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Some copula-based results on optimal number of cold standby components in parallel systems

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Abstract. This paper deals with a system's profit and efficiency functions consisting of a two-parallel components supported by $(n-2)$ cold standby redundancy. It is supposed that the components are not repairable and the failed component is immediately replaced with one of the cold standby components. Because the lifetimes of active components are dependent, their dependency is modelled by Farlie-Gumbel-Morgenstern (FGM) copula function. Finding the optimal n according to the highest profit function of the system and efficiency values is studied while the effect of the dependence parameter is considered. It is concluded that though increasing n grants higher system reliability it is not always wise as long as the redundancy maintenance costs. Due to the complexity of the formulas, for the large values of n, the results are analyzed numerically and graphically. Finally, some examples are given to illustrate the results.

Keywords: Cold standby redundancy; Copula function; Profit function; Reliability; System efficiency.

1 Introduction

Utilizing the redundant units in the system has always seemed a proper tool to improve the system's structure of more reliable and available systems. The pioneering works like Boland et al. (1988) and Boland et al. (1991) investigated the redundancy allocation in coherent systems. But, to optimize the system's profit, one also needs to consider the costs of spare unites and maintenance. Gupta et al. (1986) analyzed the cost function of server two-unit cold standby. Kong and Frangopol (2004) used a life-cycle cost analysis of deteriorating structures to introduce the optimum maintenance scenario. Ram et al. (2013) employed cost profit analysis for a highly reliable complex system. Ram and Kumar (2015) used cost profit analysis to investigate a standby system incorporating waiting time to repair. Singh and Gulati (2014) studied the

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reliability and cost analysis of a two-unit standby redundant complex system under different repair facility. Singh and Rawal (2014) focused on the standby complex system and used profit function to discuss the available maintenance.

The present paper deals with the average profit function of a parallel $(2,n-2)$ system. The parallel $(2,n-2)$ system, as was discussed by Papageorgiou and Kokolakis (2004) and Papageorgiou and Kokolakis (2007) is a system of two initially active units which are supported by $n-2$ cold standbys of the same type. Once one active unit fails, it is replaced by one of the standbys instantly and the process is continued to the final standby. Here, we intend to find the optimum number of units in the system while installing the cold standbys costs. To model the reliability of a system consisting of time-dependent units, we have specifically employed Farlie-Gumbel-Morgenstern (FGM) copula which is known for being well-fitted to data of weak dependence (see, Nelsen (2006)). Domma and Giordano (2013) obtained a closed-form expression for stress-strength reliability using the dependence through FGM copula and Shih and Emura (2019) extended their results using the generalized FGM copula. Yongjin et al. (2018) studied the effect of degrees of dependence among components on system reliability using FGM copula. Zhang and Zhang (2022) proposed a copula-based approach to study the allocation problem of hot standbys in series systems composed of two heterogeneous and dependent components.

The rest of the paper is organized as follows. Section 2 contains some preliminary concepts. Section 3 discusses the main topics related to parallel $(2,n-2)$ system structure and defines the profit function of the system. A criteria is proposed for evaluating the system efficiency. Also, some upper and lower bounds are provided which facilitate to find the optimum strategy for the system structure. In Section 4, some illustrative examples are given in which FGM is applied for modelling the dependence concept of components lifetimes in the system when the component lifetime follows exponential distribution.

2 Preliminaries

Suppose that (X,Y) has distribution function (df) F with absolutely continuous marginal dfs F_1 and F_2. The random vector (X,Y) is said to be positive quadrant dependent (PQD) (negative quadrant dependent (NQD)) if and only if for every $x,y \in R$,

$$F(x,y) \geq (\leq) F_1(x)F_2(y).$$

The notion of 2-copula, denoted by $C(.,.)$, was first introduced by Sklar (1959) which models bivariate distribution of (X,Y) by its marginals as follows

$$F(x,y) = C(F_1(x), F_2(y)). \tag{1}$$

The survival copula is also defined by Nelsen (2006) as

$$\begin{aligned} \bar{F}(x,y) &= P(X > x, Y > y) \\ &= \bar{F}_1(x) + \bar{F}_2(y) - 1 + C(1-\bar{F}_1(x), 1-\bar{F}_2(y)) \\ &= \hat{C}(\bar{F}_1(x), \bar{F}_2(y)), \end{aligned}$$

where $\bar{F}_i(x) = 1 - F_i(x)$ for $i = 1,2$.

The renowned FGM family of bivariate dfs is of the form

$$F(x,y) = F_1(x)F_2(y)\left[1+\alpha\bar{F}_1(x)\bar{F}_2(y)\right],\ -1 \le \alpha \le 1,$$

where α is the dependence parameter, $\alpha \ge 0$ models PQD and $\alpha \le 0$ models NQD. According to (1) for $u, v \in [0,1]$, the FGM 2-copula is given by

$$C(u,v) = uv\left[1+\alpha(1-u)(1-v)\right],\ -1 \le \alpha \le 1.$$

A well-known version of multivariate FGM family had been introduced by Nelsen (2006) as

$$F(x_1,x_2,\dots,x_n) = \prod_{i=1}^{n} F_i(x_i)\left[1+\sum_{k=2}^{n}\sum_{1\le i_1<i_2<\dots<i_k\le n}\alpha_{i_1 i_2 \dots i_k}\prod_{j=1}^{k}\bar{F}_{i_j}(x_{i_j})\right].$$

Applying n-dimensional version of Sklar's Theorem was given in Schweizer and Sklar (1983), the FGM n-copula family introduced by Johnson and Kotz (1972) is as

$$C(u_1,u_2,\dots,u_n) = \prod_{i=1}^{n} u_i\left[1+\sum_{k=2}^{n}\sum_{1\le i_1<i_2<\dots<i_k\le n}\alpha_{i_1 i_2 \dots i_k}\prod_{j=1}^{k}(1-u_{i_j})\right], \tag{2}$$

in which $(2^n - n - 1)$ parameters must satisfy the following conditions

$$1+\sum_{k=2}^{m}\sum_{s_1\le r_1<r_2<\dots<r_k\le s_m}\alpha_{r_1 r_2 \dots r_k}\prod_{j=1}^{n}\xi_{r_j}\ge 0,\ |\xi_{r_j}|\le 1,$$

for $1 \le s_1 < s_m \le n$ and $m = 2,3,\dots,n$. Here, each k-margin, $2 \le k < n$, of an FGM n-copula is an FGM k-copula (see, Conway (1983)) for more applications of FGM n-copula).

This study chooses the FGM copula to model the dependence structure of component's lifetimes. The main reasons for this choice are modelling weak dependence which is the property of the lifetime data, the capability of modelling both positive and negative dependence, and well-fitting to real reliability data (see, Amini et al. (2011)).

3 Main results

In this section, we suppose a parallel system contains n components that operated initially by two components and the remaining $(n-2)$ components are cold standbys, which is known as the parallel $(2,n-2)$ system. We study the structures and some properties of the systems and define the mean profit and efficiency functions up to time t. Later, a lower bound for the profit function is introduced that simplifies the effect of the number of redundancies.

