, \lambda_1}_{n_1^*}, \underbrace{\lambda_2, \ldots, \lambda_2}_{n_2^*}) \succeq^w (\underbrace{\mu_1, \ldots, \mu_1}_{n_1^*}, \underbrace{\mu_2, \ldots, \mu_2}_{n_2^*}) \Rightarrow Y_{n^*:n^*}(n_1^*, n_2^*) \leq_{rh} X_{n^*:n^*}(n_1^*, n_2^*). \tag{23}$$

Also, from Theorem 4, we get

$$(n_1, n_2) \succeq_w (n_1^*, n_2^*) \Rightarrow X_{n^*:n^*}(n_1^*, n_2^*) \leq_{rh} X_{n:n}(n_1, n_2). \tag{24}$$

Thus, the proof of the theorem follows after combining the inequalities given by (23) and (24). $\square$

Below, we consider an example to illustrate Theorem 5.

Example 2. Consider $\lambda = (3, 2)$, $\mu = (6, 5)$, $(n_1, n_2) = (2, 10)$, $(n_1^*, n_2^*) = (3, 4)$, $\psi(x) = e^{\frac{1}{d}(1-e^x)}$, $x > 0$. Also, let the baseline distribution be $F(x) = 1 - \left(\frac{x}{b}\right)^{-a}$, $x \geq b > 0$, $a > 0$. It is not hard to see that, for $d = 0.2$, $a = 5$ and $b = 1$, all the conditions of Theorem 5 are satisfied (see Table 1). Furthermore, we plot the graph of $F_{X_{12:12}(2,10)}(x)/F_{Y_{7:7}(3,4)}(x)$ in Figure 3. This shows that the result in Theorem 5 holds.

Remark 3. *Let us consider (i) Independence copula with generator $\psi(x) = e^{-x}$, $x > 0$ and (ii) Ali-Mikhail-Haq copula with generator $\psi(x) = \frac{1-\sigma}{e^x - \sigma}$, $\sigma \in [-1, 0)$, $x > 0$. One can easily check that the above two copulas satisfies all the conditions of Theorems 3– 5 (see Table 1).*

Now, we derive conditions such that the star order holds between $X_{n^*:n^*}(n_1^*, n_2^*)$ and $Y_{n^*:n^*}(n_1^*, n_2^*)$. Denote $\lambda_{2:2} = \max\{\lambda_1, \lambda_2\}$, $\lambda_{1:2} = \min\{\lambda_1, \lambda_2\}$, $\mu_{2:2} = \max\{\mu_1, \mu_2\}$, and $\mu_{1:2} = \min\{\mu_1, \mu_2\}$.
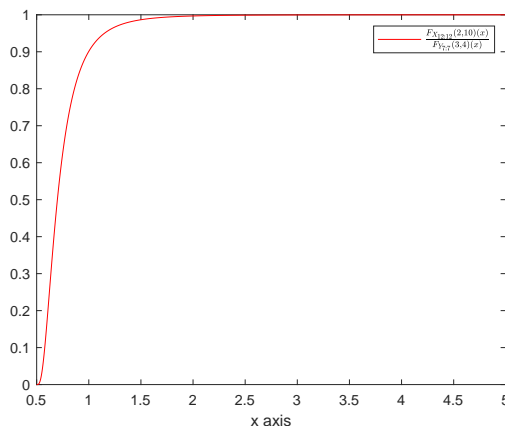
Figure 3: Plot of the ratio of two distribution functions $F_{X_{12:12}(2,10)}(x)/F_{Y_{7:7}(3,4)}(x)$ in Example 2.

Theorem 6. *Under the set-up as in Assumption 1, with $\tilde{r}_1 = \tilde{r}_2 = \tilde{r}$ and $\psi_1 = \psi_2 = \psi$, we have*

$$\frac{\lambda_{2:2}}{\lambda_{1:2}} \geq \frac{\mu_{2:2}}{\mu_{1:2}} \Rightarrow Y_{n^*:n^*}(n_1^*, n_2^*) \leq_* X_{n^*:n^*}(n_1^*, n_2^*),$$

provided $\frac{\psi}{\psi'}$ is decreasing and convex, $\frac{x\tilde{r}'(x)}{\tilde{r}(x)}$ is decreasing, and $x\tilde{r}(x)$ is increasing.

Proof. Under the assumption, $\tilde{r}_1 = \tilde{r}_2 = \tilde{r}$ gives $F_1 = F_2 = F$. The distribution functions of $X_{n^*:n^*}(n_1^*, n_2^*)$ and $Y_{n^*:n^*}(n_1^*, n_2^*)$ are, respectively, given by

$$F_{X_{n^*:n^*}(n_1^*, n_2^*)}(x) = \psi\left[n_1^*\phi\left(F\left(x\lambda_1\right)\right) + n_2^*\phi\left(F\left(x\lambda_2\right)\right)\right]$$

and

$$F_{Y_{n^*:n^*}(n_1^*, n_2^*)}(x) = \psi\left[n_1^*\phi\left(F\left(x\mu_1\right)\right) + n_2^*\phi\left(F\left(x\mu_2\right)\right)\right].$$

To obtain the required result, we consider two cases.
Case I: $\lambda_1 + \lambda_2 = \mu_1 + \mu_2$.

