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Research Article

Stability of impulsive fractional stochastic integro-differential equations with state dependent delay and Poisson jumps by using Mainardi's function

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Abstract

In this work, the stability results for a nonlinear mathematical model are derived, and the power system is realized by utilizing fractional calculus theory. The fixed point theorem is used to establish sufficient conditions for the existence of a mild solution and the stability of a nonlinear impulsive fractional stochastic integro-differential equation with state-dependent delays with Mainardi's function in a Hilbert space. Numerical simulations are provided to validate the obtained theoretical results. The proposed

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model supports (i) predicting the instability of synchronization between generators and the lines and (ii) stabilizing the disturbance that occurs in synchronization among generators and the lines.

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1 Introduction

At present, stochastic differential equations in finite and infinite dimensions have received a lot of attention because they are used in explaining a variety of phenomena in ecology, physics, electrical engineering, population dynamics biology, medicine, and other scientific, and engineering fields; for details, see [6, 7, 10, 13, 16]. Exploring fractional differential calculus [1, 4, 9, 11, 15, 17, 18, 20, 21, 24] is recognized as a crucial tool, raising the research in a different area of science, engineering, control theory, and many others. The fractional order of derivatives and integrals are two crucial aspects that fortify to enhance real-life applications, and variables in the power system change throughout time. Historically, power systems have been designed to account for system behavior. In [14], the author described random perturbation in power systems in terms of the model of a stochastic differential equation by $\frac{dx(\mathfrak{s})}{d\mathfrak{s}} = f(x,\mathfrak{s}) + \frac{d\eta(x,\mathfrak{s},\xi)}{d\mathfrak{s}}$, where x represents the state variable, and $f(x,\mathfrak{s})$ is the differential form of the trajectory function $\psi(x,\mathfrak{s})$, which is the solution of differential equation. Moreover, $\eta(x,\mathfrak{s},\xi)$ represents the stochastic variable. When fared to stochastic disturbance, the synchronization of power generators is a crucial prerequisite for the correct functioning of a power system. The fluctuations in frequency and phase angle differences between the generators are sufficiently modest. Desynchronization can occur due to severe variations, resulting in a widespread power outage. Indeed, evaluating the performance of a nonlinear stochastic system driven by Brownian motion is not enough to evaluate the stability of electrical signals in the synchronization of generators and various lines. Hence, a Poisson jump has been added to fulfill the requirement. For the sort of wide-band and excessive peak-to-average signals, it is extremely observed that memory outcomes inside the electricity amplifier become better understood. The output of an electricity amplifier is not most effective depending on the present-day input pattern but on the previous sample. Hence, the proposed model has been upgraded into a fractional order system.

Balasubramaniam et al. [2] investigated controllability for neutral stochastic differential inclusions with infinite delay in abstract space. Tan [22] studied the exponential stability of fractional stochastic differential equations. Ma, Arthi, and Anthoni [12] admitted the exponential stability behavior of neutral stochastic integro-differential equation with fractional Brownian motion and impulsive effects. Bahuguna, Sakthivel, and Chandha [4] inquired about the existence and asymptotic stability for a neutral stochastic fractional differential equation with impulses driven by the Poisson jump. The latest studies by Suganya et al.[8, 21] investigated the existence result of fractional stochastic differential and integro-differential systems with state-dependent delay in Hilbert spaces utilizing the fixed point theorems. To the best of the authors' knowledge, there are no papers available in the literature with state-dependent delay for power systems. Based on the above, the authors were inspired to consider the impulsive neutral fractional stochastic integro-differential equation (INFSIDE) driven by the Poisson jump with state-dependent delay of the following form:

