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On the remaining lifespan of devices

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Abstract. Let $X(t)$ denote the remaining lifespan of a device. Two-dimensional degenerate diffusion processes $(X(t),Y(t))$, where $Y(t)$ is a variable that influences the remaining lifespan, are proposed to model the evolution of $X(t)$ over time. These processes are defined in such a way that $X(t)$ will be strictly decreasing as time increases: $dX(t) = \rho[X(t),Y(t)]dt$, where ρ is a strictly negative function and $\{Y(t),t \geq 0\}$ is a diffusion process. Next, optimal control problems for such diffusion processes, in which the final time is the random time when the device is considered to be worn out, are considered. This type of problems is known as homing problems. The dynamic programming equation satisfied by the value function is derived and particular problems are solved explicitly for diffusion processes $\{Y(t),t \geq 0\}$ that are important for applications. To do so, we must solve non-linear partial differential equations, subject to the appropriate boundary conditions.

Keywords: Brownian motion; Degradation; Diffusion processes; First-passage time; Optimal control.

1 Introduction

Let $D(t)$ be the amount of degradation, or wear, of a device at time t. Numerous papers have been written on models for the evolution of degradation over time. Shahraki et al. (2017) wrote a review paper in which they gave 126 relatively recent references on degradation modelling. The models proposed by various authors include gamma, Wiener and inverse Gaussian processes.

Whitmore and Schenkelberg (1997), Nicolai and Dekker (2007), and Ye et al. (2013), in particular, used one-dimensional Wiener processes as models for the stochastic process $\{D(t),t \geq 0\}$. More recently, Zhang et al. (2017) as well as Zhai et al. (2018) proposed random-effects Wiener processes to model degradation. In Zhou et al. (2021), a generalized Wiener process is used. Ghamlouch et al. (2015) considered a jump-diffusion process for the evolution of the *health indicator* of a system.

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All these papers are of good quality. Indeed, depending on the choice for their infinitesimal parameters, one-dimensional diffusion (or jump-diffusion) processes can produce good results. However, in general degradation $D(t)$ should be a strictly increasing function of t, while the *remaining useful lifetime* (RUL) should decrease as time t increases. As is well known, any diffusion process both increases and decreases in *any* time interval.

Rishel (1991) has pointed out that in order to obtain a strictly increasing degradation with models based on diffusion processes, one must consider $(n+1)$-dimensional degenerate diffusion processes defined by a system of stochastic differential equations of the following form (see Øksendal (2003)):

$$\mathrm{d}X(t) = \rho[X(t), Y_1(t), \ldots, Y_n(t)]\,\mathrm{d}t, \tag{1}$$

$$\mathrm{d}Y_i(t) = f[X(t), Y_1(t), \ldots, Y_n(t)]\,\mathrm{d}t + \sigma[X(t), Y_1(t), \ldots, Y_n(t)]\,\mathrm{d}B_i(t) \tag{2}$$

for $i = 1, 2, \ldots, n$, where $Y_1(t), \ldots, Y_n(t)$ are variables that influence the wear or degradation $X(t)$, ρ is a function that is always positive, and $B_1(t), \ldots, B_n(t)$ are independent standard Brownian motions. If $X(t)$ represents the remaining lifespan, rather than the degradation, of the device at time t, then the function ρ should be always negative.

The author has written a number of papers on the optimal control of *wear processes*; see, in particular, Lefebvre and Gaspo (1996a) and Lefebvre and Gaspo (1996b), and Lefebvre (2000). He also computed in Lefebvre (2010) the mean first-passage time to zero for wear processes.

In this paper, we will propose in Section 2 two degenerate two-dimensional diffusion processes that could be used to model the remaining lifespan (or degradation) of various devices. Then, in Section 3, optimal control problems for such processes will be set up and solved explicitly. In these problems, the optimizer tries to maximize a certain performance criterion from the initial time $t_0 = 0$ until the time τ when the device is considered to be worn out. This time τ is a random variable.

Remark. The function ρ must be strictly increasing (or decreasing) in the interval $[0, \tau]$, not necessarily for any t. Moreover, if the probability that the remaining lifespan $X(t)$ will increase (or the wear will decrease) in $[0, \tau]$ is negligible, then the proposed model is considered to be acceptable in practice.

Optimal control problems for which the final time is a random variable were called *homing problems* by Whittle (1982). He also considered the case when the cost criterion is risk-sensitive; Whittle (1990).

Finally, we will make a few concluding remarks in Section 4.

2 Integrated diffusion processes as models for the remaining lifespan

First, we recall an important result on n-dimensional stochastic processes; see Lefebvre (2007).