For convenience, we assume $\lambda_1 + \lambda_2 = \mu_1 + \mu_2 = 1$. For this case, it is clear that $(\lambda_1, \lambda_2) \succeq^m$ (μ_1, μ_2). Now, take $\lambda_2 = \lambda \geq \lambda_1$, $\mu_2 = \mu \geq \mu_1$. Hence, $\lambda_1 = 1 - \lambda$ and $\mu_1 = 1 - \mu$. Based on this, the distribution functions of $X_{n^*:n^*}(n_1^*, n_2^*)$ and $Y_{n^*:n^*}(n_1^*, n_2^*)$ can be written in the following form

$$F_{X_{n^*:n^*}(n_1^*, n_2^*)}(x) \stackrel{def}{=} F_\lambda(x) = \psi\left[n_1^*\phi\left(F\left(x(1-\lambda)\right)\right) + n_2^*\phi\left(F\left(x\lambda\right)\right)\right]$$

and

$$F_{Y_{n^*:n^*}(n_1^*, n_2^*)}(x) \stackrel{def}{=} F_\mu(x) = \psi\left[n_1^*\phi\left(F\left(x(1-\mu)\right)\right) + n_2^*\phi\left(F\left(x\mu\right)\right)\right].$$

Now, according to Lemma 3, we have to show that $\frac{F_\lambda'(x)}{xf_\lambda(x)}$ is decreasing in $x \in \mathbb{R}^+$, for $\lambda \in (1/2, 1]$. The derivative of F_λ, with respect to λ is given by

$$\begin{aligned} F_\lambda'(x) &= \left[-xn_1^*\tilde{r}(x(1-\lambda))\frac{\psi[\phi\left(F\left(x(1-\lambda)\right)\right)]}{\psi'[\phi\left(F\left(x(1-\lambda)\right)\right)]} + xn_2^*\tilde{r}(x\lambda)\frac{\psi[\phi\left(F\left(x\lambda\right)\right)]}{\psi'[\phi\left(F\left(x\lambda\right)\right)]}\right] \\ &\quad \times \psi'\left[n_1^*\phi\left(F\left(x(1-\lambda)\right)\right) + n_2^*\phi\left(F\left(x\lambda\right)\right)\right]. \end{aligned} \tag{25}$$

Also, the probability density function corresponding to F_λ is

$$f_\lambda(x) = \left[(1-\lambda)n_1^*\tilde{r}(x(1-\lambda))\frac{\psi[\phi(F(x(1-\lambda)))]}{\psi'[\phi(F(x(1-\lambda)))]} + \lambda n_2^*\tilde{r}(x\lambda)\frac{\psi[\phi(F(x\lambda))]}{\psi'[\phi(F(x\lambda))]}\right]$$
$$\times \psi'\left[n_1^*\phi(F(x(1-\lambda))) + n_2^*\phi(F(x\lambda))\right]. \tag{26}$$

Therefore,

$$\frac{F_\lambda'(x)}{xf_\lambda(x)} = \left(\lambda + \left[\frac{n_2^*\tilde{r}(x\lambda)\dfrac{\psi[\phi(F(x\lambda))]}{\psi'[\phi(F(x\lambda))]}}{n_1^*\tilde{r}(x(1-\lambda))\dfrac{\psi[\phi(F(x(1-\lambda)))]}{\psi'[\phi(F(x(1-\lambda)))]}} - 1\right]^{-1}\right)^{-1}.$$

Thus, it suffices to show that $L(x) = \left(\tilde{r}(x\lambda)\frac{\psi[\phi(F(x\lambda))]}{\psi'[\phi(F(x\lambda))]}\right) / \left(\tilde{r}(x(1-\lambda))\frac{\psi[\phi(F(x(1-\lambda)))]}{\psi'[\phi(F(x(1-\lambda)))]}\right)$ is decreasing in $x \in \mathbb{R}^+$, for $\lambda \in (1/2, 1]$. The derivative of $L(x)$ with respect to x is obtained as

$$L'(x) \overset{sign}{=} \frac{\lambda\tilde{r}'(x\lambda)}{\tilde{r}(x\lambda)} + \lambda\tilde{r}(x\lambda)\left[\frac{\psi[\phi(F(x\lambda))]}{\psi'[\phi(F(x\lambda))]}\right]' - \frac{(1-\lambda)\tilde{r}'(x(1-\lambda))}{\tilde{r}((1-\lambda)x)}$$
$$- (1-\lambda)\tilde{r}(x(1-\lambda))\left[\frac{\psi[\phi(F(x(1-\lambda)))]}{\psi'[\phi(F(x(1-\lambda)))]}\right]'.$$

Under the assumptions made, $\frac{x\tilde{r}'(x)}{\tilde{r}(x)}$ is decreasing and $x\tilde{r}(x)$ is increasing. Therefore, for $\lambda \in (1/2, 1]$,

$$\frac{x\lambda\tilde{r}'(x\lambda)}{\tilde{r}(x\lambda)} \leq \frac{x(1-\lambda)\tilde{r}'(x(1-\lambda))}{\tilde{r}(x(1-\lambda))} \leq 0 \text{ and } x\lambda\tilde{r}(x\lambda) \geq x(1-\lambda)\tilde{r}(x(1-\lambda)) \geq 0. \tag{27}$$

Also, since $\frac{\psi}{\psi'}$ is decreasing and convex, we have

$$\left[\frac{\psi[\phi(F(x\lambda))]}{\psi'[\phi(F(x\lambda))]}\right]' \leq \left[\frac{\psi[\phi(F(x(1-\lambda)))]}{\psi'[\phi(F(x(1-\lambda)))]}\right]' \leq 0. \tag{28}$$

Now, combining (27) and (28), we get $L'(x) \leq 0$, for $x \in \mathbb{R}^+$.

Case II. $\lambda_1 + \lambda_2 \neq \mu_1 + \mu_2$.