$$\begin{aligned}
&\frac{d}{d\mathfrak{s}}\left[J^{1-q}\big(x(\mathfrak{s})-\mathcal{L}(\mathfrak{s},x_{\tau(\mathfrak{s},x_{\mathfrak{s}})})-\varphi(0)+\mathcal{L}(0,\varphi)\big)\right]\\
&=A\left[x(\mathfrak{s})-\mathcal{L}(\mathfrak{s},x_{\tau(\mathfrak{s},x_{\mathfrak{s}})})\right]+J_{\mathfrak{s}}^{1-q}f(\mathfrak{s},x_{\tau(\mathfrak{s},x_{\mathfrak{s}})})\\
&+\sigma(\mathfrak{s},x_{\tau(\mathfrak{s},x_{\mathfrak{s}})})\frac{dW(\mathfrak{s})}{d\mathfrak{s}}\\
&+\int_Z h(\mathfrak{s},x_{\tau(\mathfrak{s},x_{\mathfrak{s}})},\eta)\tilde{N}(d\mathfrak{s},d\eta),\quad \mathfrak{s}\in[0,\hat{a}]=J,\mathfrak{s}\neq\mathfrak{s}_k \qquad (1)
\end{aligned}$$

$$x(\mathfrak{s})=\varphi(\mathfrak{s})\in\mathbb{B},\ \mathfrak{s}\in(-\infty,0] \qquad (2)$$

$$\Delta_{\mathfrak{s}_k}^{1-q}x(\mathfrak{s}_k)=I_k(x(\mathfrak{s}_k)),\mathfrak{s}=\mathfrak{s}_k,\qquad k=1,2,\ldots,m, \qquad (3)$$

where $0<q<1, J^{1-q}$ is the $(1-q)$ order Riemann–Liouville fractional integral operator, $x(\cdot)$ takes values in a separable Hilbert space $\tilde{\mathbb{V}}$, and $A: D(A)\subset\tilde{\mathbb{V}}\to\tilde{\mathbb{V}}$ is an infinitesimal generator of strongly continuous semigroup of a bounded linear operator. The impulsive moments $\mathfrak{s}_k$ satisfy the condition $0 < \mathfrak{s}_1 < \mathfrak{s}_2 < \cdots < \mathfrak{s}_k < \cdots = \infty$, $I_k : \tilde{\mathbb{V}} \to \tilde{\mathbb{V}}$, $\Delta I^{1-q}_{\mathfrak{s}_k} x(\mathfrak{s}_k) = I^{1-q}_{\mathfrak{s}_k^+} x(\mathfrak{s}_k^+) - I^{1-q}_{\mathfrak{s}_k^-} x(\mathfrak{s}_k^-)$, $I^{1-q}_{\mathfrak{s}_k^+}(x(\mathfrak{s}_k^+)$ and $I^{1-q}_{\mathfrak{s}_k^-} x(\mathfrak{s}_k^-)$ are the right and left limits at $\mathfrak{s}_k, k = 1, 2, \ldots, m$ respectively. For any $\mathfrak{s} \in [0, \hat{a}]$ and any history function x, the element $\mathcal{PC}$ is defined by $x_{\mathfrak{s}}(\theta) = x(\mathfrak{s} + \theta)$, $-\varsigma \leq \theta \leq 0$ and belongs to the phase $\mathbb{B}$. The functions $\mathcal{L}, f : J \times \mathbb{B} \to \tilde{\mathbb{V}}, \tau : J \times \mathbb{B} \to (-\infty, T]$ are measurable in $\tilde{\mathbb{V}}$-norm, and $\sigma : J \times \mathbb{B} \to L_2^0$ is a measurable mapping, where $L_2^0 = L_2(Q^{\frac{1}{2}}\tilde{\mathbb{K}}, \tilde{\mathbb{V}})$ is the space of all Hilbert–Schmidt operators for a separable Hilbert space from $Q^{\frac{1}{2}}$ into $\tilde{\mathbb{V}}$ with norm $\|\varphi\|^p_{L_2^0} = Tr(\varphi Q \varphi^*)$.

In (1), $\tilde{N}(d\mathfrak{s}, d\eta) = N(d\mathfrak{s}, d\eta) - \gamma d\mathfrak{s}(d\eta)$ denotes the compensated Poisson measure independent of $W(\mathfrak{s})$, and $N(d\mathfrak{s}, d\eta)$ represents the Poisson counting measure associated with γ.