Proposition 1. *Let* $\{\mathbf{X}(t), t \geq 0\}$ *be an* n*-dimensional stochastic process defined by*

$$\mathrm{d}\mathbf{X}(t) = (\mathbf{A}\mathbf{X}(t) + \mathbf{a})\,\mathrm{d}t + \mathbf{N}^{1/2}\,\mathrm{d}\mathbf{B}(t),$$

where $\{\mathbf{B}(t), t \geq 0\}$ *is an* n*-dimensional standard Brownian motion,* $\mathbf{A}$ *is a square matrix of order* n*,* $\mathbf{a}$ *is an* n*-dimensional vector, and* $\mathbf{N}^{1/2}$ *is a positive definite square matrix of order* n*. Then, given that* $\mathbf{X}(t_0) = \mathbf{x}$*, we may write that*

$$\mathbf{X}(t) \sim \mathrm{N}(\mathbf{m}(t), \mathbf{K}(t)) \quad \text{for } t \geq t_0,$$

where

$$\mathbf{m}(t) := \Phi(t)\left(\mathbf{x} + \int_{t_0}^{t} \Phi^{-1}(u)\,\mathbf{a}\,du\right)$$

and

$$\mathbf{K}(t) := \Phi(t)\left(\int_{t_0}^{t} \Phi^{-1}(u)\mathbf{N}[\Phi^{-1}(u)]'\,du\right)\Phi'(t),$$

where the symbol prime *denotes the* transpose *of the matrix, and the function* $\Phi(t)$ *is given by*

$$\Phi(t) := e^{\mathbf{A}(t-t_0)} = \sum_{n=0}^{\infty} \mathbf{A}^n \frac{(t-t_0)^n}{n!}.$$

In two dimensions, the system (1), (2) can be written as follows:

$$\mathrm{d}X(t) = \rho[X(t), Y(t)]\,\mathrm{d}t, \tag{3}$$
$$\mathrm{d}Y(t) = f[X(t), Y(t)]\,\mathrm{d}t + \sigma[X(t), Y(t)]\,\mathrm{d}B(t). \tag{4}$$

We will now give two examples of degenerate two-dimensional diffusion processes that could serve as models for the remaining lifespan $X(t)$ of devices, and for which Proposition 1 can be used to obtain the distribution of $X(t)$.

Example 1. Consider first the two-dimensional degenerate diffusion process $(X(t), Y(t))$ defined by

$$\begin{aligned} \mathrm{d}X(t) &= cY(t)\,\mathrm{d}t, \\ \mathrm{d}Y(t) &= \mu\,\mathrm{d}t + \sigma\,\mathrm{d}B(t), \end{aligned} \tag{5}$$

where c is a negative constant and $\{B(t), t \geq 0\}$ is a standard Brownian motion, so that $\{Y(t), t \geq 0\}$ is a Wiener process with drift coefficient $\mu \in \mathbb{R}$ and diffusion coefficient $\sigma > 0$. Moreover $\{X(t), t \geq 0\}$ is an integrated Brownian motion multiplied by c. Assuming that $(X(0), Y(0)) = (x, y)$, we find (see Lefebvre (2007), p. 212) that

$$\mathbf{m}(t) = \begin{bmatrix} x + cyt + \frac{1}{2}c\mu t^2 \\ y + \mu t \end{bmatrix}$$

and

$$\mathbf{K}(t) = \begin{bmatrix} c^2\sigma^2 t^3/3 & c\sigma^2 t^2/2 \\ c\sigma^2 t^2/2 & \sigma^2 t \end{bmatrix}.$$

Now, in theory, the Wiener process $\{Y(t), t \geq 0\}$ could become negative before $X(t) = 0$, so that the remaining lifespan would start to increase. Since $Y(t) \sim \mathrm{N}(y + \mu t, \sigma^2 t)$, we have

$$p_1(t) := P[Y(t) < 0] = \Phi\left(\frac{-y - \mu t}{\sigma\sqrt{t}}\right),$$

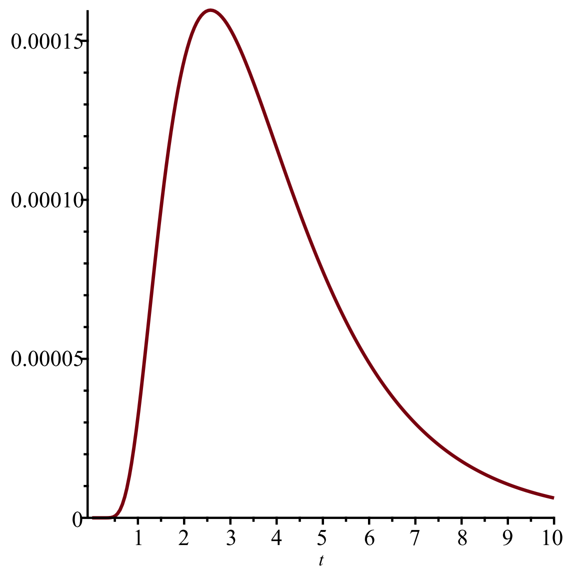

Figure 1: Value of $p_1(t) := \Phi\left(\frac{-3-t}{\sqrt{t}}\right)$ in the interval [0,10].

where here $\Phi(\cdot)$ denotes the distribution function of a $\mathrm{N}(0,1)$ random variable. Hence, if y is large enough and $\mu > 0$, the above probability becomes rapidly negligible. For instance, if $y = 3$ and $\mu = \sigma = 1$, we see in Figure 1 that $p_1(t) := \Phi\left(\frac{-3-t}{\sqrt{t}}\right)$ is smaller than approximately 0.00016. Moreover, since $Y(t)$ has a Gaussian distribution, the parameters μ and σ are easy to estimate.