In this case, we can take $\lambda_1 + \lambda_2 = k(\mu_1 + \mu_2)$, where k is a scalar. Hence, $(k\mu_1, k\mu_2) \preceq^m (\lambda_1, \lambda_2)$. Let us consider n^* dependent nonnegative random variables sharing Archimedean copula with generator ψ, such that $Z_i \sim F(k\mu_1 x)$, for $i = 1, \ldots, n_1^*$ and $Z_j \sim F(k\mu_2 x)$, for $j = n_1^* + 1, \ldots, n^*$. Here, $n_1^* + n_2^* = n^*$. Then, from Case I, we have $Z_{n^*:n^*}(n_1^*, n_2^*) \leq_* X_{n^*:n^*}(n_1^*, n_2^*)$. Furthermore, star order is scale invariant, and hence we obtain $Y_{n^*:n^*}(n_1^*, n_2^*) \leq_* X_{n^*:n^*}(n_1^*, n_2^*)$. $\qquad\square$

Remark 4. *Let us consider* $\psi(x) = e^{-x}$, $x > 0$ *in Theorem* 6. *Then, Theorem* 6 *reduces to (Zhang et al., 2019, Theorem* 3.14*) when* $\alpha_1 = \alpha_2 = 1$.

Using the fact that the star order implies the Lorenz order, the following result is a direct consequence of Theorem 6. Furthermore, since the Lorenz order is mainly used to compare the income distributions, the following corollary is more interesting from the point of its applications in the study of incomes.

Corollary 2. *Under the set-up as in Theorem 6,*

$$\frac{\lambda_{2:2}}{\lambda_{1:2}} \geq \frac{\mu_{2:2}}{\mu_{1:2}} \Rightarrow Y_{n^*:n^*}(n_1^*, n_2^*) \leq_{Lorenz} X_{n^*:n^*}(n_1^*, n_2^*),$$

provided $\frac{\psi}{\psi'}$ is decreasing and convex, $\frac{x\tilde{r}'(x)}{\tilde{r}(x)}$ is decreasing, and $x\tilde{r}(x)$ is increasing.

In the following theorem, an interesting result has been developed to compare $Y_{n^*:n^*}(n_1^*, n_2^*)$ and $X_{n^*:n^*}(n_1^*, n_2^*)$ according to the dispersive order.

Theorem 7. *Under the same assumptions of Theorem 6, let us consider $\frac{\psi}{\psi'}$ be decreasing and convex, $\tilde{r}(x)$, $\frac{x\tilde{r}'(x)}{\tilde{r}(x)}$ be decreasing, and $x\tilde{r}(x)$ be increasing. Also, let $\lambda_2 \leq \mu_2 \leq \mu_1 \leq \lambda_1$. Then,*

$$(\underbrace{\lambda_1, \ldots, \lambda_1}_{n_1^*}, \underbrace{\lambda_2, \ldots, \lambda_2}_{n_2^*}) \succeq^w (\underbrace{\mu_1, \ldots, \mu_1}_{n_1^*}, \underbrace{\mu_2, \ldots, \mu_2}_{n_2^*}) \Rightarrow Y_{n^*:n^*}(n_1^*, n_2^*) \leq_{disp} X_{n^*:n^*}(n_1^*, n_2^*).$$

Proof. Ahmed et al. (1986) established that, for two continuous random variables X and Y, if $X \leq_* Y$ holds, then $X \leq_{st} Y \Rightarrow X \leq_{disp} Y$. Therefore, from Corollary 1(*ii*), we have $Y_{n^*:n^*}(n^*, n^*) \leq_{st} X_{n^*:n^*}(n^*, n^*)$. Hence, considering Theorem 6, we can conclude that the required result holds. □

Remark 5. *It is notable that, in Theorem 7, if we take $\psi(x) = e^{-x}$, $x > 0$, then Theorem 7 reduces to (Zhang et al., 2019, Theorem 3.15(i)) when $\alpha_1 = \alpha_2 = 1$.*

Remark 6. *Let $F(x) = \left(\frac{x}{a}\right)^l$, $0 < x \leq a$. For this baseline distribution, it can be easily checked that $\tilde{r}(x)$, $\frac{x\tilde{r}'(x)}{\tilde{r}(x)}$ are decreasing and $x\tilde{r}(x)$ is increasing. Therefore, we can consider the power distribution as the baseline distribution function in Theorems 6 and 7.*

The following theorem provides some conditions for comparing two largest order statistics $X_{n:n}(n_1, n_2)$ and $Y_{n^*:n^*}(n_1^*, n_2^*)$ according to the likelihood ratio order. This can be proved by the similar approach of Theorem 3.12 of Torrado (2017). Here, we consider $\psi(x) = e^{-x}$, $x > 0$, as the generator of the Independence copula. Also, denote $u(x) = x\tilde{r}(x)$, $\eta(x) = -\frac{xf'(x)}{f(x)}$ and $v(x) = \frac{x\tilde{r}'(x)}{\tilde{r}(x)}$.

Theorem 8. *Let $X_1, \ldots, X_n$ be n nonnegative dependent random variables sharing independence copula with generator ψ, such that $X_i \sim F(x\lambda_1)$, for $i = 1, \ldots, n_1^*$ and $X_j \sim F(x\lambda_2)$, for $j = n_1^* + 1, \ldots, n^*$. Also, let $Y_1, \ldots, Y_n$ be n dependent nonnegative random variables sharing Independence copula with generator ψ, such that $Y_i \sim F(x\mu_1)$, for $i = 1, \ldots, n_1$ and $Y_j \sim F(x\mu_2)$, for $j = n_1 + 1, \ldots, n$. Assume that $v(x)$ is decreasing and that $\eta(x)$, $x^2\tilde{r}(x)$, and $u(x)v'(x)$ are increasing. Suppose $\mu_1 \leq \lambda_1 \leq \lambda_2 \leq \mu_2$ and $1 \leq n_1 \leq n_1^* \leq n_2^* \leq n_2$, $n = n_1 + n_2$, $n^* = n_1^* + n_2^*$. Then, $(n_1, n_2) \succeq^w (n_1^*, n_2^*)$ and*

$$(\underbrace{\lambda_1, \ldots, \lambda_1}_{n_1^*}, \underbrace{\lambda_2, \ldots, \lambda_2}_{n_2^*}) \succeq^w (\underbrace{\mu_1, \ldots, \mu_1}_{n_1^*}, \underbrace{\mu_2, \ldots, \mu_2}_{n_2^*}) \Rightarrow X_{n:n}(n_1, n_2) \leq_{lr} Y_{n^*:n^*}(n_1^*, n_2^*).$$

