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A probabilistic model for the structure functions of coherent systems

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Abstract. The probabilistic idea for the structure-function of a coherent system is suggested in the literature. In this paper, a primary model for this idea is defined. This model leads to an extension of the class of coherent systems with binary (deterministic) structure functions. Also, in the stochastic comparison of the coherent systems with probabilistic structure functions, it is shown that the concept of survival signature can successfully be used in this model. The well-known idea of the system signature is not usable here.

Keywords: Coherent systems; Probabilistic structure function; Survival signature; System signature.

1 Introduction

The mathematical and statistical theory of system reliability is based on the central concept of "structure function", which is a binary function and describes deterministically the state of a system when the states of its components are given. In practical uses and real world applications, it is somewhat restricted as for various reasons the functioning of system components does not always provide absolute certainty that the system will function. In other words, except for the failures of system components, other random factors may cause to system failure. For example in a car system sometimes we have seen that the main and key components of the car are functioning but the car does not work. Some unconsidered factors such as the road, weather or driver circumstances, can lead to uncertainty about whether or not the system meets the actual requirements. In a computer system, there is always some uncertainty about the quality of the compatibility between the computer with the new components and softwares. Generally, in the most of real-world systems, because of lack of our perfect knowledge and our uncertainty about the quality of the system functioning, a generalization of structure function from binary function to a probability may have substantial advantages for realistic system reliability quantification.

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This idea was suggested by Coolen and Coolen-Maturi (2016a). A specific model for probabilistic structure function is not given in their work.

In this paper, among various factors that may have effects on system performance, we consider the quality of the links between system components that we think it is an important factor in system functioning. The perfect required performance of a single component when it is putted in a system, is not only depend on its functioning but also to its relationship and link with the other system components. In the following section, we explain our model in detail. Illustrative examples are also given. Finally in Section 3, we show that the survival signature a concept defined in Coolen and Coolen-Maturi (2013), can successfully be used in our model for probabilistic structure function. Particularly its application in stochastic comparison among the coherent systems with the probabilistic structure functions is studied and show that the system signature is deficient here.

2 Probabilistic structure function

In this section, for the sake of completeness we first review the binary structure function and then present our probabilistic model for the structure function. Consider a system consists of n components and assume that all components and the system are in a functioning or failed state. In a fixed point of time, let the state vector $\mathbf{X} = (X_1, \ldots, X_n) \in \{0,1\}^n$, with

$$X_i = \begin{cases} 1, & \text{if } i\text{th component is working,} \\ 0, & \text{otherwise.} \end{cases}$$

The structure function $\phi : \{0,1\}^n \to \{0,1\}$ is defined as

$$\phi(\mathbf{X}) = \phi(X_1, \ldots, X_n) = \begin{cases} 1, & \text{if the system is working,} \\ 0, & \text{otherwise.} \end{cases}$$

The system is said to be coherent if $\phi(\mathbf{X})$ is not decreasing in any X_i and all components be relevant, that is

$$\phi(1_i, \mathbf{X}) - \phi(0_i, \mathbf{X}) = 1,$$

at least for one $\mathbf{X} \in \{0,1\}^{n-1}$.
Obviously $\phi(1, \ldots, 1) = 1$ and $\phi(0, \ldots, 0) = 0$.
Also $\phi(\mathbf{X}) = X_i \phi(1_i, \mathbf{X}) + (1 - X_i)\phi(0_i, \mathbf{X})$, $i = 1, \ldots, n$ (pivotal decomposition).
If $\phi(\mathbf{X}) = 1(0)$ then $\mathbf{X}$ is said to be a path(cut) vector and the corresponding subset $P = \{1 \le i \le n | X_i = 1\}$($C = \{1 \le i \le n | X_i = 0\}$) is called a path(cut) set of the system. Note that an arbitrary vector $\mathbf{X} \in \{0,1\}^n$ always is a path vector or a cut vector but not both. Whereas $P \subseteq \{1, \ldots, n\}$ can be both a path and a cut set.