Remark. To obtain an appropriate model for the degradation of the device, one has simply to assume that the constant c in Eq. (5) is positive instead of negative.

Example 2. Next, we consider the degenerate two-dimensional diffusion process $\mathbf{Z}(t) = (X(t), Y(t))$ defined by the system of stochastic differential equations

$$\begin{aligned} \mathrm{d}X(t) &= cY(t)\,\mathrm{d}t, \\ \mathrm{d}Y(t) &= -\alpha Y(t)\,\mathrm{d}t + \sigma\,\mathrm{d}B(t), \end{aligned}$$

where $c < 0$ and $\sigma > 0$ are constants, $\alpha \neq 0$ and $\{B(t), t \geq 0\}$ is a standard Brownian motion. Hence, if $\alpha > 0$, $\{Y(t), t \geq 0\}$ is an Ornstein-Uhlenbeck process and $\{X(t), t \geq 0\}$ is its integral times the constant c. We suppose that $(X(t_0), Y(t_0)) = (x, y)$, with $x \geq 0$ and $y > 0$.

We have

$$\mathbf{A} = \begin{pmatrix} 0 & c \\ 0 & -\alpha \end{pmatrix}, \quad \mathbf{a} = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \quad \text{and} \quad \mathbf{N}^{1/2} = \begin{pmatrix} 0 & 0 \\ 0 & \sigma \end{pmatrix}.$$

We calculate

$$\mathbf{A}^2 = \begin{pmatrix} 0 & c \\ 0 & -\alpha \end{pmatrix} \begin{pmatrix} 0 & c \\ 0 & -\alpha \end{pmatrix} = \begin{pmatrix} 0 & -c\alpha \\ 0 & \alpha^2 \end{pmatrix},$$

$$\mathbf{A}^3 = \begin{pmatrix} 0 & c \\ 0 & -\alpha \end{pmatrix} \begin{pmatrix} 0 & -c\alpha \\ 0 & \alpha^2 \end{pmatrix} = \begin{pmatrix} 0 & c\alpha^2 \\ 0 & -\alpha^3 \end{pmatrix}$$

and

$$\mathbf{A}^4 = \begin{pmatrix} 0 & c \\ 0 & -\alpha \end{pmatrix} \begin{pmatrix} 0 & c\,\alpha^2 \\ 0 & -\alpha^3 \end{pmatrix} = \begin{pmatrix} 0 & -c\,\alpha^3 \\ 0 & \alpha^4 \end{pmatrix}.$$

Hence, we deduce that we can write that

$$\mathbf{A}^n = \begin{pmatrix} 0 & (-1)^{n-1}\,c\,\alpha^{n-1} \\ 0 & (-1)^n\,\alpha^n \end{pmatrix} \quad \text{for } n \in \mathbb{N}.$$

It follows that

$$\begin{aligned} \Phi(t) &= \sum_{n=0}^{\infty} \mathbf{A}^n \frac{(t-t_0)^n}{n!} \\ &= \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} + \sum_{n=1}^{\infty} \begin{pmatrix} 0 & (-1)^{n-1}\,c\,\alpha^{n-1} \\ 0 & (-1)^n\,\alpha^n \end{pmatrix} \frac{(t-t_0)^n}{n!} \\ &= \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} + \begin{pmatrix} 0 & c\,(t-t_0)\,e^{-\alpha(t-t_0)} \\ 0 & e^{-\alpha(t-t_0)} - 1 \end{pmatrix} \\ &= \begin{pmatrix} 1 & c\,(t-t_0)\,e^{-\alpha(t-t_0)} \\ 0 & e^{-\alpha(t-t_0)} \end{pmatrix}. \end{aligned}$$

Next,

$$\begin{aligned} \Phi^{-1}(t) &= e^{\alpha(t-t_0)} \begin{pmatrix} e^{-\alpha(t-t_0)} & -c\,(t-t_0)\,e^{-\alpha(t-t_0)} \\ 0 & 1 \end{pmatrix} \\ &= \begin{pmatrix} 1 & -c\,(t-t_0) \\ 0 & e^{\alpha(t-t_0)} \end{pmatrix}. \end{aligned}$$