3.2 Orderings between the smallest order statistics

In the previous subsection, we focus on the conditions, under which the largest order statistics are comparable according to various stochastic orders. Here, we develop conditions such that the

usual stochastic, hazard rate, star, and Lorenz orders hold between the smallest order statistics. The reliability functions of $X_{1:n^*}(n_1^*, n_2^*)$ and $Y_{1:n^*}(n_1^*, n_2^*)$ are, respectively, given by

$$\bar{F}_{X_{1:n^*}(n_1^*,n_2^*)}(x) = \psi_1 \left[\sum_{i=1}^{n^*} \phi_1 \left(\bar{F}_i (x\lambda_i) \right) \right] = \psi_1 \left[n_1^* \phi_1 \left(\bar{F}_1 (x\lambda_1) \right) + n_2^* \phi_1 \left(\bar{F}_2 (x\lambda_2) \right) \right] \tag{29}$$

and

$$\bar{F}_{Y_{1:n^*}(n_1^*,n_2^*)}(x) = \psi_2 \left[\sum_{i=1}^{n^*} \phi_2 \left(\bar{F}_i (x\mu_i) \right) \right] = \psi_2 \left[n_1^* \phi_2 \left(\bar{F}_1 (x\mu_1) \right) + n_2^* \phi_2 \left(\bar{F}_2 (x\mu_2) \right) \right]. \tag{30}$$

Li et al. (2016) compared two series systems with same number of components in the sense of the usual stochastic order when the corresponding random variables follow the dependent scale model with same baseline distribution function.

Here, we consider multiple-outlier models with different baseline distribution functions and prove the usual stochastic order between $X_{1:n}(n_1, n_2)$ and $Y_{1:n^*}(n_1^*, n_2^*)$. This is presented in Theorem 7. To obtain this result, let us first prove the following two theorems. In the following theorems, it is assumed that the samples are heterogeneous and taken from the multiple-outlier dependent scale models with different baseline F_1 and F_2. We now consider the following assumption.

Theorem 9. *Under the set-up as in Assumption 1, with $r_1(x) \leq (\geq) r_2(x)$ and $n_1^* \leq (\geq) n_2^*$,*

$$(\underbrace{\lambda_1, \ldots, \lambda_1}_{n_1^*}, \underbrace{\lambda_2, \ldots, \lambda_2}_{n_2^*}) \succeq_w (\underbrace{\mu_1, \ldots, \mu_1}_{n_1^*}, \underbrace{\mu_2, \ldots, \mu_2}_{n_2^*}) \Rightarrow X_{1:n^*}(n_1^*, n_2^*) \leq_{st} Y_{1:n^*}(n_1^*, n_2^*),$$

provided $\lambda, \mu \in \mathscr{E}_+ (\mathscr{D}_+)$, $\phi_2 \circ \psi_1$ is super-additive, ψ_1 or ψ_2 is log-convex, and $r_1(x)$ or $r_2(x)$ is increasing.

Proof. Let us denote $C(\lambda, \psi_1, x) = \bar{F}_{X_{1:n^*}(n_1^*,n_2^*)}(x)$ and $D(\mu, \psi_2, x) = \bar{F}_{Y_{1:n^*}(n_1^*,n_2^*)}(x)$ (given in (29) and (30)). According to Lemma 4, the super-additivity property of $\phi_2 \circ \psi_1$ provides $C(\mu, \psi_1, x) \leq D(\mu, \psi_2, x)$. In order to prove the desired result, we need to show that $C(\lambda, \psi_1, x) \leq C(\mu, \psi_1, x)$. This is equivalent to show that $C(\lambda, \psi_1, x)$ is decreasing and Schur-concave with respect to λ. Taking derivative of $C(\lambda, \psi_1, x)$ with respect to λ_i, for $i = 1, 2$, we get

$$\frac{\partial C(\lambda, \psi_1, x)}{\partial \lambda_i} = -n_i^* x r_i(x\lambda_i) \frac{\psi_1 \left[\phi_1 \left(\bar{F}_i (x\lambda_i) \right) \right]}{\psi_1' \left[\phi_1 \left(\bar{F}_i (x\lambda_i) \right) \right]} \psi_1' \left[\sum_{i=1}^{n^*} \phi_1 \left(\bar{F}_i (x\lambda_i) \right) \right]. \tag{31}$$

Equation (31) shows that $C(\lambda, \psi_1, x)$ is decreasing in λ_1. The other part can be proved in the similar argument as of Theorem 1.

$\square$

The following corollary is immediate from Theorem 9.

Corollary 3. *Let the set-up as in Assumption 1 hold with $\psi_1 = \psi_2 = \psi$ and $n_1^* \leq (\geq) n_2^*$. Also, let ψ be log-convex and let $\lambda, \mu \in \mathscr{E}_+ (\mathscr{D}_+)$. Then,*

(i) $(\underbrace{\lambda_1,\ldots,\lambda_1}_{n_1^*},\underbrace{\lambda_2,\ldots,\lambda_2}_{n_2^*}) \succeq_w (\underbrace{\mu_1,\ldots,\mu_1}_{n_1^*},\underbrace{\mu_2,\ldots,\mu_2}_{n_2^*}) \Rightarrow X_{1:n^*}(n_1^*,n_2^*) \leq_{st} Y_{1:n^*}(n_1^*,n_2^*),$ *provided $r_1(x)$*
   or $r_2(x)$ is increasing with $r_1(x) \leq (\geq) r_2(x)$.