If $P(C) \subseteq \{1, \ldots, n\}$ is a path(cut) set and $Q \subset P(C)$ is not a path(cut) set then $P(C)$ is a minimal path(cut) set of the system. It is known that if $P_1, \ldots, P_r(C_1, \ldots, C_s)$ are all minimal path(cut) sets of the system, then

$$\phi(\mathbf{X}) = \max_{1 \le i \le r} \min_{j \in P_i} X_j = \min_{1 \le i \le s} \max_{j \in C_i} X_j. \tag{1}$$

If $p_i = EX_i = P(X_i = 1)$ is the reliability of component i then

$$h(\mathbf{p}) = h(p_1, \ldots, p_n) = E\phi(\mathbf{X}) = P(\phi(\mathbf{X}) = 1)$$

is the reliability function of the system. For more details on the reliability of coherent systems with binary structure function, see Barlow and Proschan (1975).

Now, we define our model for the structure function as a probability.
Here we still assume that the binary state for the system components. We also take into account the quality of the links between the components as an effective factor in system performance. This factor may cause to failure of the system even if all its components are working. We consider a particular link for each system component and assume that the component is working perfectly, if both the component and its specific link are functioning. Therefore, given the states of components, we consider the state of the system to be a conditional probability as follows

$$\phi_{\mathbf{a}}(\mathbf{x}) = P_{\mathbf{a}}(\text{system is functioning}|\mathbf{X} = \mathbf{x}),$$

where $\mathbf{a} = (a_1, \ldots, a_n)$ with

$$a_i = P(\text{the link of component } i \text{ is functioning}), i = 1, \ldots, n.$$

As usual we assume that the failure of the components leads to the failure of the system definitely. Let

$$S = \begin{cases} 1, & \text{if system is functioning,} \\ 0, & \text{otherwise,} \end{cases}$$

then $\phi_{\mathbf{a}}(\mathbf{x}) = P_{\mathbf{a}}(S = 1|\mathbf{X} = \mathbf{x})$.

Model assumption:
For a given vector $\mathbf{a} = (a_1, \ldots, a_n)$, we assume that
A1. $\phi_{\mathbf{a}}(\mathbf{x})$ is increasing in x_i and a_i, $i = 1, \ldots, n$.
A2. The component i is relevant, that is $\phi_{\mathbf{a}}(1_i, \mathbf{x}) > 0$ and $\phi_{\mathbf{a}}(0_i, \mathbf{x}) = 0$ at least for one $\mathbf{x} \in \{0,1\}^{n-1}$, $i = 1, \ldots, n$.

Under the above conditions, we call the system as a coherent system. In a coherent system we have $\phi_{\mathbf{a}}(0, \ldots, 0) = 0$. Also $\phi_{\mathbf{a}}(1, \ldots, 1) > 0$, which is not necessary equal to 1. It means that even if all components of the system are working, there exists a positive probability of system failure. But if $\mathbf{x}$ is a cut vector we have $\phi_{\mathbf{a}}(\mathbf{x}) = \phi(\mathbf{x}) = 0$ in which $\phi(\mathbf{x})$ is the binary structure function. In fact

$$\phi_{\mathbf{a}}(\mathbf{x}) = h(\mathbf{a}.\mathbf{x}),$$

where

$$\mathbf{a}.\mathbf{x} = (a_1x_1, \ldots, a_nx_n),$$

and $h(\mathbf{p}) = E\phi(\mathbf{X})$ is the reliability function of the system with binary structure function.
In particular case when $\mathbf{a} = (1, 1, \ldots, 1)$, we have $\phi_{\mathbf{a}}(\mathbf{x}) = h(\mathbf{x}) = \phi(\mathbf{x})$.
Note that $\phi_{\mathbf{a}}(1, \ldots, 1) = h(a_1, \ldots, a_n) \leq 1$.
A3. We assume that the links between components are functioning independently and are independent of the states of components. Also assume that X_i's, $i = 1, \ldots, n$ are independent.