We have

$$\mathbf{m}(t) = \Phi(t) \left[\begin{pmatrix} x \\ y \end{pmatrix} + \begin{pmatrix} 0 \\ 0 \end{pmatrix} \right] = \begin{pmatrix} x + c\,(t-t_0)\,e^{-\alpha(t-t_0)}\,y \\ e^{-\alpha(t-t_0)}\,y \end{pmatrix}$$

and

$$\begin{aligned} &\Phi^{-1}(u)\,\mathbf{N}\,[\Phi^{-1}(u)]' \\ &= \begin{pmatrix} 1 & -c\,(u-t_0) \\ 0 & e^{\alpha(u-t_0)} \end{pmatrix} \begin{pmatrix} 0 & 0 \\ 0 & \sigma^2 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ -c\,(u-t_0) & e^{\alpha(u-t_0)} \end{pmatrix} \\ &= \begin{pmatrix} 0 & -c\,\sigma^2\,(u-t_0) \\ 0 & \sigma^2\,e^{\alpha(u-t_0)} \end{pmatrix} \begin{pmatrix} 1 & 0 \\ -c\,(u-t_0) & e^{\alpha(u-t_0)} \end{pmatrix} \\ &= \begin{pmatrix} c^2\,\sigma^2\,(u-t_0)^2 & -c\,\sigma^2\,(u-t_0)\,e^{\alpha(u-t_0)} \\ -c\,\sigma^2\,(u-t_0)\,e^{\alpha\,(u-t_0)} & \sigma^2\,e^{2\alpha(u-t_0)} \end{pmatrix}, \end{aligned}$$

so that we obtain the symmetric matrix

$$\begin{aligned} \mathbf{M}(t) &:= \int_{t_0}^{t} \Phi^{-1}(u)\,\mathbf{N}\,[\Phi^{-1}(u)]'\,\mathrm{d}u \\ &= \begin{pmatrix} c^2\,\sigma^2\,\dfrac{(t-t_0)^3}{3} & -c\,\sigma^2\,\dfrac{1 + [\alpha\,(t-t_0) - 1]\,e^{\alpha(t-t_0)}}{\alpha^2} \\ - & \sigma^2\,\dfrac{e^{2\alpha(t-t_0)} - 1}{2\,\alpha} \end{pmatrix}. \end{aligned}$$

Finally, we calculate

$$\begin{aligned}
\mathbf{K}(t) &:= \Phi(t)\,\mathbf{M}(t)\,\Phi'(t)\\
&= \begin{pmatrix} 1 & c\,(t-t_0)\,e^{-\alpha(t-t_0)} \\ 0 & e^{-\alpha(t-t_0)} \end{pmatrix} \mathbf{M}(t) \begin{pmatrix} 1 & 0 \\ c\,(t-t_0)\,e^{-\alpha(t-t_0)} & e^{-\alpha(t-t_0)} \end{pmatrix}\\
&= \begin{pmatrix} K_{11} & K_{12} \\ K_{12} & K_{22} \end{pmatrix},
\end{aligned}$$

where

$$\begin{aligned}
K_{11} &:= \frac{c^2\,\sigma^2\,(t-t_0)}{3\,\alpha^2}\Big[6-(9/2)\,\alpha\,(t-t_0)+\alpha^2\,(t-t_0)^2\\
&\qquad - 6\,e^{-\alpha(t-t_0)} - (3/2)\,(t-t_0)\,\alpha\,e^{-2\alpha(t-t_0)}\Big],
\end{aligned}$$

$$K_{12} := \frac{c\,\sigma^2}{2\,\alpha^2}\left[2-\alpha\,(t-t_0)-2\,e^{-\alpha(t-t_0)}-\alpha\,(t-t_0)\,e^{-2\alpha(t-t_0)}\right]$$

and

$$K_{22} := \frac{\sigma^2}{2\,\alpha}\left[1-e^{-2\alpha(t-t_0)}\right].$$

The probability

$$p_2(t) := \Phi\left(\frac{-e^{-\alpha t}\,y}{\frac{\sigma}{\sqrt{2\,\alpha}}\sqrt{1-e^{-2\alpha t}}}\right) \tag{6}$$

that $Y(t) < 0$ is negligible if $\alpha < 0$, as can be seen in Figure 2. However, if $\alpha > 0$, $Y(0) = y$ must be large for $p_2(t)$ to be very small. Finally, since $Y(t)$ has again a Gaussian distribution, the parameters α and σ can be easily estimated.

In the next section, optimal control problems for degenerate two-dimensional diffusion processes that can be used to model the evolution of the remaining lifespan of devices over time will be set up and solved explicitly.

3 Optimal control

In this section, we consider controlled versions of the two-dimensional diffusion processes defined in the previous section (see Rishel (1991)):

$$\begin{aligned}
\mathrm{d}X_u(t) &= \rho[X_u(t),Y_u(t),u(t)]\,\mathrm{d}t,\\
\mathrm{d}Y_u(t) &= f[X_u(t),Y_u(t),u(t)]\,\mathrm{d}t+\sigma[X_u(t),Y_u(t),u(t)]\,\mathrm{d}B(t),
\end{aligned}$$

where $u(t)\,(=u[X_u(t),Y_u(t)])$ is the control variable, which is assumed to be a continuous function.