The rest of the paper is organized as follows: Section 2 provides some basic definitions and consents. In section 3, the results on the existence and uniqueness of mild solution exponential stability are discussed. In section 4, numerical simulations are demonstrated. Finally, in section 5, the conclusion is provided.

2 Preliminaries

The collection of all strongly measurable, p-integrable, $\tilde{\mathbb{V}}$-valued random variables denoted by $L_p(\Omega, \mathcal{F}, \mathbb{P}, \tilde{\mathbb{V}}) \equiv L_p(\Omega, \tilde{\mathbb{V}})$ is a Banach space equipped with the norm $\|x(\cdot)\|_{L_p} = (E\|x(\cdot, w)\|^p)^{\frac{1}{p}}$, where the expectation E is defined by $E(h) = \int_\Omega h(w) d\mathbb{P}$. Let $C_{1-q}(J, \tilde{\mathbb{V}}) = \{x : \mathfrak{s}^{1-q} x(t) \in C(J, \tilde{\mathbb{V}})\}$ be a Banach space with pth norm. Then $\left(\|x\|^p_{C_{1-q}}\right)^{\frac{1}{p}} = \left\{\sup_{t\in J} \mathfrak{s}^{1-q}\|x(\mathfrak{s})\|^p_{\mathbb{B}}\right\}^{\frac{1}{p}}$. Consider the piecewise continuous Banach space $PC_{1-q}(J, \tilde{\mathbb{V}}) = \{x : (\mathfrak{s} - \mathfrak{s}_k)^{1-q} x(\mathfrak{s}) \in C((\mathfrak{s}_k, \mathfrak{s}_{k+1}], \tilde{\mathbb{V}})$ and $\lim_{\mathfrak{s} \to \mathfrak{s}_k} (\mathfrak{s} - \mathfrak{s}_k)^{1-q} x(\mathfrak{s})$ exists, $k = 1, 2, \ldots, m\}$ for investigating impulsive condition with the norm

$$\|x\|^p_{PC_{1-q}(J, \tilde{\mathbb{V}})} = \max\{\sup_{\mathfrak{s} \in (\mathfrak{s}_k, \mathfrak{s}_{k+1}]} (\mathfrak{s} - \mathfrak{s}_k)^{p(1-q)} \|x(\mathfrak{s})\|^p : k = 0, 1, 2, \ldots, m\}.$$

Let $\tilde{Z}$ be the closed subspace of all continuous process $x \in \mathcal{PC}\left(J, L_p(\Omega, \tilde{\mathbb{V}})\right)$ consisting of $\mathcal{F}_{\mathfrak{s}}$-adapted measurable process, $\mathcal{F}_0$ adapted process $\phi \in L_p(\Omega, \mathbb{B})$.

Let $\|\cdot\|_{\tilde{Z}}$ be a semi-norm defined by $\|x\|_{\tilde{Z}} = \left(\sup_{\mathfrak{s}\in J} \|x_{\mathfrak{s}}\|_{\mathbb{B}}^{p}\right)^{\frac{1}{p}}$, where $\|x_{\mathfrak{s}}\|_{\mathbb{B}} \leq \overline{L}E\,\|\phi\|_{\mathbb{B}} + \overline{N}sup\{E\,\|x(s\| \,:\, 0 \leq s \leq \hat{a})\}$, with $\overline{L} = \sup_{\mathfrak{s}\in J} L(t)$ and $\overline{N} = \sup_{\mathfrak{s}\in J} N(\mathfrak{s})$. It is easy to verify that $\tilde{Z}$ furnished with the above-defined norm topology is a Banach space.