(ii) $(\underbrace{\lambda_1,\ldots,\lambda_1}_{n_1^*},\underbrace{\lambda_2,\ldots,\lambda_2}_{n_2^*}) \succeq_w (\underbrace{\mu_1,\ldots,\mu_1}_{n_1^*},\underbrace{\mu_2,\ldots,\mu_2}_{n_2^*}) \Rightarrow X_{1:n^*}(n_1^*,n_2^*) \leq_{st} Y_{1:n^*}(n_1^*,n_2^*),$ *provided $r(x)$*
   is increasing, where $r_1(x) = r_2(x) = r(x)$.

The next result reveals that the smallest order statistics $X_{1:n^*}(n_1^*,n_2^*)$ dominates $X_{1:n}(n_1,n_2)$ in the sense of the usual stochastic order under the condition that (n_1^*,n_2^*) is weakly submajorized by (n_1,n_2). The following assumption will be helpful to prove the next two results.

Theorem 10. *Let Assumption 2 hold. Then, for $\lambda = (\lambda_1,\lambda_2) \in \mathscr{E}_+$ and $F_1 \leq F_2$, we have*

$$(n_1,n_2) \succeq_w (n_1^*,n_2^*) \Rightarrow X_{1:n}(n_1,n_2) \leq_{st} X_{1:n^*}(n_1^*,n_2^*).$$

Proof. Using the similar approach of Theorem 2, one can obtain the required result.
$\square$

The following remark follows from Theorems 9 and 10.

Remark 7. *Let Assumption 3 hold, with $r_1(x) \leq r_2(x)$. Also, let λ, $\mu \in \mathscr{E}_+$. Then, for $(n_1,n_2) \succeq_w (n_1^*,n_2^*)$,*

$$(\underbrace{\lambda_1,\ldots,\lambda_1}_{n_1^*},\underbrace{\lambda_2,\ldots,\lambda_2}_{n_2^*}) \succeq_w (\underbrace{\mu_1,\ldots,\mu_1}_{n_1^*},\underbrace{\mu_2,\ldots,\mu_2}_{n_2^*}) \Rightarrow X_{1:n}(n_1,n_2) \leq_{st} Y_{1:n^*}(n_1^*,n_2^*),$$

provided $\phi_2 \circ \psi_1$ is super-additive, ψ_1 or ψ_2 is log-convex and $r_1(x)$ or $r_2(x)$ is increasing.

As an illustration of Remark 7, we present the following example.

Example 3. Take $\lambda = (2,6)$, $\mu = (1,3)$, $(n_1,n_2) = (4,8)$, $(n_1^*,n_2^*) = (6,7)$, $\psi_1(x) = e^{-x^{\frac{1}{d_1}}}$ and $\psi_2(x) = e^{-x^{\frac{1}{d_2}}}$, $x > 0$, d_1, $d_2 \in [1,\infty)$. Also, let $F_1(x) = \left(\frac{x}{a}\right)^l$, $0 < x \leq a$ and $F_2(x) = 1 - e^{-x}$, $x > 0$. It can be seen that, for $d_1 = 9$, $d_2 = 10$ $a = 400$ and $l = 2$, all the conditions of Remark 7 are satisfied (see Table 1). Now, we plot the graph of $\bar{F}_{X_{1:12}(4,8)}(x) - \bar{F}_{Y_{1:13}(6,7)}(x)$, given in Figure 4. The figure suggests that $X_{1:12}(4,8) \leq_{st} Y_{1:13}(6,7)$ holds.

Next, we present a counterexample, which shows that the stated usual stochastic order in Remark 7 does not hold if the conditions $r_1(x) \leq r_2(x)$ and r_2 is increasing, are dropped out.

Counterexample 2. Take $\lambda = (1.2,3.6)$, $\mu = (1.4,3)$, $(n_1,n_2) = (2,11)$, $(n_1^*,n_2^*) = (3,9)$, $\psi_1(x) = e^{-x^{\frac{1}{4.5}}}$, and $\psi_2(x) = e^{-x^{\frac{1}{5}}}$, $x > 0$. Also, suppose $F_1(x) = 1 - e^{-x}$ and $F_2(x) = 1 - (1+2x)^{-0.5}$, $x > 0$. Clearly, all the conditions of Remark 7 are satisfied except $r_1 \leq r_2$ and r_2 is increasing. Now, the graphs of $\bar{F}_{X_{1:13}}(2,11)(x)$ and $\bar{F}_{Y_{1:12}}(3,9)(x)$ are depicted in Figure 5. It reveals that the usual stochastic order in Remark 7 does not hold.

Upon using Remark 7, one can easily conclude the following corollary.
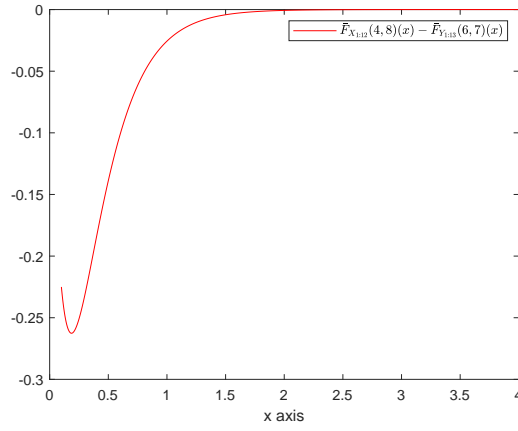
Figure 4: Plot of $\bar{F}_{X_{1:12}(4,8)}(x) - \bar{F}_{Y_{1:13}(6,7)}(x)$ as in Example 3.
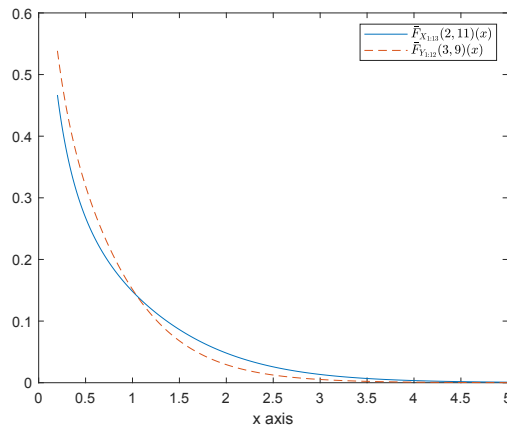