Minimal path(cut) sets

If $\phi_{\mathbf{a}}(\mathbf{x}) > (=)0$ we call $\mathbf{x}$ as a path(cut) vector. The path(cut) sets and the minimal path(cut) sets are defined as before. Although $\phi_{\mathbf{a}}(\mathbf{x})$ is simply satisfied in the pivotal decomposition but the Equation (1) does not hold true for $\phi_{\mathbf{a}}(\mathbf{x})$.

Lemma 1. *For $\mathbf{p} = (p_1, \ldots, p_n)$ and $\mathbf{a} = (a_1, \ldots, a_n)$ the system reliability is given by*

$$h_{\mathbf{a}}(\mathbf{p}) = E_{\mathbf{a}}S = P_{\mathbf{a}}(S = 1) = \sum_{\mathbf{x}_p} \phi_{\mathbf{a}}(\mathbf{x}_p) P(\mathbf{X} = \mathbf{x}_p).$$

Also

$$1 - h_{\mathbf{a}}(\mathbf{p}) = P_{\mathbf{a}}(S = 0) = \sum_{\mathbf{x}_p} (1 - \phi_{\mathbf{a}}(\mathbf{x}_p)) P(\mathbf{X} = \mathbf{x}_p) + \sum_{\mathbf{x}_c} P(\mathbf{X} = \mathbf{x}_c),$$

where $\mathbf{x}_p(\mathbf{x}_c)$ is a path(cut) vector of the system.

Proof. We have

$$h_{\mathbf{a}}(\mathbf{p}) = P_{\mathbf{a}}(S = 1) = \sum_{\mathbf{x}_p} P_{\mathbf{a}}(S = 1 | \mathbf{X} = \mathbf{x}_p) P(\mathbf{X} = \mathbf{x}_p) = \sum_{\mathbf{x}_p} \phi_{\mathbf{a}}(\mathbf{x}_p) P(\mathbf{X} = \mathbf{x}_p).$$

The proof of the second equality given for unreliability of the system is similar. □

Remark 1. *The Lemma 1 shows clearly that the system failure is not only dependent to the component failures but also on the failure of the links between components. The first sum in $1 - h_{\mathbf{a}}(\mathbf{p})$ gives in fact the contribution of the link failures between the system components to the failure of the system and the second sum is the same for the component failures.*

Remark 2. *As mentioned before, $\phi_{\mathbf{a}}(\mathbf{x})$ reduces to the binary structure function $\phi(\mathbf{x})$ when $\mathbf{a} = (1, \ldots, 1)$, that is all links between components are functioning. Therefore the class of coherent systems with binary structure functions is a subclass of coherent systems with probabilistic structure functions. Obviously $\phi_{\mathbf{a}}(\mathbf{x}) \leq \phi(\mathbf{x})$ and therefore $h_{\mathbf{a}}(\mathbf{p}) \leq h(\mathbf{p})$.*

In our model, in fact

$$h_{\mathbf{a}}(\mathbf{p}) = E(\phi_{\mathbf{a}}(\mathbf{X})) = E(\phi(\mathbf{a}.\mathbf{X})) = h(\mathbf{a}.\mathbf{p}) = h(a_1 p_1, \ldots, a_n p_n).$$

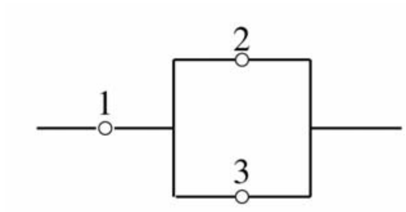
Example 1. We now give some examples.