Let $(\mathbb{B}, \|\cdot\|)$ be the semi-normed linear space of phase space of measurable mapping $\mathcal{F}_0 : (-\infty, 0] \to \tilde{\mathbb{V}}$ satisfying the axioms to Renu and Dwijendra [19] as follows:

A) Suppose that $X : (-\infty, \hat{a}] \to \tilde{\mathbb{V}}, \hat{a} > 0$ is such that $x_0 \in \mathbb{B}$ and $x \in \mathcal{P}C(J, \tilde{\mathbb{V}})$ for each $\mathfrak{s} \in J$. Then the following condition hold:
(i) $x_{\mathfrak{s}} \in \mathbb{B}$, (ii) $\|x(\mathfrak{s})\| \leq \mathbb{L}\,\|x_{\mathfrak{s}}\|_{\mathbb{B}}$, (iii) $\|x(\mathfrak{s})\|_{\mathbb{B}} \leq \mathbb{H}E\,\|x_0\|_{\mathbb{B}} + \tilde{N}(\mathfrak{s})sup\{E\,\|x(\varsigma)\| \,:\, 0 \leq \varsigma \leq \hat{a}\}$, where $\mathbb{L} > 0$ is a constant, $\tilde{N}, \mathbb{H} : [0,\infty) \to [0,\infty)$, $\mathbb{N}$ is continuous, $\mathbb{H}$ is locally bounded, and $\mathbb{L}, \tilde{N}$, and $\mathbb{H}$ are independent of $x(\cdot)$.

B) The function $t \to \phi_t$ is well-defined from the set $\mathcal{R}(\tau^-) = \{\tau(\varsigma, \hat{\psi}) : (\varsigma, \hat{\psi} \in [0, \hat{a}] \times \mathbb{B}\}$ into $\mathbb{B}$, and there exists a continuous and bounded function $J^{\phi} : \mathcal{R}(\tau^-) \to (0,\infty)$ is satisfying $\|\phi_{\mathfrak{s}}\|_{\mathbb{B}} \leq J^{\phi}(\mathfrak{s})\,\|\phi\|_{\mathbb{B}}$ for every $\mathfrak{s} \in \mathcal{R}(\tau^-)$.

C) The space $\mathbb{B}$ is complete.

Lemma 1. [19] Choose $x : (-\infty, \hat{a}] \to \tilde{\mathbb{V}}$ as a function such that $x_0 = \phi$ and $x|J \in PC(J, \tilde{\mathbb{V}})$. The norm

$$\|x_t\|_{\mathbb{B}} \leq (\mathbb{H}_{\hat{a}} + J^{\phi})\,\|\phi\|_{\tilde{\mathbb{V}}} + \tilde{N}_{\hat{a}} \sup\{\|x(\tau)\| : \tau \in \mathfrak{s}, [0, \max\{0, \mathfrak{s}\}], t \in \mathcal{R}(\tau^-) \cup [0, \hat{a}]\},$$

where $J^{\phi} = \sup_{\mathfrak{s}\in\mathcal{R}(\tau^-)} J^{\phi}(\mathfrak{s})$, $\mathbb{H}_{\hat{a}} = \sup_{\mathfrak{s}\in[0,\hat{a}]} \mathbb{H}(\mathfrak{s})$, $\tilde{N}_{\hat{a}} = \sup_{\mathfrak{s}\in[0,\hat{a}]} \tilde{N}(\mathfrak{s})$.

Definition 1. [3] The Mainardi's function is defined by $M_q(\tilde{Z}) = \sum_{n=0}^{\infty} \frac{(-\tilde{Z})^n}{n!\Gamma(1-qn-q)}, 0 < q < 1, \tilde{Z} \in C$, it is clear that

$$\int_0^{\infty} M_q(\hat{r}) d\hat{r} = 1, \quad 0 < q < 1.$$

On the other hand, $M_q(\tilde{Z})$ satisfies the following equalities:

$$\int_0^\infty \frac{q}{\hat{r}^{q+1}} M_q(\frac{1}{\hat{r}^q}) e^{-\mu \hat{r}} d\hat{r} = e^{-\mu q} \quad \text{and}$$

$$\int_0^\infty \hat{r}^q M_q(\hat{r}) d\hat{r} = \frac{\gamma(q+1)}{\gamma(q\beta+1)}, \quad \beta > -1, 0 < q < 1. \tag{4}$$