Figure 5: Plots of $\bar{F}_{X_{1:13}(2,11)}(x)$ and $\bar{F}_{Y_{1:12}(3,9)}(x)$ as in Counterexample 2

Corollary 4. *Let Assumption 3 hold with $\psi_1 = \psi_2 = \psi$. Also, let $\lambda,\ \mu \in \mathscr{E}_+$, ψ is log-convex and $(n_1, n_2) \succeq_w (n_1^*, n_2^*)$. Then,*

(i) $(\underbrace{\lambda_1, \ldots, \lambda_1}_{n_1^*}, \underbrace{\lambda_2, \ldots, \lambda_2}_{n_2^*}) \succeq_w (\underbrace{\mu_1, \ldots, \mu_1}_{n_1^*}, \underbrace{\mu_2, \ldots, \mu_2}_{n_2^*}) \Rightarrow X_{1:n}(n_1, n_2) \leq_{st} Y_{1:n^*}(n_1^*, n_2^*)$, *provided $r_1(x)$ or $r_2(x)$*
   is increasing and $r_1(x) \leq r_2(x)$.

(ii) $(\underbrace{\lambda_1, \ldots, \lambda_1}_{n_1^*}, \underbrace{\lambda_2, \ldots, \lambda_2}_{n_2^*}) \succeq_w (\underbrace{\mu_1, \ldots, \mu_1}_{n_1^*}, \underbrace{\mu_2, \ldots, \mu_2}_{n_2^*}) \Rightarrow X_{1:n}(n_1, n_2) \leq_{st} Y_{1:n^*}(n_1^*, n_2^*)$, *provided $r(x)$ is*
   increasing, where $r_1 = r_2 = r$.

Next, we provide three consecutive theorems, which deal with the hazard rate ordering between the smallest order statistics.

Theorem 11. *Let Assumption 1 hold with $n_1^* \leq (\geq) n_2^*$, $\psi_1 = \psi_2 = \psi$, $F_1 = F_2 = F$, $\tilde{r}_1 = \tilde{r}_2 = \tilde{r}$ and $r_1 = r_2 = r$. Also, suppose that ψ is log-concave, that $\frac{1-\psi}{\psi'}$ is decreasing, that $[\frac{1-\psi}{\psi'}]' \frac{\psi}{\psi'}$ is increasing, and that $\lambda,\ \mu \in \mathscr{E}_+ (\mathscr{D}_+)$. Then,*

$$(\underbrace{m_1, \ldots, m_1}_{n_1^*}, \underbrace{m_2, \ldots, m_2}_{n_2^*}) \succeq_w (\underbrace{v_1, \ldots, v_1}_{n_1^*}, \underbrace{v_2, \ldots, v_2}_{n_2^*}) \Rightarrow X_{1:n^*}(n_1^*, n_2^*) \leq_{hr} Y_{1:n^*}(n_1^*, n_2^*),$$

where $m_i = \log \lambda_i$ and $v_i = \log \mu_i$, $i = 1,\ 2$, provided $r(x)$ is increasing, $x\tilde{r}(x)$ is increasing and convex.

Proof. Denote by f the probability density function corresponding to the distribution function F. The hazard rate function of $X_{1:n^*}(n_1^*, n_2^*)$ is given by

$$r_{X_{1:n^*}(n_1^*,n_2^*)}(x) \overset{def}{=} \mathscr{E}(\mathbf{m}) = \frac{\psi'[z]}{\psi[z]} \left[\sum_{i=1}^{n^*} \frac{e^{m_1} f(xe^{m_1})}{\psi'[\phi(\bar{F}(xe^{m_1}))]} \right]$$

$$= \frac{\psi'[z]}{\psi[z]} \left[\sum_{i=1}^{n^*} \frac{e^{m_i} \tilde{r}(xe^{m_i})[1 - \psi[\phi(\bar{F}(xe^{m_i}))]]}{\psi'[\phi(\bar{F}(xe^{m_i}))]} \right],$$

where $z = n_1^* \phi(\bar{F}(xe^{m_1})) + n_2^* \phi(\bar{F}(xe^{m_2}))$, $m_i = \log \lambda_i$, for $i = 1, 2$ and $\mathbf{m} = (m_1, m_2)$. Also, f is the probability density function of F. The partial derivative of $\mathscr{E}(\mathbf{m})$ with respect to m_i for $i = 1, 2$ is given by

$$\frac{\partial \mathscr{E}(\mathbf{m})}{\partial m_i} = -n_i^* x e^{m_i} \tilde{r}(xe^{m_i}) \frac{d}{dz} \left[\frac{\psi'(z)}{\psi(z)} \right] \left[\frac{1 - \psi[\phi[\bar{F}(xe^{m_i})]]}{\psi'[\phi(\bar{F}(xe^{m_i}))]} \right] \left[\sum_{i=1}^{n^*} \frac{e^{m_i} f(xe^{m_i})}{\psi'[\phi(\bar{F}(xe^{m_i}))]} \right]$$

$$- n_i^* r(xe^{m_i}) \left[x[e^{m_i}]^2 \tilde{r}(xe^{m_i}) \right] \frac{\psi'(z)}{\psi(z)} \left[\frac{\psi(v)}{\psi'(v)} \left[\frac{d}{dv} \left[\frac{1 - \psi(v)}{\psi'(v)} \right] \right] \right]_{v = \phi(\bar{F}(xe^{m_i}))}$$

$$+ n_i^* \frac{d}{dw} [w\tilde{r}(w)]_{w=xe^{m_i}} \frac{1 - \psi[\phi[\bar{F}(xe^{m_i})]]}{\psi'[\phi[\bar{F}(xe^{m_i})]]} \frac{\psi'(z)}{\psi(z)}. \tag{32}$$