- Series system: We have $\phi_{\mathbf{a}}(\mathbf{x}) = a_1 a_2 \cdots a_n x_1 x_2 \cdots x_n$ and $h_{\mathbf{a}}(\mathbf{p}) = a_1 a_2 \cdots a_n p_1 p_2 \cdots p_n$.
- Parallel system: In this system we have

$$\phi_{\mathbf{a}}(\mathbf{x}) = 1 - \prod_1^n (1 - a_i x_i)$$

and

$$h_{\mathbf{a}}(\mathbf{p}) = 1 - \prod_1^n (1 - a_i p_i).$$



- 2-out-of-3:G system: In this system we have

$$\phi_{\mathbf{a}}(\mathbf{x}) = a_1a_2x_1x_2 + a_1a_3x_1x_3 + a_2a_3x_2x_3 - 2a_1a_2a_3x_1x_2x_3$$

  and

$$h_{\mathbf{a}}(\mathbf{p}) = a_1a_2p_1p_2 + a_1a_3p_1p_3 + a_2a_3p_2p_3 - 2a_1a_2a_3p_1p_2p_3.$$

- Series-parallel system: For this system we have

$$\phi_{\mathbf{a}}(\mathbf{x}) = a_1a_2x_1x_2 + a_1a_3x_1x_3 - a_1a_2a_3x_1x_2x_3$$

  and

$$h_{\mathbf{a}}(\mathbf{p}) = a_1a_2p_1p_2 + a_1a_3p_1p_3 - a_1a_2a_3p_1p_2p_3.$$

3 Stochastic comparison

In this section, for the sake of completeness, we first review the concepts of system signature and survival signature and find their relationship. We then define the survival signature of a coherent system with probabilistic structure function and show that it can successfully be used in our model.

The concept of the system signature was introduced by Samaniego (1985). It is a very useful tool and has a wide range of applications in the study of reliability analysis of coherent systems. Let $T = \phi(T_1, \ldots, T_n)$ be the lifetime of a coherent system where T_i is the lifetime of component i. When T_i's are independent and identically distributed (i.i.d.), it is shown that

$$P(T > t) = \sum_{i=1}^{n} s_i P(T_{i:n} > t), \tag{2}$$

where $T_{i:n}$ is the i^{th} ordered component lifetime, $s_i = P(T = T_{i:n})$ and the probability vector $\mathbf{s} = (s_1, \ldots, s_n)$ is the system signature (see, Samaniego (1985)).

Although the system signature is an important tool and has many applications in reliability studies of coherent systems with binary structure functions but it is not the case for the coherent systems with probabilistic structure functions as we have seen in the previous section that the system failure is not determined by the failure of components deterministically. Note that the Equation (2) does not hold for the systems with probabilistic structure functions.

This section shows that how in coherent systems with probabilistic structure functions, the "survival signature" (a concept introduced by Coolen and Coolen-Maturi (2013)), plays the role of the system signature in coherent systems with binary structure functions.

As given in Coolen and Coolen-Maturi (2013), the survival signature for a coherent system of order n with components of r different types is defined in which the system has m_k components of type k, $k = 1, \dots, r$ and also the components of the same type are exchangeable and the components of different types are independent. For $i_k = 0, \dots, m_k$ and $k = 1, \dots, r$ this measure is defined as follow

$$\bar{s}(i_1, \dots, i_r) = P(\phi(\mathbf{X}) = 1 | \text{exactly } i_k \text{ components of type } k \text{ are working})$$

$$= \left[\prod_{k=1}^{r} \binom{m_k}{i_k}\right]^{-1} \sum_{\mathbf{x} \in S_{i_1,\dots,i_r}} \phi(\mathbf{x}),$$

where $S_{i_1,\dots,i_r} = \{\mathbf{x} | \sum_{j=1}^{m_k} x_j^k = i_k,\ k = 1, \dots, r.\}$

Also the system reliability is given by

$$P(T > t) = \sum_{i_1=0}^{m_1} \cdots \sum_{i_r=0}^{m_r} \bar{s}(i_1, \dots, i_r) \prod_{k=1}^{r} P(C_t^k = i_k),$$

where $C_t^k \in \{0, 1, \dots, m_k\}$ is the number of components of type k that function at time t.