Lemma 2. [3] An H-valued stochastic process $x(\mathfrak{s}) : \mathfrak{s} \in (\infty, \hat{a}]$ is said to be a mild solution of the system (1)–(3) if $\mathfrak{s}$ satisfies the following integral equation:

$$x(\mathfrak{s}) = \begin{cases} S_q(\mathfrak{s})[\varphi(0) - \mathcal{L}(0,\varphi)] + \mathcal{L}(\mathfrak{s}, x_{\tau(\mathfrak{s},x_\mathfrak{s})}) + \int_0^\mathfrak{s} S_q(\mathfrak{s}-\varsigma) f(\varsigma, x_{\tau(\varsigma,x_\varsigma)}) d\varsigma \\ + \int_0^\mathfrak{s} (\mathfrak{s}-\varsigma)^{q-1} T_q(\mathfrak{s}-\varsigma)\sigma(\varsigma, x_{\tau(\varsigma,x_\varsigma)}) dW(\varsigma) \\ + \int_0^\mathfrak{s} (\mathfrak{s}-\varsigma)^{q-1} \int_Z T_q(\mathfrak{s}-\varsigma) h(\varsigma, x_{\tau(\varsigma,x_\varsigma)}, \eta)\lambda(d\eta) d\varsigma, \quad \mathfrak{s} \in [0, \mathfrak{s}_1], \\ \vdots \qquad\qquad \vdots \qquad\qquad \vdots \\ T_q(\mathfrak{s}-\mathfrak{s}_k)(\mathfrak{s}-\mathfrak{s}_k)^{q-1}\left[x(\mathfrak{s}_k^-) + I_k(x(\mathfrak{s}_k^-))\right] + \mathcal{L}(\mathfrak{s}_k, x_{\tau(\mathfrak{s}_k, x_{\mathfrak{s}_k})}) \\ + \int_{\mathfrak{s}_k}^\mathfrak{s} S_q(\mathfrak{s}-\varsigma) f(\varsigma, x_{\tau(\varsigma,x_\varsigma)}) d\nu \\ + \int_{\mathfrak{s}_k}^\mathfrak{s} (\mathfrak{s}-\varsigma)^{q-1} T_q(\mathfrak{s}-\varsigma)\sigma(\varsigma, x_{\tau(\varsigma,x_\varsigma)}) dW(\varsigma) \\ + \int_{\mathfrak{s}_k}^\mathfrak{s} \int_Z (\mathfrak{s}-\varsigma)^{q-1} T_q(\mathfrak{s}-\varsigma) h(\varsigma, x_{\tau(\varsigma,x_\varsigma)}, \eta)\lambda(d\eta) d\varsigma, \quad \mathfrak{s} \in (\mathfrak{s}_k, \mathfrak{s}_{k+1}], \end{cases} \tag{5}$$

where $S_q(\mathfrak{s})x = \int_0^\infty M_q(\hat{r}) T(\mathfrak{s}^q \hat{r}) x d\hat{r}, \mathfrak{s} \geq 0, x \in \tilde{\mathbb{H}}$, and
$T_q(\mathfrak{s})x = \int_0^\infty q\hat{r} M_q(\hat{r}) T(\mathfrak{s}^q \hat{r}) x d\hat{r}$.

The following assumptions would have to be fulfilled to achieve the main result.

(A_1) The analytic semi-group $T(\mathfrak{s})$ generated by A is compact for $t > 0$, and there exists $\tilde{M} > 0$ such that

$$\sup_{\mathfrak{s}\geq 0} \|T(\mathfrak{s})\| \leq \tilde{M}, \qquad \mathfrak{s} \geq 0. \tag{6}$$

(A_2) The function $f : J \times \mathbb{B} \to \tilde{\mathbb{V}}$ is continuous, and there exist constants $\hat{C}_1, \hat{C}_2 > 0$ such that

$$E\|f(\mathfrak{s}, x) - f(\mathfrak{s}, y)\|^p \leq \hat{C}_1 \|x - y\|_\mathbb{B}^p$$
$$\text{and } E\|f(\mathfrak{s}, x)\|^p \leq \hat{C}_2(\|x\|_\mathbb{B}^p + 1).$$