3.1 System profit and efficiency functions

Let T_i be the lifetime of the ith component of the parallel $(2,n-2)$ system and S_i be standby duration of the cold standby component, for $i = 1,2,\dots,n$. Consequently, $S_1 = S_2 = 0$. Once the first failure occurs, the third (the first standby) component is replaced and starts its operation at $S_3 = \min\{T_1, T_2\}$. The forth (the second standby) component is replaced upon failure times of $\max\{T_1, T_2\}$ or $S_3 + T_3$, whichever occurs first i.e.

$$S_4 = \min\{\max\{T_1, T_2\}, T_3 + S_3\}, \tag{3}$$

Also, for $j = 5$ we have

$$S_5 = \min\{\max\{T_1, T_2, T_3 + S_3\}, T_4 + S_4\}. \tag{4}$$

Similarly, for $j \geq 6$

$$S_j = \min\{\max\{T_1, T_2, T_3 + S_3, \ldots, T_{j-2} + S_{j-2}\}, T_{j-1} + S_{j-1}\}, \quad j = 6, \ldots, n. \tag{5}$$

See, Papageorgiou and Kokolakis (2007) for more details. Suppose that T_i's $i = 1, 2, \ldots, n$ are iid with CDF F_T and F_{S_j} is the CDF of S_j. Clearly, T_i's and S_j's are not independent and S_j is independent of T_i for $i \geq j$. Let T_s be the system lifetime. Then as given in Papageorgiou and Kokolakis (2007),

$$T_s = \max\{T_1, T_2, T_3 + S_3, \ldots, T_n + S_n\}.$$

Here, we have supposed that there exists positive or negative dependence between the units lifetimes and the standby periods. Then the reliability function of the proposed parallel $(2, n-2)$ system is given by

$$\begin{aligned} \bar{F}_{sys,n}(x) &= P(T_s > x) \\ &= 1 - C(F_{T_1}(x), F_{T_2}(x), F_{T_3+S_3}(x), \ldots, F_{T_n+S_n}(x)), \end{aligned} \tag{6}$$

where $C(.)$ denotes an appropriate copula function for modelling the dependence between units lifetimes. The cdf of $T_j + S_j$ is given by

$$F_{T_j+S_j}(u) = \int_0^u F_{S_j}(u-s) f_T(s) ds, \quad j = 3, \ldots, n.$$

For $j = 3$, we have

$$\begin{aligned} F_{T_3+S_3}(x) &= P(T_3 + S_3 \leq x) \\ &= \int_0^x F_{S_3}(x-u) f_T(u) du \\ &= F_T(x) - \int_0^x \bar{F}_T{}^2(x-u) f_T(u) du. \end{aligned} \tag{7}$$

Let k_1 be the income for reliability of system per unit time and k_2 be the storage and maintenance cost per unit time of standby components. Then, the expected profit function of parallel $(2, n-2)$ system up to time $t > 0$ is given by

$$B_n(t) = In_n(t) - Co_n(t), \tag{8}$$

where

$$In_n(t) = k_1 \int_0^t \bar{F}_{sys,n}(x) dx, \tag{9}$$

denotes the system average income in $[0,t]$ and

$$Co_n(t) = k_2 \sum_{j=3}^{n} \int_0^t \bar{F}_{S_j}(x)dx, \tag{10}$$

is the average cost of the system during the interval $[0,t]$. Ram and Singh (2010) considered a similar criteria for the expected profit of complex systems to do the cost analysis. Ram et al. (2013) also used the same expected profit for cost analyzing of a standby system with including the waiting time to repair in their study.

For a further rational comparisons of systems, one needs to consider the system efficiency as well. We propose the following criteria for evaluating the system efficiency, for $n \geq 3$

$$E_n(t) \;\; = \;\; \frac{In_n(t)}{In_n(t)+Co_n(t)}, \quad t>0. \tag{11}$$

By (11), it is obvious that $0 \leq E_n(t) \leq 1$ for $t > 0$. Also, $E_n(t)$ tends to 1 and 0 as $Co_n(t)$ tends to 0 and $+\infty$, respectively. For achieving the simple computations, we consider the lower bound profit function using this strategy for the reliability of system and its standby components in the next subsection.

3.2 Upper and lower bounds

As n is increased, the main formulas (8) and (11) become too complex to follow. In order to consider the best number of components in the system, we can provide a simple lower bound for the profit function. First, we obtain upper bound for reliability function of S_k, $k \geq 3$. For $k = 3$, we have $\bar{F}_{S_3}(u) = [\bar{F}_T(u)]^2$. According to the basic probability and convolution theorem, from (3), we obtain

$$\begin{aligned}
\bar{F}_{S_4}(x) &= P(\min\{\max\{T_1,T_2\},T_3+S_3\} > x) \\
&\leq P(T_3+S_3 > x) \\
&= 1-\int_{t_3=0}^{x} P(T_3+S_3 \leq x | T_3 = t_3) dF_{T_3}(t_3) \\
&= \bar{F}_T(x) + \int_{t_3=0}^{x} \bar{F}_T^2(x-t_3) f_T(t_3) dt_3.
\end{aligned} \tag{12}$$

Similarly from (4), we obtain

$$\begin{aligned}
\bar{F}_{S_5}(x) &= P(\min\{\max\{T_1,T_2,T_3+S_3\},T_4+S_4\} > x) \\
&\leq P(T_4+S_4 > x) \\
&= 1-\int_{t_4=0}^{x} P(T_4+S_4 \leq x | T_4 = t_4) dF_{T_4}(t_4) \\
&= \bar{F}_T(x) + \int_{t_4=0}^{x} \bar{F}_{S_4}(x-t_4) f_T(t_4) dt_4 \\
&\leq \bar{F}_T(x) + \int_{t_4=0}^{x} \left[\bar{F}_T(x-t_4) + \int_{t_3=0}^{x-t_4} \bar{F}_T^2(x-t_3-t_4) f_T(t_3) dt_3 \right] f_T(t_4) dt_4 \\
&= \bar{F}_T(x) + \int_{t_4=0}^{x} \bar{F}_T(x-t_4) f_T(t_4) dt_4 + \int_{t_4=0}^{x} \int_{t_3=0}^{x-t_4} \bar{F}_T^2(x-t_3-t_4) f_T(t_3) f_T(t_4) dt_3 dt_4,
\end{aligned} \tag{13}$$

where second inequality in (13) is obtained by (12). Also, using (13), an upper bound for the reliability function of S_6 and S_7 are given by

$$\begin{aligned}\bar{F}_{S_6}(x) &\leq P(T_5+S_5>x)\\ &\leq \bar{F}_T(x)+\int_{t_5=0}^{x}\left[\bar{F}_T(x-t_5)+\int_{t_4=0}^{x-t_5}[\bar{F}_T(x-t_5-t_4)\right.\\ &\qquad \left.+\int_{t_3=0}^{x-t_4-t_5}\bar{F}_T^2(x-t_3-t_4-t_5)f_T(t_3)dt_3\Bigg]f_T(t_4)dt_4\right]f_T(t_5)dt_5.\end{aligned}$$

and

$$\begin{aligned}\bar{F}_{S_7}(x) &\leq P(T_6+S_6>x)\\ &\leq \bar{F}_T(x)+\int_{t_6=0}^{x}\left[\bar{F}_T(x-t_6)+\int_{t_5=0}^{x-t_6}\left[\bar{F}_T(x-t_6-t_5)+\int_{t_4=0}^{x-t_5-t_6}[\bar{F}_T(x-t_6-t_5-t_4)\right.\right.\\ &\qquad \left.\left.+\int_{t_3=0}^{x-t_4-t_5-t_6}\bar{F}_T^2(x-t_3-t_4-t_5-t_6)f_T(t_3)dt_3\Bigg]f_T(t_4)dt_4\right]f_T(t_5)dt_5\right]f_T(t_6)dt_6,\end{aligned}$$

respectively. Continuing the same process, from (5), an explicit possible upper bound for $\bar{F}_{S_k}$ can be achieved which is stated in the next lemma. For convenience of notations, throughout the paper, we suppose that $\sum_{r=i}^{j}a_r=0$ for $i>j$.