Under the above stated assumptions it is shown that the $\bar{s}$, as like as the system signature is not dependent on the joint distribution of the components (see, Coolen and Coolen-Maturi (2013)). For more details on the survival signatures and their applications, see Coolen and Coolen-Maturi (2016b) and Coolen and Coolen-Maturi (2021).

3.1 Relationship between system signature and survival signature

In this subsection we consider the relationship between the system signature and survival signature in coherent systems with binary structure functions. For simplicity, we assume that all n components of the system are of the same type. That is $r = 1$. Therefore we have

$$\bar{s}(i) = P(\phi(\mathbf{X}) = 1 | \text{exactly } i \text{ components are working}) = \frac{\sum_{|\mathbf{x}|=i} \phi(\mathbf{x})}{\binom{n}{i}},$$

where $|\mathbf{x}| = \sum_1^n x_k$. Note that $\bar{s}(0) = 0$ and $\bar{s}(n) = 1$.

The following lemma gives the relationship between s_i and $\bar{s}(i)$.

Lemma 2. *We have*

$$\sum_{k=i+1}^{n} s_k = \bar{s}(n-i) = \frac{\sum_{|\mathbf{x}|=n-i} \phi(\mathbf{x})}{\binom{n}{i}}.$$

Proof. In view of the definition of system signature which is a probability vector, and definition of survival signature, we have

$$\begin{aligned}\sum_{k=i+1}^{n} s_k &= P(\text{system failure occurs after } i\text{th failure of components}) \\ &= P(\text{system is working until } i\text{th failure of components,}) \\ &= P(\phi(\mathbf{X}) = 1 | \sum X_k = n-i) = \bar{s}(n-i).\end{aligned}$$

□

Note that for $i = 1, \ldots, n$ we have:

$$s_i = \bar{s}(n-i+1) - \bar{s}(n-i).$$

Lemma 3. *Consider two coherent systems with structure functions $\phi_1(\mathbf{X})$ and $\phi_2(\mathbf{X})$. Let $\bar{s}_1$, $\bar{s}_2$, $\mathbf{s}_1$ and $\mathbf{s}_2$ denote their survival signatures and system signatures, respectively. Then*

$$\bar{s}_1(i) \leq \bar{s}_2(i), i = 1, 2, \ldots, n \quad \textit{if and only if} \quad \mathbf{s}_1 \leq_{st} \mathbf{s}_2.$$

Proof. In view of the definition of usual stochastic order, the proof follows from Lemma 2. □

To see the definition of the stochastic order $\mathbf{s}_1 \leq_{st} \mathbf{s}_2$, and other stochastic orders, we refer the reader to Shaked and Shanthikumar (2007).

The Lemma 3 shows that in stochastic ordering of coherent systems with binary structure functions, the system signature and survival signature are two equivalent tools. We see in the sequel that, it is not case for the coherent systems with probabilistic structure functions.

Definition 1. *In a coherent system with probabilistic structure function $\phi_{\mathbf{a}}(\mathbf{x})$, we define the survival signature as follow*

$$\bar{s}_{\mathbf{a}}(i) = P_{\mathbf{a}}(\textit{system is functioning}|\textit{the number of working components is } i).$$

The following lemma gives an expression for $\bar{s}_{\mathbf{a}}(i)$ in a coherent system with i.i.d. components. It is useful for determining the reliability function of the system.

Lemma 4. *In a coherent system with i.i.d. components and with probabilistic structure function $\phi_{\mathbf{a}}(\mathbf{x})$, we have*

$$\bar{s}_{\mathbf{a}}(i) = \frac{\sum_{\mathbf{x}:|\mathbf{x}|=i} \phi_{\mathbf{a}}(\mathbf{x})}{\binom{n}{i}}.$$