(A_3) The function $\mathcal{L} : J \times \mathbb{B} \to \tilde{\mathbb{V}}$ is continuous, and there exist constants $\hat{C}_3, \hat{C}_4 > 0$ such that

$$E\|\mathcal{L}(\mathfrak{s}, x) - \mathcal{L}(\mathfrak{s}, y)\|^p \leq \hat{C}_3\|x - y\|_{\mathbb{B}}^p$$
$$\text{and } E\|\mathcal{L}(\mathfrak{s}, x)\|^p \leq \hat{C}_4(\|x\|_{\mathbb{B}}^p + 1).$$

(A_4) The function $\sigma : J \times \mathbb{B} \to L_2^0$ satisfies

(a) for each $\mathfrak{s} \in J,\ \sigma(\mathfrak{s}, \cdot) : \mathbb{B} \to L_2^0$ is continuous and for each $x \in \mathbb{B},\ \sigma(\cdot, x) : J \to L_2^0$ is strongly measurable.

(b) There are a positive integrable function, $m \in L^1([0, a])$ and a continuous nondecreasing function $\varrho_\sigma : [0, \infty) \to (0, \infty)$ such that for every $(\mathfrak{s}, x) \in J \times \mathbb{B}$, we have

$$\left(\int_0^{\mathfrak{s}} \left(E\|\sigma(\varsigma, x)\|_{L_2^0}^p\right)^{\frac{2}{p}} d\varsigma\right)^{\frac{p}{2}} \leq m(\mathfrak{s})\varrho_\sigma(\|x\|_{\mathbb{B}}^p),$$

$$\lim_{s\to 0} \inf \frac{\varrho_\sigma(s)}{s} = \varrho < \infty.$$

(A_5) The impulsive function $I_k : \tilde{\mathbb{V}} \to \tilde{\mathbb{V}}$ is continuous and there exist positive numbers $q_k, \overline{q}_k (k = 1, 2, \ldots)$ such that $\sum_{k=1}^{\infty} q_k < \infty$ and for all $x, y \in \tilde{\mathbb{V}}$,

$$E \left\|I_k(x) - I_k(y)\right\|^p \leq q_k \left\|x - y\right\|_{\mathbb{B}}^p \text{ and } \left\|I_k(0)\right\| = 0,$$

$$E \left\|I_k(x)\right\|^p \leq (1 + \overline{q}_k) \left\|x\right\|^p.$$

(A_6) [23] The function $h : J \times \mathbb{B} \to \tilde{\mathbb{V}}$ satisfies the following Lipschitz condition is continuous and for $x, y \in \mathbb{B}$, and there exists constants $\hat{C}_6, \hat{C}_7$ such that

$$\int_Z E \left\|h(\mathfrak{s}, x(\mathfrak{s}), \eta) - h(\mathfrak{s}, y(\mathfrak{s}), \eta)\right\|^p \lambda(d\eta)d\mathfrak{s} \leq \hat{C}_6 E \left\|x - y\right\|_{\mathbb{B}}^p,$$

$$\int_Z E \left\|h(\mathfrak{s}, x(\mathfrak{s}), \eta)\right\|^p \lambda(d\eta)d\mathfrak{s} \leq \hat{C}_7(E \left\|x\right\|_{\mathbb{B}}^p + 1).$$

(A_7)

$$4^{p-1}\tilde{N}_{\hat{a}} \max_{1\leq k\leq \tilde{n}} \Bigg\{ (\hat{C}_4 + 1) + \hat{M}^p \left(\mathfrak{s} - \mathfrak{s}_k\right)^{p-1} (\hat{C}_2 + 1)$$

$$+\frac{\hat{M}^p}{\Gamma^p(q)}\frac{\left(\mathfrak{s}-\mathfrak{s}_k\right)^{2(q-1)+1)^{\frac{p}{2}}}}{(2q-2+1)^{\frac{p}{2}}}\hat{m}(\mathfrak{s})\varrho_\sigma$$
$$+\frac{\hat{M}^p}{\Gamma^p(q)}(\hat{C}_7+1)\left[\frac{(\mathfrak{s}-\mathfrak{s}_k)^{\frac{(2pq-p-2)}{2}}}{\left(\frac{2pq-p-2}{p-2}\right)^{\frac{p-2}{2}}}+\frac{(\mathfrak{s}-\mathfrak{s}_k)^{2pq-2p+1}}{2pq-2p+1}\right]\Bigg\}<1.$$

3 Main result

Theorem 1. The existence and uniqueness of a mild solution for (1)–(3) have been obtained through hypotheses $(A_1)-(A_7)$.