Lemma 1. *Suppose the assumptions of the proposed model hold. Then, a possible upper bound for the reliability of the mth standby component, $4\leq m\leq n$, is achieved by*

$$\begin{aligned}\bar{F}_{S_m}(x) &\leq \bar{F}_{S_{m-1}+T_{m-1}}(x)\\ &\leq \bar{F}_T(x)+\int_{t_{m-1}=0}^{x}\left[\bar{F}_T(x-t_{m-1})+\int_{t_{m-2}=0}^{x-t_{m-1}}[\bar{F}_T(x-t_{m-1}-t_{m-2})+\cdots\right.\\ &\quad +\int_{t_4=0}^{x-\sum_{i=5}^{m-1}t_i}\left[\bar{F}_T\left(x-\sum_{i=4}^{m-1}t_i\right)\right.\\ &\quad \left.\left.+\int_{t_3=0}^{x-\sum_{i=4}^{m-1}t_i}\bar{F}_T^2\left(x-\sum_{i=3}^{m-1}t_i\right)f_T(t_3)dt_3\right]f_T(t_4)dt_4\cdots\Bigg]f_T(t_{m-2})dt_{m-2}\right]f_T(t_{m-1})dt_{m-1}. \qquad (14)\end{aligned}$$

As T_j+S_j is stochastically smaller than $\max\{T_1,T_2,T_3+S_3,\ldots,T_n+S_n\}$, for $1\leq j\leq n$, we have the following possible lower bound for the system reliability:

$$\begin{aligned}\bar{F}_{sys,n}(x) &= P(\max\{T_1,T_2,T_3+S_3,\ldots,T_n+S_n\}>x)\\ &\geq P(T_j+S_j>x),\quad j=1,2,...,n.\end{aligned}$$

Especially,

$$\bar{F}_{sys,n}(x)\geq\bar{F}_T(x). \qquad (15)$$

Also, using Bonferroni inequality and Lemma 1, we have

$$
\begin{aligned}
\bar{F}_{sys,n}(x) &= 1 - P(\max\{T_1, T_2, T_3+S_3, \ldots, T_n+S_n\} \le x) \\
&\le \sum_{m=1}^{n} \bar{F}_{T_m+S_m}(x) \\
&\le n\bar{F}_T(x) \\
&+ \sum_{m=3}^{n} \left[\int_{t_m=0}^{x} \left[\bar{F}_T(x-t_m) + \int_{t_{m-1}=0}^{x-t_m} [\bar{F}_T(x-t_m-t_{m-1}) + \cdots \right.\right. \\
&+ \int_{t_4=0}^{x-\sum_{i=5}^{m} t_i} \left[\bar{F}_T\left(x - \sum_{i=4}^{m} t_i\right) \right. \\
&\left.\left.\left.\left. + \int_{t_3=0}^{x-\sum_{i=4}^{m} t_i} \bar{F}_T^2\left(x - \sum_{i=3}^{m} t_i\right) f_T(t_3)dt_3 \right] f_T(t_4)dt_4 \cdots \right] f_T(t_{m-1})dt_{m-1} \right] f_T(t_m)dt_m \right]. \qquad (16)
\end{aligned}
$$

where $S_1 = S_2 = 0$.

Utilizing the upper bound for the kth component survival in (14) and the lower bound for system survival in (15), a possible lower bound for the profit function of system in (8) is achieved which is stated in the following proposition.

Proposition 1. *A possible lower bound for the profit function of parallel* $(2, n-2)$ *system,* $n \ge 4$*, at time* $t > 0$ *is obtained by*

$$
\begin{aligned}
B_n(t) &= k_1 \int_{u=0}^{t} \bar{F}_{sys,n}(u)du - k_2 \sum_{m=3}^{n} \int_{u=0}^{t} \bar{F}_{S_m}(u)du \\
&\ge k_1 \int_0^t \bar{F}_T(u)du - k_2 \int_0^t \bar{F}_T^2(u)du - k_2(n-3) \int_0^t \bar{F}_T(u)du \\
&- k_2 \sum_{m=4}^{n} \int_{u=0}^{t} \left[\int_{t_{m-1}=0}^{u} \left[\bar{F}_T(u-t_{m-1}) + \int_{t_{m-2}=0}^{u-t_{m-1}} [\bar{F}_T(u-t_{m-1}-t_{m-2}) + \cdots \right.\right. \\
&+ \int_{t_4=0}^{u-\sum_{i=5}^{m-1} t_i} \left[\bar{F}_T\left(u - \sum_{i=4}^{m-1} t_i\right) \right. \\
&\left.\left.\left.\left. + \int_{t_3=0}^{u-\sum_{i=4}^{m-1} t_i} \bar{F}_T^2\left(u - \sum_{i=3}^{m-1} t_i\right) f_T(t_3)dt_3 \right] f_T(t_4)dt_4 \cdots \right] f_T(t_{m-2})dt_{m-2} \right] f_T(t_{m-1})dt_{m-1} \right] du.
\end{aligned}
$$

It is observed that the effect of dependence parameter is not appeared in the lower bound of profit function given in Proposition 1.

Using the inequalities in (14), (15) and (16), we obtain a lower bound for the propose system efficiency evaluation in (11), which is stated in the next proposition.