From (32), it is easy to see that $\mathscr{E}(\mathbf{m})$ is increasing with respect to $\mathbf{m}$. Now, we only need to show the Schur-convexity of $\mathscr{E}(\mathbf{m})$ with respect to $\mathbf{m}$. This is equivalent to show that for $1 \leq i \leq j \leq n^*$,

$$\left[\frac{\partial \mathscr{E}(\mathbf{m})}{\partial m_i} - \frac{\partial \mathscr{E}(\mathbf{m})}{\partial m_j} \right] \leq (\geq) 0, \text{ for } \mathbf{m} \in \mathscr{E}_+ (\mathscr{D}_+). \tag{33}$$

Utilizing the assumptions made, the rest of the proof follows from the similar arguments of Theorem 9. Thus, it is omitted for the sake of conciseness.

□

The following theorem demonstrates that under some conditions, the hazard rate ordering between $X_{1:n}(n_1,n_2)$ and $X_{1:n^*}(n_1^*,n_2^*)$ exists.

Theorem 12. *Let Assumption 2 hold with $\psi_1 = \psi$ and $\tilde{r}_1 = \tilde{r}_2 = \tilde{r}$. Then, for $\lambda \in \mathscr{E}_+$, we have*

$$(n_1,n_2) \succeq_w (n_1^*,n_2^*) \Rightarrow X_{1:n}(n_1,n_2) \leq_{hr} X_{1:n^*}(n_1^*,n_2^*),$$

provided $x\tilde{r}(x)$ is increasing, ψ'/ψ and $\frac{1-\psi}{\psi'}$ are decreasing.

Proof. The required result can be proved if we show that $r_{X_{1:n}(n_1,n_2)}(x) \geq r_{X_{1:n^*}(n_1^*,n_2^*)}(x)$ and equivalently,

$$\frac{\psi'[\sum_{i=1}^{n} \phi(\bar{F}(x\lambda_i))]}{\psi[\sum_{i=1}^{n} \phi(\bar{F}(x\lambda_i))]} \left[\sum_{i=1}^{n} \frac{\lambda_i \tilde{r}(x\lambda_i)[1-\psi[\phi(\bar{F}(x\lambda_i))]]}{\psi'[\phi(\bar{F}(x\lambda_i))]} \right] \geq \frac{\psi'[\sum_{i=1}^{n^*} \phi(\bar{F}(x\lambda_i))]}{\psi[\sum_{i=1}^{n^*} \phi(\bar{F}(x\lambda_i))]} \left[\sum_{i=1}^{n^*} \frac{\lambda_i \tilde{r}(x\lambda_i)[1-\psi[\phi(\bar{F}(x\lambda_i))]]}{\psi'[\phi(\bar{F}(x\lambda_i))]} \right]. \tag{34}$$

To prove inequality (34), it is sufficient to show that the following two inequalities hold:

$$(n_1^* - n_1)\phi(\bar{F}(x\lambda_1)) \leq (n_2 - n_2^*)\phi(\bar{F}(x\lambda_2)) \tag{35}$$

and

$$(n_1^* - n_1)\frac{\lambda_1 \tilde{r}(x\lambda_1)[1-\psi[\phi(\bar{F}(x\lambda_1))]]}{\psi'[\phi(\bar{F}(x\lambda_1))]} \geq (n_2 - n_2^*)\frac{x\lambda_2 \tilde{r}(x\lambda_2)[1-\psi[\phi(\bar{F}(x\lambda_2))]]}{\psi'[\phi(\bar{F}(x\lambda_2))]}. \tag{36}$$

Furthermore, $(n_1,n_2) \succeq_w (n_1^*,n_2^*) \Rightarrow (n_1 + n_2) \geq (n_1^* + n_2^*) \Rightarrow (n_2 - n_2^*) \geq (n_1^* - n_1) \geq 0$. Also, $\lambda_1 \leq \lambda_2 \Rightarrow \phi(\bar{F}(x\lambda_2)) \geq \phi(\bar{F}(x\lambda_1)) \geq 0$. With the help of decreasing property of $\frac{1-\psi}{\psi'}$, we obtain

$$\frac{1-\psi(w)}{\psi'(w)}\Big|_{w=\phi[\bar{F}(x\lambda_2)]} \leq \frac{1-\psi(w)}{\psi'(w)}\Big|_{w=\phi[\bar{F}(x\lambda_1)]} \leq 0. \tag{37}$$

Since $x\tilde{r}(x)$ is increasing,

$$x\lambda_1 \tilde{r}(x\lambda_1) \leq x\lambda_2 \tilde{r}(x\lambda_2). \tag{38}$$

Thus, the proof is completed from (37), (38), and the given assumptions. □

The next theorem states that if the scale parameters are connected with the weakly submajorized order and the sample size pairs (n_1,n_2) and (n_1^*,n_2^*) have weakly submajorized order, then the smallest order statistics of $X_{1:n}(n_1,n_2)$ is dominated by $Y_{1:n^*}(n_1^*,n_2^*)$ according to the hazard rate order.