We define the *first-passage time*

$$\tau(x,y) = \inf\{t>0 : X(t)=d \mid X(0)=x, Y(0)=y\},$$

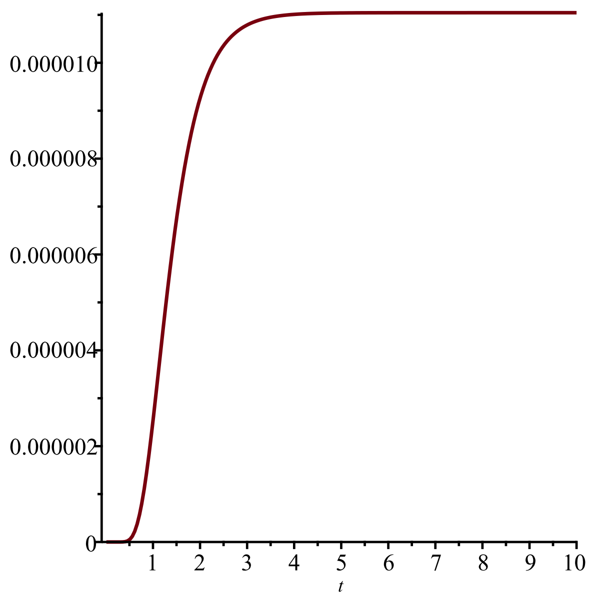

Figure 2: Value of the function $p_2(t)$ defined in Eq. (6) in the interval [0,10], when $\alpha = -1$, $\sigma = 1$ and $y = 3$.

where $x > d$ and $y > 0$, and $d \geq 0$ is a value for which the device of interest is considered to be worn out. The random variable $\tau(x,y)$ represents the remaining useful lifetime of the device, denoted by RUL.

We look for the value $u^*(t)$ of the control variable that maximizes the expected value of the performance criterion

$$J(x,y) := \int_0^{\tau(x,y)} \{r[u(t)] - c[X_u(t), Y_u(t)]\} \, \mathrm{d}t,$$

where $r[u(t)] > 0$ and $c[X_u(t), Y_u(t)] > 0$ are respectively the return and the cost per unit time.

To find the optimal control $u^*(t)$, we will use *dynamic programming*. First, we define the *value function*:

$$F(x,y) = \sup_{u(t),\, 0 \leq t < \tau(x,y)} E[J(x,y)]. \tag{7}$$

That is, $F(x,y)$ is the expected reward (or expected cost, if it is negative) obtained when the optimizer chooses the optimal value of $u(t)$ in the interval $[0, \tau(x,y))$.

Let $u(0) = u_0$. Making use of Bellman's principle of optimality, we can write that

$$\begin{aligned} F(x,y) &= \sup_{u(t),\,0\le t<\tau(x,y)} E\left[\int_0^{\Delta t} \{r[u(t)] - c[X_u(t), Y_u(t)]\}\,\mathrm{d}t \right.\\ &\quad \left. + F\left[x + \rho(x,y,u_0)\,\Delta t, y + f(x,y,u_0)\,\Delta t + \sigma(x,y,u_0)\,B(\Delta t)\right]\right]\\ &\quad + o(\Delta t)\\ &= \sup_{u(t),\,0\le t\le \Delta t} E\left[[r(u_0) - c(x,y)]\,\Delta t \right.\\ &\quad \left. + F\left[x + \rho(x,y,u_0)\,\Delta t, y + f(x,y,u_0)\,\Delta t + \sigma(x,y,u_0)\,B(\Delta t)\right]\right]\\ &\quad + o(\Delta t). \end{aligned} \tag{8}$$

Next, assuming that the value function $F(x,y)$ is differentiable with respect to x and twice differentiable with respect to y, we deduce from Taylor's formula for functions of two variables that

$$\begin{aligned} &F\left[x + \rho(x,y,u_0)\,\Delta t, y + f(x,y,u_0)\,\Delta t + \sigma(x,y,u_0)\,B(\Delta t)\right]\\ &= F(x,y) + \rho(x,y,u_0)\,\Delta t\,F_x(x,y)\\ &\quad + \left[f(x,y,u_0)\,\Delta t + \sigma(x,y,u_0)\,B(\Delta t)\right] F_y(x,y)\\ &\quad + \tfrac{1}{2}\left[f(x,y,u_0)\,\Delta t + \sigma(x,y,u_0)\,B(\Delta t)\right]^2 F_{yy}(x,y) + o(\Delta t). \end{aligned}$$

Since, as is well known, $E[B(\Delta t)] = 0$ and $E[B^2(\Delta t)] = V[B(\Delta t)] = \Delta t$, Eq. (8) implies that

$$\begin{aligned} 0 &= \sup_{u(t),\,0\le t\le \Delta t} \Big\{[r(u_0) - c(x,y)]\,\Delta t + \rho(x,y,u_0)\,\Delta t\,F_x(x,y)\\ &\qquad + f(x,y,u_0)\,\Delta t\,F_y(x,y) + \tfrac{1}{2}\sigma^2(x,y,u_0)\,\Delta t\,F_{yy}(x,y)\Big\}\\ &\qquad + o(\Delta t). \end{aligned} \tag{9}$$

Finally, dividing both sides of Eq. (9) by Δt and letting Δt decrease to zero, we obtain the following proposition.