Proposition 2. *A lower bound for the propose system efficiency evaluation in (11) of parallel*

$(2,n-2)$ system, $n\geq 3$, at time $t>0$ is given by

$$\begin{aligned}
E_n(t) &= \frac{In_n(t)}{In_n(t)+Co_n(t)}\\
&= \frac{k_1\int_{x=0}^{t}\bar{F}_{sys,n}(x)dx}{k_1\int_{x=0}^{t}\bar{F}_{sys,n}(x)dx+k_2\sum_{m=3}^{n}\int_{x=0}^{t}\bar{F}_{S_m}(x)dx}\\
&\geq \frac{k_1\int_{x=0}^{t}\bar{F}_T(x)dx}{\int_{x=0}^{t}\left[(nk_1+k_2(n-3))\bar{F}_T(x)+k_2[\bar{F}_T(x)]^2+(k_1+k_2)\sum_{m=3}^{n-1}\phi_m^F(x)+k_1\phi_n^F(x)\right]dx}\\
&= \left[n+(n-3)\frac{k_2}{k_1}+\frac{\int_{x=0}^{t}\left[k_2[\bar{F}_T(x)]^2+(k_1+k_2)\sum_{m=3}^{n-1}\phi_m^F(x)+k_1\phi_n^F(x)\right]dx}{k_1\int_{x=0}^{t}\bar{F}_T(x)dx}\right]^{-1} \qquad (17)
\end{aligned}$$

where

$$\begin{aligned}
\phi_m^F(x) &= \int_{t_m=0}^{x}\left[\bar{F}_T(x-t_m)+\int_{t_{m-1}=0}^{x-t_m}\left[\bar{F}_T(x-t_m-t_{m-1})+\cdots+\int_{t_4=0}^{x-\sum_{i=5}^{m}t_i}\left[\bar{F}_T\left(x-\sum_{i=4}^{m}t_i\right)\right.\right.\right.\\
&\left.\left.\left.+\int_{t_3=0}^{x-\sum_{i=4}^{m}t_i}\bar{F}_T^2\left(x-\sum_{i=3}^{m}t_i\right)f_T(t_3)dt_3\right]f_T(t_4)dt_4\cdots\right]f_T(t_{m-1})dt_{m-1}\right]f_T(t_m)dt_m. \qquad (18)
\end{aligned}$$

For special case $n=3$, by Proposition 2, we have an explicit simple expression for the bound of $E_3(t)$, which is stated below

$$\begin{aligned}
E_3(t) &= \frac{In_3(t)}{In_3(t)+Co_3(t)}\\
&\geq \frac{k_1\int_{x=0}^{t}\bar{F}_T(x)dx}{\int_{x=0}^{t}\left[3k_1\bar{F}_T(x)+k_2[\bar{F}_T(x)]^2+k_1\phi_3^F(x)\right]dx}\\
&= \left[3+\frac{1}{k_1\int_{x=0}^{t}\bar{F}_T(x)dx}\int_{x=0}^{t}\left(k_2[\bar{F}_T(x)]^2+k_1\int_{t_3=0}^{x}\bar{F}_T^2(x-t_3)f_T(t_3)dt_3\right)dx\right]^{-1}. \qquad (19)
\end{aligned}$$

The next section provides some examples for the values of profit and efficiency functions.

4 Illustrative examples

In this section, the results illustrated by two examples, in first example we obtain exact expression and in the second we find a lower bound for $E_n(t)$. The FGM copula model is considered the positive and negative dependencies between components lifetimes and the marginal distributions are assumed to be exponential.

Example 1. (Exact expression for $E_n(t)$)

Let components lifetimes be exponentially distributed, i.e. $F_T(x)=1-e^{-\lambda x}$, $x>0$. As $S_3=\min\{T_1,T_2\}$, then we have

$$\bar{F}_{S_3}(x)=[\bar{F}_T(x)]^2=e^{-2\lambda x}, \qquad (20)$$

also from (7),

$$F_{T_3+S_3}(x) = \int_0^x F_{S_3}(x-u) f_T(u) du$$
$$= 1 + \mathrm{e}^{-2\lambda x} - 2\mathrm{e}^{-\lambda x} = \left(1 - \mathrm{e}^{-\lambda x}\right)^2. \qquad (21)$$

Utilizing FGM 3-copula model as the dependence structure of units lifetimes, from (6) we reach the following expressions for the system reliability

$$\bar{F}_{sys,3}(x) = 1 - C(F_T(x), F_T(x), F_{T_3+S_3}(x))$$
$$= 1 - F_T^2(x) F_{T_3+S_3}(x) \left[1 + \alpha_1 \bar{F}_T^2(x) + \alpha_2 \bar{F}_T(x) \bar{F}_{T_3+S_3}(x) + \alpha_3 \bar{F}_T(x) \bar{F}_{T_3+S_3}(x) + \alpha_4 \bar{F}_T^2(x) \bar{F}_{T_3+S_3}(x)\right].$$

Since by assumption T_1 and T_2 are independent, $\alpha_1 = 0$. By assuming $\alpha = \alpha_2 = \alpha_3 = \alpha_4$ for simplicity, then we have

$$\bar{F}_{sys,3}(x) = 1 - F_T^2(x) F_{T_3+S_3}(x) \left[1 + \alpha(2\bar{F}_T(x)\bar{F}_{T_3+S_3}(x) + \bar{F}_T^2(x)\bar{F}_{T_3+S_3}(x))\right]. \qquad (22)$$

Substituting (20) and (21) into (22), we get

$$\bar{F}_{sys,3}(x) = \mathrm{e}^{-\lambda x}(\mathrm{e}^{-\lambda x} - 2)\left((\mathrm{e}^{-\lambda x} - 1)^2(\mathrm{e}^{-4\lambda x}\alpha - 4\mathrm{e}^{-2\lambda x}\alpha - 1) - 1\right). \qquad (23)$$

From (9) and (23), we obtain

$$In_3(t) = k_1 \int_0^t \bar{F}_{sys,3}(x) dx$$
$$= k_1 \int_0^t \mathrm{e}^{-\lambda x}(\mathrm{e}^{-\lambda x} - 2)\left((\mathrm{e}^{-\lambda x} - 1)^2(\mathrm{e}^{-4\lambda x}\alpha - 4\mathrm{e}^{-2\lambda x}\alpha - 1) - 1\right) dx$$
$$= \frac{k_1}{840\lambda}\Big[1750 + 157\alpha - 3360\mathrm{e}^{-\lambda t} + 2520\mathrm{e}^{-2\lambda t} - 1120\mathrm{e}^{-3\lambda t}(2\alpha + 1)$$
$$+ 210\mathrm{e}^{-4\lambda t}(20\alpha + 1) - 2352\alpha\mathrm{e}^{-5\lambda t} - 140\alpha\mathrm{e}^{-6\lambda t} + 480\alpha\mathrm{e}^{-7\lambda t} - 105\alpha\mathrm{e}^{-8\lambda t}\Big]. \qquad (24)$$

Also, from (10) and (20)

$$Co_3(t) = k_2 \int_0^t \bar{F}_{S_3}(x) dx = \frac{k_2}{2\lambda}\left(1 - \mathrm{e}^{-2\lambda t}\right). \qquad (25)$$

Take $A_3(t;\alpha,\lambda) = \frac{In_3(t)}{Co_3(t)}$, then by substituting (24) and (25), we have

$$A_3(t;\alpha,\lambda) = \frac{In_3(t)}{Co_3(t)}$$
$$= \frac{\lambda k_1}{420\lambda k_2\left(1 - \mathrm{e}^{-2\lambda t}\right)}\Big[1750 + 157\alpha - 3360\mathrm{e}^{-\lambda t} + 2520\mathrm{e}^{-2\lambda t} - 1120\mathrm{e}^{-3\lambda t}(2\alpha + 1)$$
$$+ 210\mathrm{e}^{-4\lambda t}(20\alpha + 1) - 2352\alpha\mathrm{e}^{-5\lambda t} - 140\alpha\mathrm{e}^{-6\lambda t} + 480\alpha\mathrm{e}^{-7\lambda t} - 105\alpha\mathrm{e}^{-8\lambda t}\Big]. \qquad (26)$$