Proof. We have

$$\begin{aligned}\bar{s}_{\mathbf{a}}(i) &= P_{\mathbf{a}}(S=1|\sum_{j=1}^{n} X_j = i)\\ &= \frac{\sum_{\mathbf{x}:|\mathbf{x}|=i} P_{\mathbf{a}}(S=1,\mathbf{X}=\mathbf{x})}{P(\sum X_j = i)}\\ &= \frac{\sum_{\mathbf{x}:|\mathbf{x}|=i} P_{\mathbf{a}}(S=1|\mathbf{X}=\mathbf{x})P(\mathbf{X}=\mathbf{x})}{P(\sum X_j = i)}\\ &= \frac{\sum_{\mathbf{x}:|\mathbf{x}|=i} \phi_{\mathbf{a}}(\mathbf{x}) p^i (1-p)^{n-i}}{\binom{n}{i} p^i (1-p)^{n-i}},\end{aligned}$$

where p is the common reliability of components. □

Lemma 4 shows that the survival signature $\bar{s}_{\mathbf{a}}(i)$ is not dependent on component reliabilities. It is easy to see that the above lemma also holds true when the lifetimes of the system components are exchangeable. Note that when $\mathbf{a} = (1,\ldots,1)$ then $\bar{s}_{\mathbf{a}}(i)$ reduces to $\bar{s}(i)$, the usual survival signature for the coherent systems with binary structure functions.

Now in the next theorem, we obtain the reliability function of the system.

Theorem 1. *Under the assumptions of Lemma 4, we have*

$$h_{\mathbf{a}}(p) = \sum_{i=1}^{n} \bar{s}_{\mathbf{a}}(i) \binom{n}{i} p^i (1-p)^{n-i}. \tag{3}$$

Proof. We have

$$\begin{aligned}h_{\mathbf{a}}(p) &= P_{\mathbf{a}}(S=1)\\ &= \sum_{i=1}^{n} P_{\mathbf{a}}(S=1|\sum_{j} X_j = i)\binom{n}{i} p^i (1-p)^{n-i}\\ &= \sum_{i=1}^{n} \bar{s}_{\mathbf{a}}(i)\binom{n}{i} p^i (1-p)^{n-i}.\end{aligned}$$

□

The following lemma compares the coherent systems with probabilistic structure functions in usual stochastic order.

Lemma 5. *Consider two coherent systems with probabilistic structure functions* $\phi^{(1)}_{\mathbf{a}_1}(\mathbf{X})$ *and* $\phi^{(2)}_{\mathbf{a}_2}(\mathbf{X})$*. Let* $\bar{s}^{(1)}_{\mathbf{a}_1}$, $\bar{s}^{(2)}_{\mathbf{a}_2}$, $\mathbf{s}_1$, $\mathbf{s}_2$, $h^{(1)}_{\mathbf{a}_1}(p)$ *and* $h^{(2)}_{\mathbf{a}_2}(p)$ *denote their survival signatures, system signatures and reliability functions, respectively. If*

$$\bar{s}^{(1)}_{\mathbf{a}_1}(i) \le \bar{s}^{(2)}_{\mathbf{a}_2}(i), i = 1,\ldots,n$$

then

$$h^{(1)}_{\mathbf{a}_1}(p) \le h^{(2)}_{\mathbf{a}_2}(p).$$

Proof. From Theorem 1, the proof is immediate. □

We note that

$$\bar{s}^{(1)}_{\mathbf{a}_1}(i) \leq \bar{s}^{(2)}_{\mathbf{a}_2}(i), i = 1, \ldots, n$$

is not necessary equivalent to $\mathbf{s}_1 \leq_{st} \mathbf{s}_2$, unless $\mathbf{a}_1 = \mathbf{a}_2 = (1, \ldots, 1)$.

The Equation (3) is a similar version of the Equation (2). For example in a series system with i.i.d. components and in view of Example 1, we have $\bar{s}_{\mathbf{a}}(i) = 0$ for $i = 1, \ldots, n-1$ and $\bar{s}_{\mathbf{a}}(n) = (a_1 a_2 \cdots a_n)/\binom{n}{i}$. Hence $h_{\mathbf{a}}(p) = a_1 a_2 \cdots a_n p^n$.

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