Proposition 2. *The value function $F(x,y)$ defined in Eq. (7) satisfies the* dynamic programming equation *(DPE)*

$$\begin{aligned} 0 &= \sup_{u_0} \Big\{ r(u_0) - c(x,y) + \rho(x,y,u_0)\,F_x(x,y) + f(x,y,u_0)\,F_y(x,y)\\ &\qquad + \tfrac{1}{2}\sigma^2(x,y,u_0)\,F_{yy}(x,y)\Big\}, \end{aligned} \tag{10}$$

subject to the boundary condition

$$F(d,y) = 0 \quad \forall y > 0. \tag{11}$$

Rishel (1991) has considered the following particular cases for the optimal control of wear processes: first, he assumed that

$$\left.\begin{array}{rcl} \rho[X_u(t),Y_u(t),u(t)] & = & \rho_0[X_u(t),Y_u(t)]\,u^2(t), \\ f[X_u(t),Y_u(t),u(t)] & = & f_0[X_u(t),Y_u(t)]\,u^2(t), \\ \sigma[X_u(t),Y_u(t),u(t)] & = & \sigma_0[X_u(t),Y_u(t)]\,u(t), \\ r[u(t)] & = & r_0\,u(t), \end{array}\right\}$$

where r_0 is positive constant. He proved that the optimal control can then be expressed as follows:

$$u_0^* = u^*(x,y) = \frac{2\,c(x,y)}{r}.$$

Next, he chose

$$\left.\begin{array}{rcl} \rho[X_u(t),Y_u(t),u(t)] & = & \rho_0[X_u(t),Y_u(t)]\,u(t), \\ f[X_u(t),Y_u(t),u(t)] & = & f_0[X_u(t),Y_u(t)]\,u(t), \\ \sigma[X_u(t),Y_u(t),u(t)] & = & \sigma_0[X_u(t),Y_u(t)]\,u(t), \\ r[u(t)] & = & r_0\,u(t). \end{array}\right\}$$

Substituting these functions into the DPE (10), we obtain (after differentiating with respect to u_0) that the optimal control is given, in terms of the value function, by

$$u^*(x,y) = -\frac{\rho_0(x,y)\,F_x + f_0(x,y)\,F_y + r_0}{\sigma_0^2(x,y)F_{yy}}.$$

Remark. There is a minus sign missing in Eq. (26) of Rishel's paper.

It follows that the value function satisfies the partial differential equation (PDE)

$$[\rho_0(x,y)\,F_x + f_0(x,y)\,F_y + r_0]^2 + 2\,c(x,y)\,\sigma_0^2(x,y)\,F_{yy} = 0, \tag{12}$$

subject to the boundary condition (11). This time, Rishel did not find an explicit expression for $u^*(x,y)$, but Lefebvre and Gaspo (1996a) first generalized Rishel's result and then solved three particular problems.

In this paper, we will solve three new problems for important diffusion processes $\{Y(t), t \geq 0\}$ when $X(t)$ represents the remaining lifespan of a device.

Problem I. Suppose that $f_0(x,y) \equiv f_{00}$ and $\sigma_0^2(x,y) = \sigma_{00}^2 y^\nu$, where $\sigma_{00} > 0$ and $\nu \in \{0,1,2\}$. Then, Eq. (4) becomes

$$\mathrm{d}Y(t) = f_{00}\,\mathrm{d}t + \sigma_{00}\,Y^{\nu/2}(t)\,\mathrm{d}B(t).$$

If $\nu = 0$, $\{Y(t), t \geq 0\}$ is a Wiener process with drift f_{00} and dispersion parameter σ_{00}. As we have seen in Section 2, although this diffusion process is Gaussian, the probability that it will become negative is negligible if f_{00} is large enough and $Y(0) = y$ is not too small.

When $\nu = 1$ (and $\sigma_{00}^2 = 2$), $\{Y(t), t \geq 0\}$ is a *squared Bessel process of dimension* $\delta = f_{00}$. It can be shown that if $\delta \geq 2$ and $y > 0$, then the process cannot attain the origin. Therefore, it is a good model for the remaining lifespan (or the wear) of a device.

In the case when $\nu = 2$, we have the following proposition.