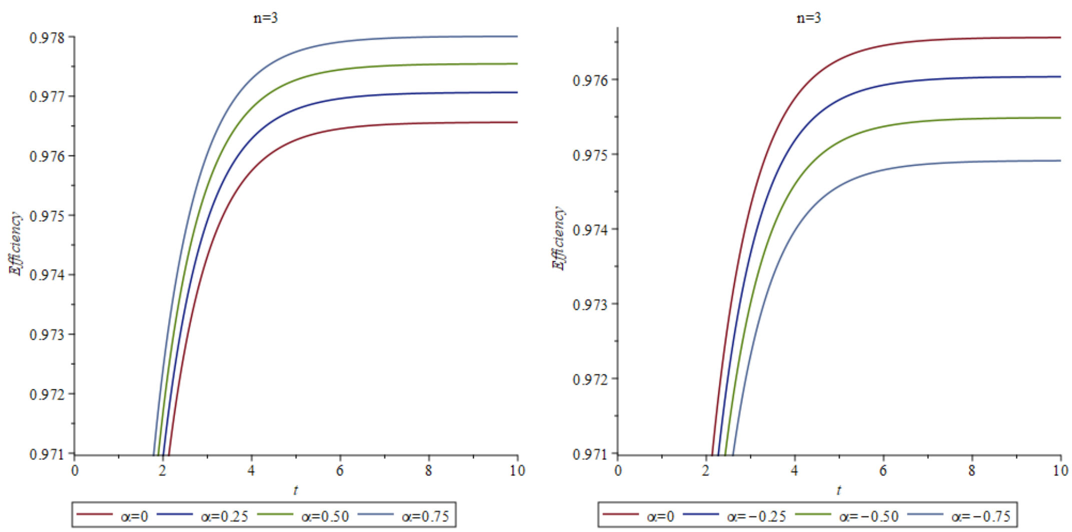

Figure 1: The plot of $E_3(t)$ for $k_1 = 10$, $k_2 = 1$, $\lambda = 1$ and some selected values of α.

Then, from (11) and (26), we can obtain the system efficiency for $n = 3$ as a function of t, α, λ, k_1 and k_2. The plot of $E_3(t)$ has been displayed in Figure 1 for some selected values of α. From Figure 1, it is observed that $E_3(t)$ is increasing in t and the results indicate that it is also increasing in α.

Now we obtain expression for $E_4(t)$. According to the properties of survival copula and using FGM 2-copula model as the dependence structure of units lifetimes, for $n = 4$ we have

$$\begin{aligned}\bar{F}_{S_4}(x) &= \hat{C}(\bar{F}_{\max\{T_1,T_2\}}(x), \bar{F}_{T_3+S_3}(x)) \\ &= \bar{F}_{\max\{T_1,T_2\}}(x) + \bar{F}_{T_3+S_3}(x) - 1 + C(1 - \bar{F}_{\max\{T_1,T_2\}}(x), 1 - \bar{F}_{T_3+S_3}(x)) \\ &= \bar{F}_{\max\{T_1,T_2\}}(x) + \bar{F}_{T_3+S_3}(x) - 1 + F_{\max\{T_1,T_2\}}(x) F_{T_3+S_3}(x) \left[1 + \alpha \bar{F}_{\max\{T_1,T_2\}}(x) \bar{F}_{T_3+S_3}(x)\right]. \end{aligned} \tag{27}$$

From (21) and (27) after simplification, we get

$$\bar{F}_{S_4}(x) = \mathrm{e}^{-2\lambda x}\left[\alpha\left(\mathrm{e}^{-6\lambda x} - 8\mathrm{e}^{-5\lambda x} + 26\mathrm{e}^{-4\lambda x} - 44\mathrm{e}^{-3\lambda x} + 41\mathrm{e}^{-2\lambda x} - 20\mathrm{e}^{-\lambda x} + 4\right) + \left(2 - \mathrm{e}^{-\lambda x}\right)^2\right]. \tag{28}$$

From (7) and (28), we obtain

$$\begin{aligned}
F_{T_4+S_4}(x) &= \int_0^x F_{S_4}(x-u) f_T(u) du \\
&= \int_0^u \lambda \Big[(41\alpha+1)\, e^{-4\lambda(x-u)} - (20\alpha+4)\, e^{-3\lambda(x-u)} + (4\alpha+4)\, e^{-2\lambda(x-u)} \\
&+ \alpha \left(e^{-8\lambda(x-u)} - 8e^{-7\lambda(x-u)} + 26e^{-6\lambda(x-u)} - 44e^{-5\lambda(x-u)} \right) \Big] e^{-\lambda u} du \\
&= \alpha e^{-\lambda x} \left(-\frac{1}{7} e^{-7\lambda x} + \frac{28}{21} e^{-6\lambda x} - \frac{182}{35} e^{-5\lambda x} + 11 e^{-4\lambda x} + \frac{1435}{105} e^{-3\lambda x} - 10 e^{-2\lambda x} + 4 e^{-\lambda x} - \frac{71}{105} \right) \\
&+ \frac{2}{3} e^{-4\lambda x} - e^{-2\lambda x} \left(2 - e^{-\lambda x} \right)^2 .
\end{aligned} \tag{29}$$

From (2) and FGM 4-copula, we acquire the following formula for

$$\begin{aligned}
\bar{F}_{sys,4}(x) &= 1 - C(F_T(x), F_T(x), F_{T_3+S_3}(x), F_{T_4+S_4}(x)) \\
&= 1 - F_T^2(x) F_{T_3+S_3}(x) F_{T_4+S_4}(x) \Big[1 + \alpha_1 \bar{F}_T^2(x) + \alpha_2 \bar{F}_T(x) \bar{F}_{T_3+S_3}(x) + \alpha_3 \bar{F}_T(x) \bar{F}_{T_3+S_3}(x) \\
&+ \alpha_4 \bar{F}_T(x) \bar{F}_{T_4+S_4}(x) + \alpha_5 \bar{F}_T(x) \bar{F}_{T_4+S_4}(x) + \alpha_6 \bar{F}_{T_3+S_3}(x) \bar{F}_{T_4+S_4}(x) + \alpha_7 \bar{F}_T^2(x) \bar{F}_{T_3+S_3}(x) \\
&+ \alpha_8 \bar{F}_T^2(x) \bar{F}_{T_4+S_4}(x) + \alpha_9 \bar{F}_T(x) \bar{F}_{T_3+S_3}(x) \bar{F}_{T_4+S_4}(x) + \alpha_{10} \bar{F}_T(x) \bar{F}_{T_3+S_3}(x) \bar{F}_{T_4+S_4}(x) \\
&+ \alpha_{11} \bar{F}_T^2(x) \bar{F}_{T_3+S_3}(x) \bar{F}_{T_4+S_4}(x) \Big].
\end{aligned} \tag{30}$$