Theorem 13. *Let Assumption 3 hold with $\psi_1 = \psi_2 = \psi$, $r_1 = r_2 = r$ and $\tilde{r}_1 = \tilde{r}_2 = \tilde{r}$. Then, for $\lambda, \mu \in \mathscr{E}_+$ and $(n_1,n_2) \succeq_w (n_1^*,n_2^*)$,*

$$(\underbrace{m_1,\ldots,m_1}_{n_1^*},\underbrace{m_2,\ldots,m_2}_{n_2^*}) \succeq_w (\underbrace{v_1,\ldots,v_1}_{n_1^*},\underbrace{v_2,\ldots,v_2}_{n_2^*}) \Rightarrow X_{1:n}(n_1,n_2) \leq_{hr} Y_{1:n^*}(n_1^*,n_2^*),$$

provided ψ is log-concave, $\frac{1-\psi}{\psi'}$ is decreasing, $[\frac{1-\psi}{\psi'}]'\frac{\psi}{\psi'}$ and $r(x)$ are increasing, $x\tilde{r}(x)$ is increasing and convex, where $m_i = \log \lambda_i$ and $v_i = \log \mu_i$, $i = 1, 2$.

Proof. The proof of the theorem follows from Theorems 11 and 12. Thus, it is omitted.   □

To illustrate Theorem 13, we now consider the following example.

Example 4. Set $\lambda = (e^{0.5}, e^{0.6})$, $\mu = (e^{0.2}, e^{0.3})$, $(n_1, n_2) = (2, 11)$, $(n_1^*, n_2^*) = (3, 7)$, and $\psi(x) = e^{\frac{1}{b}(1-e^x)}$, $x > 0$, $b \in (0, 1]$. Furthermore, let $F(x) = \left(\frac{x}{a}\right)^l$, $0 < x \leq a$. It can be easily shown that for $b = 0.99$, $a = 1000$, and $l = 2$, all the conditions of Theorem 13 are satisfied (see Table 1). Now, we plot the ratio $\frac{\bar{F}_{Y_{1:10}(3,7)}(x)}{\bar{F}_{X_{1:13}(2,11)}(x)}$ in Figure 6, which is consistent with the result in Theorem 13.
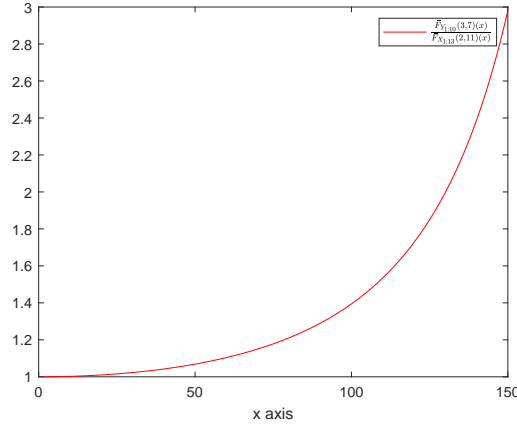
Figure 6: Plot of $\bar{F}_{Y_{1:10}(3,7)}(x)/\bar{F}_{X_{1:13}(2,11)}(x)$ as in Example 4.

Remark 8. *Consider the Independence copula with generator $\psi(x) = e^{-x}$, $x > 0$, which satisfies all the conditions of Theorems 11–13 (see Table 1).*

In the next theorem, we develop some conditions under which two smallest order statistics are comparable according to the star order.

Theorem 14. *Under the set-up as in Assumption 1, with $\tilde{r}_1 = \tilde{r}_2 = \tilde{r}$ and $\psi_1 = \psi_2 = \psi$,*

$$\frac{\lambda_{2:2}}{\lambda_{1:2}} \geq \frac{\mu_{2:2}}{\mu_{1:2}} \Rightarrow Y_{1:n^*}(n_1^*, n_2^*) \leq_* X_{1:n^*}(n_1^*, n_2^*),$$

provided $\frac{\psi}{\psi'}$ is decreasing and convex and $\frac{xr'(x)}{r(x)}$ and $xr(x)$ are decreasing.

Proof. The distribution functions of $X_{1:n^*}(n_1^*, n_2^*)$ and $Y_{1:n^*}(n_1^*, n_2^*)$ are, respectively, given by

$$F_{X_{1:n^*}}(n_1^*, n_2^*)(x) = 1 - \psi\left[n_1^*\phi\left(\bar{F}(x\lambda_1)\right) + n_2^*\phi\left(\bar{F}(x\lambda_2)\right)\right]$$

and

$$F_{Y_{1:n^*}}(n_1^*, n_2^*)(x) = 1 - \psi\left[n_1^*\phi\left(\bar{F}(x\mu_1)\right) + n_2^*\phi\left(\bar{F}(x\mu_2)\right)\right].$$

Now, the rest of the proof follows using similar arguments as in Theorem 6. Thus, it is omitted.   □

The following result is a direct consequence of Theorem 14.

Corollary 5. *Under the assumptions as in Theorem 14,*

$$\frac{\lambda_{2:2}}{\lambda_{1:1}} \geq \frac{\mu_{2:2}}{\mu_{1:2}} \Rightarrow Y_{1:n^*}(n_1^*, n_2^*) \leq_{Lorenz} X_{1:n^*}(n_1^*, n_2^*),$$

provided $\frac{\psi}{\psi'}$ is decreasing and convex and $\frac{xr'(x)}{r(x)}$ and $xr(x)$ are decreasing.

Remark 9. *Suppose $F(x) = 1 - (\frac{x}{b})^a$, $0 < x \leq b$, $a > 0$. For this baseline distribution, it is easy to check that $\frac{xr'(x)}{r(x)}$ and $xr(x)$ are decreasing. Therefore, one can consider this distribution function as the baseline distribution of Theorem 14.*

4 Concluding remarks

In this paper, we discussed some comparison results between the lifetimes of both parallel and series systems consisting of multiple-outlier dependent scale components in the sense of the usual stochastic, reversed hazard rate, dispersive order, hazard rate, likelihood ratio, star, and Lorenz orders. The dependence structure has been modeled by Archimedean copulas. Sufficient conditions have been established for the purpose of the comparisons of the order statistics. Several examples and counterexamples are presented to illustrate the established results.

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