Proposition 3. *The diffusion process $\{Y(t), t \geq 0\}$ with infinitesimal mean $f_{00} > 0$ and infinitesimal variance $\sigma_{00}^2 y^2$ has an* exit boundary *at the origin. That is, starting from $Y(0) = y > 0$, the process can reach the origin in finite time and, if it does, it will remain equal to zero indefinitely.*

Proof. We define (see Eq. (9.7.19) in Kannan (1979), p. 279, but there is a mistake in the equation)

$$h(y) = \exp\left\{\int_a^y 2\,\frac{f_{00}}{\sigma_{00}^2 u^2}\,\mathrm{d}u\right\}$$

and

$$H(y) = \frac{2}{\sigma_{00}^2 y^2 h(y)}.$$

We calculate

$$\mu_1 := \int_0^a \int_z^a h(y) H(z)\,\mathrm{d}y\,\mathrm{d}z$$

and

$$\sigma_1 := \int_0^a \int_z^a H(y) h(z)\,\mathrm{d}y\,\mathrm{d}z.$$

We find that $\mu_1 = \infty$ and $\sigma_1 < \infty$. Then, we can state that the origin is indeed an exit boundary. ■

Assume that

$$\rho_0(x,y) = -\rho_{00}\,\frac{(x-d)}{y},$$

where $\rho_{00} > 0$. We must then solve the PDE

$$\left[-\rho_{00}\,\frac{(x-d)}{y}\,F_x + f_{00}\,F_y + r_0\right]^2 + 2\,c(x,y)\,\sigma_{00}^2\,y^{\nu}\,F_{yy} = 0, \tag{13}$$

subject to the boundary condition $F(d,y) = 0$ for any $y > 0$.

Remark. We will see that with the function $\rho_0(x,y)$ defined above, the optimally controlled process $X_u^*(t)$ will actually decrease immediately to d if $Y_u^*(t)$ decreases to zero. Hence, here the type of boundary at the origin for the diffusion process $\{Y(t), t \geq 0\}$ does not really matter.

Let us look for a solution of the form

$$F(x,y) = F_0\left[e^{(x-d)y} - 1\right] \quad \text{for } x \geq d,\ y > 0, \tag{14}$$

which satisfies the boundary condition $F(d,y) = 0$. Substituting $F(x,y)$ into Eq. (13), we obtain the following proposition.

Proposition 4. *If $\rho_{00} = f_{00}$ and the function $c(x,y)$ is given by*

$$c(x,y) = -\frac{r_0^2\,e^{-(x-d)y}}{2F_0\,\sigma_{00}^2\,y^{\nu}\,(x-d)^2}, \tag{15}$$

where $F_0 < 0$, then the function $F(x,y)$ defined in Eq. (14) is the value function in Problem I.

Remark. Notice that the value function is negative. Thus, with the cost $c(x,y)$ defined in Eq. (15), the best that the optimizer can do is to minimize the losses.

We deduce from Eq. (3) the value of the optimal control.

Corollary 1. *The optimal control in Problem I is*

$$u^*(x,y) = -\frac{r_0\, e^{-(x-d)y}}{F_0\, \sigma_{00}^2\, y^{\nu}\, (x-d)^2}. \tag{16}$$

Remarks. (i) The optimal control is positive, so that the optimizer does not try to make $\rho[X_u(t), Y_u(t)]$ become positive.

(ii) We have

$$\begin{aligned}
\mathrm{d}X_u^*(t) &= \rho[X_u^*(t), Y_u^*(t), u^*(t)]\,\mathrm{d}t \\
&= -\rho_{00}\,\frac{[X_u^*(t)-d]}{Y_u^*(t)}\, u^*[X_u^*(t), Y_u^*(t)]\,\mathrm{d}t \\
&= \rho_{00}\,\frac{r_0\, \exp\{[d - X_u^*(t)]\, Y_u^*(t)\}}{F_0\, \sigma_{00}^2\, [Y_u^*(t)]^{\nu+1}\, [X_u^*(t)-d]}\,\mathrm{d}t.
\end{aligned}$$

Therefore, as mentioned above, the optimally controlled process $X_u^*(t)$ will decrease at once to d if $Y_u^*(t)$ decreases to zero.

Problem II. Assume now that $f_0(x,y) = f_{00}\,(\kappa - y)$, where f_{00} and κ are real parameters, and $\sigma_0^2(x,y) = \sigma_{00}^2 y$. With these choices, Eq. (4) becomes

$$\mathrm{d}Y(t) = f_{00}\,[\kappa - Y(t)]\,\mathrm{d}t + \sigma_{00}\, Y^{1/2}(t)\,\mathrm{d}B(t).$$

Therefore, $\{Y(t), t \geq 0\}$ is a (generalized) Cox-Ingersoll-Ross (CIR) process. This process is used in financial mathematics as a model for the evolution of interest rates. When

$$2\, f_{00}\, \kappa \geq \sigma_{00}^2, \tag{17}$$

the process, starting from $Y(0) = y > 0$, will always remain positive.