By assumption T_1 and T_2 are independent (i.e. $\alpha_1 = 0$) and let us take $\alpha = \alpha_2 = \alpha_3 = \cdots = \alpha_{11}$ for simplicity, then substituting (28) and (29) into (30), we get

$$\begin{aligned}
\bar{F}_{sys,4}(x) &= 1 - \left(\frac{1}{49} e^{-3\lambda x} - \frac{68}{147} e^{-2\lambda x} - \frac{10646}{2205} e^{-\lambda x} - \frac{13522}{441} \right) \alpha^3 e^{-21\lambda x} \\
&+ \frac{e^{-\lambda x}}{11025} \Big[-(1451471\alpha + 1050)\, \alpha^2 e^{-19\lambda x} + (4412444\alpha + 20300)\, \alpha^2 e^{-18\lambda x} \\
&-(9599217\alpha + 176120)\, \alpha^2 e^{-17\lambda x} + (14510782\alpha + 897400)\, \alpha^2 e^{-16\lambda x} \\
&-\left(13443613\alpha^2 + 2939510\alpha + 1225 \right) \alpha e^{-15\lambda x} + \left(3264300\alpha^2 + 6296080\alpha + 19600 \right) \alpha e^{-14\lambda x} \\
&+\left(8178235\alpha^2 - 8254750\alpha - 134750 \right) \alpha e^{-13\lambda x} - \left(4024202\alpha^2 - 4416090\alpha - 507150 \right) \alpha e^{-12\lambda x} \\
&-\left(24299249\alpha^2 - 4633930\alpha + 1080100 \right) \alpha e^{-11\lambda x} + \left(62839976\alpha^2 - 9071580\alpha + 1081500 \right) \alpha e^{-10\lambda x} \\
&- \Big[\left(85278587\alpha^3 + 794360\alpha^2 - 335055\alpha \right) e^{-\lambda x} - 78298098\alpha^2 - 20302100\alpha + 2194395 \Big] \alpha e^{-8\lambda x} \\
&- \Big[\left(51848470\alpha^3 + 33413170\alpha^2 - 2201430\alpha - 3675 \right) e^{-\lambda x} - 25063072\alpha^3 - 31040240\alpha^2 + 224420\alpha + 36750 \Big] e^{-6\lambda x} \\
&- \Big[\left(8690913\alpha^3 + 19047910\alpha^2 + 1435945\alpha - 154350 \right) e^{-\lambda x} - 2058008\alpha^3 - 7960330\alpha^2 - 1564220\alpha + 349125 \Big] e^{-4\lambda x} \\
&- \Big[\left(299052\alpha^3 + 2213400\alpha^2 + 892605\alpha - 459375 \right) e^{-\lambda x} - 20164\alpha^3 - 375200\alpha^2 - 290920\alpha + 352800 \Big] e^{-2\lambda x} \\
&-\left(29820\alpha^2 + 28980\alpha - 147000 \right) e^{-\lambda x} - (7455\alpha + 25725) \Big].
\end{aligned} \tag{31}$$

Summing-up, from (20) and (28), one can obtain an expression for $Co_4(t)$, also, from (31), we can find $In_4(t)$. Consequently, substituting in (11), we have the system efficiency for $n = 4$ as a

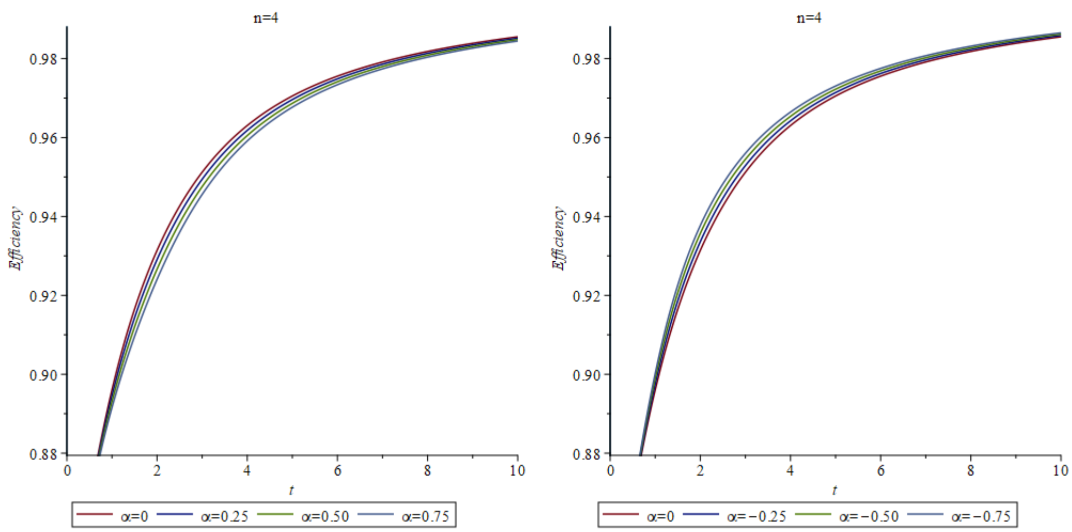

Figure 2: The plot of $E_4(t)$ for $k_1 = 10$, $k_2 = 1$, $\lambda = 1$ and some selected values of α.

function of t, α, λ, k_1 and k_2. The plot of $E_4(t)$ has been displayed in Figure 2 for some selected values of α. From Figure 2, it is observed that $E_4(t)$ is also increasing in t and decreasing in α. Also, Figures 1 and 2 indicate that $E_n(t)$ decreases as n increases.

By continuing the same process step by step, we can get an exact explicit for $E_n(t)$ when $n \geq 5$.

Example 2. (Lower bound for $E_n(t)$)

Assume that components lifetimes are exponentially distributed as $F_T(x) = 1 - \mathrm{e}^{-\lambda x}$, $x > 0$. To obtain the lower bound of $E_n(t)$, denoted by $LE_n(t)$, as appeared in (17), we should first compute $\phi_m^F(x)$ for $3 \leq m \leq n$. Let $n = 3$, then from (18), we have

$$\begin{aligned}\phi_3^F(x) &= \int_{t_3=0}^{x} \bar{F}_T^2(x-t_3) f_T(t_3) dt_3 \\ &= \int_{t_3=0}^{x} \mathrm{e}^{-2\lambda(x-t_3)} \lambda \mathrm{e}^{-\lambda t_3} dt_3 \\ &= \mathrm{e}^{-\lambda x} - \mathrm{e}^{-2\lambda x}. \end{aligned} \tag{32}$$

From (32) and (19), we have a lower bound for $E_3(t)$, as

$$LE_3(t) = \frac{2k_1(1-e^{-\lambda t})}{7k_1 + k_2 - 8k_1 e^{-\lambda} + (k_1 - k_2)e^{-2\lambda}}. \tag{33}$$

Similarly, from (18), we have

$$\begin{aligned}
\phi_4^F(x) &= \int_{t_4=0}^{x} \left[\bar{F}_T(x-t_4) + \int_{t_3=0}^{x-t_4} \bar{F}_T^2(x-t_4-t_3) f_T(t_3) dt_3\right] f_T(t_4) dt_4 \\
&= \int_{t_4=0}^{x} \left[\mathrm{e}^{-\lambda(x-t_4)} + \int_{t_3=0}^{x-t_4} \mathrm{e}^{-2\lambda(x-t_4-t_3)} \lambda \mathrm{e}^{-\lambda t_3} dt_3\right] \lambda \mathrm{e}^{-\lambda t_4} dt_4 \\
&= \mathrm{e}^{-2\lambda x} + 2\lambda x \mathrm{e}^{-\lambda x} - \mathrm{e}^{-\lambda x}.
\end{aligned} \tag{34}$$

From (17), for $n = 4$, we have

$$E_4(t) \geq \frac{k_1 \int_{x=0}^{t} \bar{F}_T(x) dx}{\int_{x=0}^{t} \left[(4k_1 + k_2(4-3))\bar{F}_T(x) + k_2[\bar{F}_T(x)]^2 + (k_1+k_2)\phi_3^F(x) + k_1\phi_4^F(x)\right] dx} \tag{35}$$

Upon substituting (32) and (34) into (35), we obtain a lower bound for $E_4(t)$, as

$$LE_4(t) = \frac{k_1\left(1-\mathrm{e}^{-\lambda t}\right)}{-2\left(3k_1+k_2\right) + k_1\lambda t)\mathrm{e}^{-\lambda t} + 2(3k_1+k_2)}.$$

Similarly, from (18), we have

$$\begin{aligned}
\phi_5^F(x) &= \int_{t_5=0}^{x} \left[\bar{F}_T(x-t_5) + \int_{t_4=0}^{x-t_5} \left[\bar{F}_T(x-t_5-t_4) + \int_{t_3=0}^{x-t_4-t_5} \bar{F}_T^2(x-t_5-t_4-t_3) f_T(t_3) dt_3\right] f_T(t_4) dt_4\right] f_T(t_5) dt_5 \\
&= \int_{t_4=0}^{x} \left[\mathrm{e}^{-\lambda(x-t_5)} + \int_{t_3=0}^{x-t_5} \left[\mathrm{e}^{-\lambda(x-t_5-t_4)} + \int_{t_3=0}^{x-t_4-t_5} \mathrm{e}^{-2\lambda(x-t_5-t_4-t_3)} \lambda \mathrm{e}^{-\lambda t_3} dt_3\right] \lambda \mathrm{e}^{-\lambda t_4} dt_4\right] \lambda \mathrm{e}^{-\lambda t_5} dt_5 \\
&= -\mathrm{e}^{-2\lambda x} + (1+\lambda^2 x^2)\mathrm{e}^{-\lambda x},
\end{aligned} \tag{36}$$

and

$$\begin{aligned}
\phi_6^F(x) &= \int_{t_6=0}^{x} \Bigg[\bar{F}_T(x-t_6) + \int_{t_5=0}^{x-t_6} \Bigg[\bar{F}_T(x-t_6-t_5) + \int_{t_4=0}^{x-t_5-t_6} \Bigg[\bar{F}_T(x-t_6-t_5-t_4) \\
&\quad + \int_{t_3=0}^{x-t_4-t_5-t_6} \bar{F}_T^2(x-t_6-t_5-t_4-t_3) f_T(t_3) dt_3\Bigg] f_T(t_4) dt_4\Bigg] f_T(t_5) dt_5\Bigg] f_T(t_6) dt_6 \\
&= \int_{t_6=0}^{x} \Bigg[\mathrm{e}^{-\lambda(x-t_6)} + \int_{t_5=0}^{x-t_6} \Bigg[\mathrm{e}^{-\lambda(x-t_6-t_5)} + \int_{t_4=0}^{x-t_5-t_6} \Bigg[\mathrm{e}^{-\lambda(x-t_6-t_5-t_4)} \\
&\quad + \int_{t_3=0}^{x-t_4-t_5-t_6} \mathrm{e}^{-2\lambda(x-t_6-t_5-t_4-t_3)} \lambda \mathrm{e}^{-\lambda t_3} dt_3\Bigg] \lambda \mathrm{e}^{-\lambda t_4} dt_4\Bigg] \lambda \mathrm{e}^{-\lambda t_5} dt_5\Bigg] \lambda \mathrm{e}^{-\lambda t_6} dt_6 \\
&= \mathrm{e}^{-2\lambda x} + (\frac{1}{3}\lambda^3 x^3 + 2\lambda x - 1)\mathrm{e}^{-\lambda x}.
\end{aligned} \tag{37}$$

Also

$$\begin{aligned}\phi_7^F(x) &= \int_{t_7=0}^{x}\Bigg[\bar{F}_T(x-t_7)+\int_{t_6=0}^{x-t_7}\Bigg[\bar{F}_T(x-t_7-t_6)+\int_{t_5=0}^{x-t_6-t_7}\Bigg[\bar{F}_T(x-t_7-t_6-t_5)\\
&+\int_{t_4=0}^{x-t_5-t_6-t_7}\Bigg[\bar{F}_T(x-t_7-t_6-t_5-t_4)+\int_{t_3=0}^{x-t_4-t_5-t_6-t_7}\bar{F}_T^2(x-t_7-t_6-t_5-t_4-t_3)f_T(t_3)dt_3\Bigg]\\
&f_T(t_4)dt_4\Bigg]f_T(t_5)dt_5\Bigg]f_T(t_6)dt_6\Bigg]f_T(t_7)dt_7\\
&= \int_{t_7=0}^{x}\Bigg[\mathrm{e}^{-\lambda(x-t_7)}+\int_{t_6=0}^{x-t_7}\Bigg[\mathrm{e}^{-\lambda(x-t_7-t_6)}+\int_{t_5=0}^{x-t_6-t_7}\Bigg[\mathrm{e}^{-\lambda(x-t_7-t_6-t_5)}\\
&+\int_{t_4=0}^{x-t_5-t_6-t_7}\mathrm{e}^{-\lambda(x-t_7-t_6-t_5-t_4)}+\int_{t_3=0}^{x-t_4-t_5-t_6-t_7}\mathrm{e}^{-2\lambda(x-t_7-t_6-t_5-t_4-t_3)}\lambda\mathrm{e}^{-\lambda t_3}dt_3\Bigg]\lambda\mathrm{e}^{-\lambda t_4}dt_4\Bigg]\\
&\lambda\mathrm{e}^{-\lambda t_5}dt_5\Bigg]\lambda\mathrm{e}^{-\lambda t_6}dt_6\Bigg]\lambda\mathrm{e}^{-\lambda t_7}dt_7\\
&= -\mathrm{e}^{-2\lambda x}+(\frac{1}{12}\lambda^4x^4+\lambda^2x^2+1)\mathrm{e}^{-\lambda x},\end{aligned}\tag{38}$$

and

$$\begin{aligned}\phi_8^F(x) &= \int_{t_8=0}^{x}\Bigg[\bar{F}_T(x-t_8)+\int_{t_7=0}^{x-t_8}\Bigg[\bar{F}_T(x-t_8-t_7)+\int_{t_6=0}^{x-t_7-t_8}\Bigg[\bar{F}_T(x-t_8-t_7-t_6)\\
&+\int_{t_5=0}^{x-t_6-t_7-t_8}\Bigg[\bar{F}_T(x-t_8-t_7-t_6-t_5)+\int_{t_4=0}^{x-t_5-t_6-t_7-t_8}\Bigg[\bar{F}_T(x-t_8-t_7-t_6-t_5-t_4)\\
&+\int_{t_3=0}^{x-t_4-t_5-t_6-t_7-t_8}\bar{F}_T^2(x-t_8-t_7-t_6-t_5-t_4-t_3)f_T(t_3)dt_3\Bigg]f_T(t_4)dt_4\Bigg]f_T(t_5)dt_5\Bigg]f_T(t_6)dt_6\Bigg]\\
&f_T(t_7)dt_7\Bigg]f_T(t_8)dt_8\\
&= \mathrm{e}^{-2\lambda x}+\left(\frac{1}{60}\lambda^5x^5+\frac{1}{3}\lambda^3x^3+2\lambda x-1\right)\mathrm{e}^{-\lambda x}.\end{aligned}\tag{39}$$