The transition density function of $\{Y(t), t \geq 0\}$ is known to be

$$p(y,t;y_0,t_0) = \gamma e^{-u-v} \left(\frac{v}{u}\right)^{q/2} I_q(2\sqrt{uv}),$$

where

$$\gamma := \frac{2\, f_{00}}{\left(1 - e^{-f_{00}(t-t_0)}\right)\sigma_{00}^2},\;\; u := \gamma y_0\, e^{-f_{00}(t-t_0)},\;\; v := \gamma y,\;\; q := \frac{2\, f_{00}\, \kappa}{\sigma_{00}^2} - 1$$

and $I_q(\cdot)$ is a modified Bessel function of the first kind of order q (see Abramowitz and Stegun (1965)).

Assume that both f_{00} and κ are negative, and that the condition in Eq. (17) holds. We try a solution of the PDE (12) of the same form as in Problem I: $F(x,y) = F_0\left[e^{(x-d)y} - 1\right]$. One can check that this function is indeed a solution of Eq. (12) that satisfies the boundary condition $F(d,y)=0$ if and only if

$$\rho_0(x,y) = -f_{00}\,(\kappa - y)\,\frac{(x-d)}{y} \quad (<0)$$

and $c(x,y)$ is the same as in Eq. (15), with again $F_0 < 0$. Moreover, the optimal control is given by the function defined in Eq. (16), with $\nu = 1$.

Problem III. In this last problem, we now assume that $X(t)$ represents the wear of the device, rather than its remaining lifespan. Therefore, the function $\rho_0(x,y)$ should be positive. Moreover, we suppose that $X(0) = x \in (0,d)$. The differential equation that we must solve is the same PDE as above, and the boundary condition is still $F(d,y) = 0$.

Let us choose

$$\rho_0(x,y) = \frac{d-x}{xy}, \quad f_0(x,y) = \frac{1}{x} \quad \text{and} \quad \sigma_0(x,y) \equiv 1.$$

We then find that the function

$$F(x,y) = F_0\left[e^{(x-d)y} - 1\right] \quad \text{for } 0 < x \leq d,\ y > 0$$

is a solution of Eq. (12) that satisfies $F(d,y) = 0$ if and only if

$$c(x,y) = -\frac{r_0^2\, e^{(d-x)y}}{2F_0\,(d-x)^2},$$

where F_0 is negative. Furthermore, the optimal control is

$$u^*(x,y) = -\frac{r_0\, e^{(d-x)y}}{F_0\,(d-x)^2}.$$

Remarks. (i) In this problem, the value function is positive. Therefore, it is a reward, rather than a cost as in the previous problems.

(ii) The diffusion process $\{Y(t), t \geq 0\}$ is a Bessel process of *dimension* or *parameter* $\alpha = 3$. For Bessel processes of dimension $\alpha \geq 2$, the origin is an *entrance boundary*, which means that if $X(0) = x > 0$, the process will never hit the origin. The transition density function of $\{Y(t), t \geq 0\}$ is (see Karlin and Taylor (1981), p. 368)

$$p(y,t;y_0,0) = \sqrt{\frac{2}{\pi t}}\,\exp\left\{-\frac{y_0^2 + y^2}{2t}\right\}\frac{y}{y_0}\,\sinh\left(\frac{y_0\, y}{t}\right)$$

for $t > 0$, $y_0 > 0$ and $y > 0$.

4 Concluding remarks

In this paper, we proposed models for the evolution of the remaining lifespan $X(t)$ of devices that are more realistic than one-dimensional diffusion processes. Indeed, the degenerate two-dimensional processes defined in the paper were such that $X(t)$ was a strictly decreasing function of time, as should be. Actually, in Section 2, we mentioned the fact that there was a positive probability that $X(t)$ would increase before reaching the value $d \geq 0$ for which the device is worn out. However, we saw that in practice this probability can be really negligible. Moreover, the aim in Section 2 was to give examples for which it is possible to compute the exact distribution of $X(t)$. This is possible, thanks to Proposition 1, when the diffusion process $\{Y(t), t \geq 0\}$ is Gaussian and the function $\rho[X(t), Y(t)]$ in Eq. (3) is affine.

In Section 3, we presented three optimal control problems for degenerate two-dimensional diffusion processes that could serve as models for the remaining lifespan of devices. These problems are particular homing problems. Obtaining exact and explicit solutions to such problems in two or more dimensions is generally a very difficult task. In Lefebvre and Gaspo (1996a), the authors used the method of similarity solutions to reduce the PDE that must be solved to an *ordinary* differential equation. Here, we proposed a simple function for the value function $F(x, y)$, and we saw that this simple function was indeed the exact solution to the problem considered, if certain conditions hold. Furthermore, the three problems solved in Section 3 were for diffusion processes $\{Y(t), t \geq 0\}$ that are very important for the applications, whereas the ones solved in Lefebvre and Gaspo (1996a) were purely mathematical problems.

When it is not possible to find the exact solution to a particular optimal control problem, one may try to make use of numerical methods to obtain the value function and the optimal control in any special case. It is also sometimes possible to determine upper bounds for both the value function and the optimal control, as Makasu (2022) did for certain homing problems.

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