Consequently, using (32), (34), (36), (37), (38) and (39) and also (17), we obtain a lower bound for $E_n(t), n=5,6,7,8$ which are given by

$$LE_5(t)=\frac{2k_1\left(\mathrm{e}^{-\lambda t}-1\right)}{((2\lambda^2t^2+8\lambda t+20)k_1+(4\lambda t+8)k_2)\mathrm{e}^{-\lambda t}+(k_2-k_1)\mathrm{e}^{-2\lambda t}-19k_1-9k_2},$$

$$LE_6(t)=\frac{3k_1\left(\mathrm{e}^{-\lambda t}-1\right)}{(k_1\lambda^3t^3+3\lambda^2t^2(2k_1+k_2)+12\lambda t(2k_1+k_2)+42k_1+24k_2)\mathrm{e}^{-\lambda t}-42k_1-24k_2},$$

$$LE_7(t)=\frac{12k_1\left(\mathrm{e}^{-\lambda t}-1\right)}{(k_1\lambda^4t^4+4\lambda^3t^3(2k_1+k_2)+24\lambda^2t^2(2k_1+k_2)+48\lambda t(3k_1+2k_2)+240k_1+144k_2)\mathrm{e}^{-\lambda t}-A},$$

and

$$LE_8(t)=\frac{60k_1\left(\mathrm{e}^{-\lambda t}-1\right)}{(t^5\lambda^5k_1+5\lambda^4t^4(2k_1+k_2)+40\lambda^3t^3(2k_1+k_2)+120\lambda^2t^2(3k_1+2k_2)+360\lambda t(3k_1+2k_2))\mathrm{e}^{-\lambda t}+B},$$

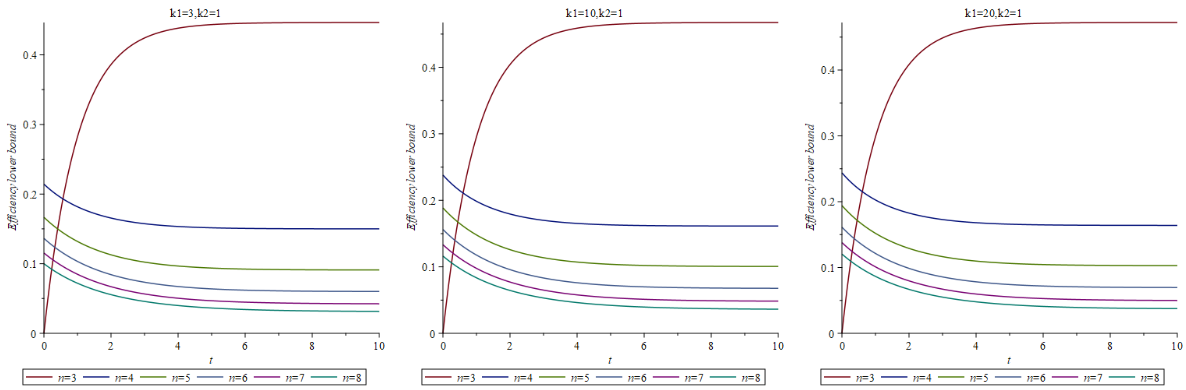

Figure 3: The plot of $LE_n(t)$ for $n=3,\ldots,8$, $k_2=1$ and some selected values of k_1.

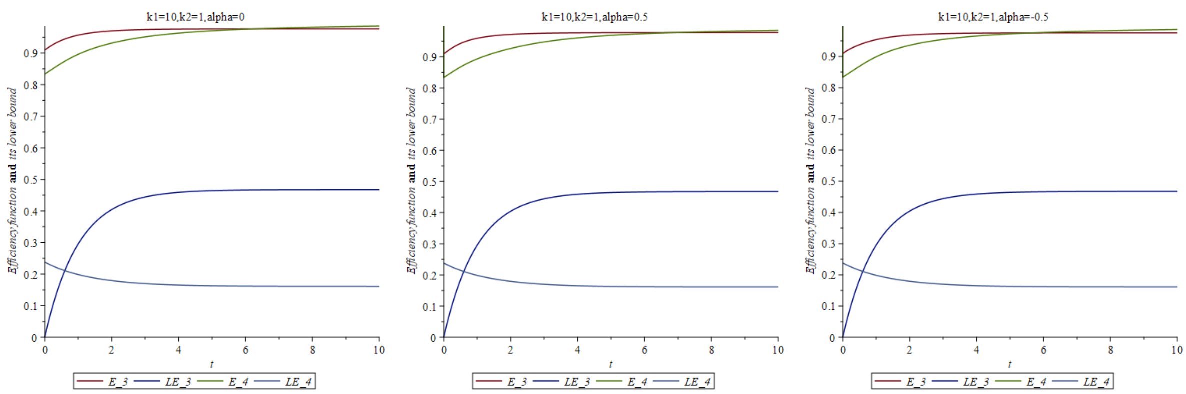

Figure 4: The plots of $E_n(t)$ and $LE_n(t)$ for $n=3$ and 4, $k_1=10$, $k_2=1$ and some selected values of α.

respectively, where

$$A = 6(k_1-k_2)\,\mathrm{e}^{-2\lambda t} + 234k_1 + 150k_2 \quad \text{and} \quad B = (1560k_1 + 1080k_2)\left(\mathrm{e}^{-\lambda t} - 1\right).$$

The plot of $LE_n(t)$ displayed in Figure 3 for $n=3,\ldots,8$, $k_2=1$ and some selected values of k_1. From this figure, it is concluded that

(i) For $n=3$, the lower bound $LE_3(t)$ is increasing in t which is obvious analytically from (33).

(ii) For $n \geq 4$, the lower bound $LE_n(t)$ is decreasing in t and n.

(iii) With the change of k_1, the monotony behavior of the graphs is almost similar.

To compare the exact value of $E_n(t)$ and its lower bound $LE_n(t)$, for $n=3$ and 4, their graphs are plotted in Figure 4. This figure shows that the difference between $E_3(t)$ and $LE_3(t)$ is decreasing in t while it increases for $n=4$.

5 Conclusion

The allocation problems of cold standby components for a $(2, n-2)$ parallel system have been studied. The components are not repairable ; once a failure occurs, it has been replaced immediately with one of the cold standby components. This causes a dependency between the lifetime of the components after the first failure. So, the copula function has been used to model the mentioned dependency. A profit function and criteria for system efficiency have been defined to find the best allocation policies of one or more standby component(s). To illustrate the proposed policy, numerical computations, and graphical analyses have been presented. Numerical results indicated that one standby component optimizes the system efficiency function.

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