Proof. Choose $\hat{B}_{\hat{a}}$ as the space of all functions $x:(-\infty,\hat{a}]$ to ensure that $x_0\in\hat{B}_{\hat{a}}$ and the condition $x\to\tilde{\mathbb{V}}$ is continuous. Let $|\cdot|_{\hat{a}}$ be the semi-norm in $\hat{B}$ specified by

$$\|x\|_{\hat{a}}=\|x_0\|_{\mathbb{B}}+\sup_{\varsigma\in[0,\hat{a}]}\left(E\|x(\varsigma)\|^p\right)^{\frac{1}{p}}=\sup_{\varsigma\in[0,\hat{a}]}\left(E\|x(\varsigma)\|^p\right)^{\frac{1}{p}}.$$

Let $\tilde{Z}_{\hat{a}}=\mathcal{P}C\big(J,L_p(\Omega;\mathbb{B})\big)$. Consider the map $\Phi:Z_{\hat{a}}\to Z_{\hat{a}}$ defined by

$$\hat{\mathcal{B}}^0_{\hat{a}}=\{x\in\hat{\mathcal{B}}_{\hat{a}};x_0=0\in\hat{\mathcal{B}}\},$$

$$(\Phi x)(\mathfrak{s})=\begin{cases} S_q(t)[\phi(0)-\mathcal{L}(0,\varphi)]+\mathcal{L}(\mathfrak{s},x_{\tau(\mathfrak{s},x_\mathfrak{s})})+\int_0^\mathfrak{s}S_q(\mathfrak{s}-\varsigma)f(\varsigma,x_{\tau(\varsigma,x_\varsigma)})d\varsigma\\ +\int_0^\mathfrak{s}(\mathfrak{s}-\varsigma)^{q-1}T_q(\mathfrak{s}-\varsigma)\sigma(\varsigma,x_{\tau(\varsigma,x_\varsigma)})dW(\varsigma)\\ +\int_0^t\int_Z(\mathfrak{s}-\varsigma)^{q-1}T_q(\mathfrak{s}-s)h(\varsigma,x_{\tau(\varsigma,x_\varsigma)},\eta)\lambda(d\eta)d\varsigma, \quad \mathfrak{s}\in[0,\mathfrak{s}_1],\\ \vdots \qquad\qquad \vdots \qquad\qquad \vdots\\ T_q(\mathfrak{s}-\mathfrak{s}_k)(\mathfrak{s}-\mathfrak{s}_k)^{q-1}\left[x(\mathfrak{s}_k^-)+I_k(x(\mathfrak{s}_k^-))\right]+\mathcal{L}(\mathfrak{s}_k,x_{\tau(\mathfrak{s}_k,x_{\mathfrak{s}_k})})\\ +\int_{\mathfrak{s}_k}^\mathfrak{s}S_q(\mathfrak{s}-\varsigma)f(\varsigma,x_{\tau(\varsigma,x_\varsigma)})d\varsigma\\ +\int_{\mathfrak{s}_k}^\mathfrak{s}(\mathfrak{s}-\varsigma)^{q-1}T_q(\mathfrak{s}-\varsigma)\sigma(\varsigma,x_{\tau(\varsigma,x_\varsigma)})dW(\varsigma)\\ +\int_{\mathfrak{s}_k}^\mathfrak{s}\int_Z(\mathfrak{s}-\varsigma)^{q-1}T_q(\mathfrak{s}-\varsigma)h(\varsigma,x_{\tau(\varsigma,x_\varsigma)},\eta)\lambda(d\eta)d\varsigma, \quad \mathfrak{s}\in(\mathfrak{s}_k,\mathfrak{s}_{k+1}]. \end{cases}$$

For $\varphi\in\tilde{Z}$, define

$$\psi(\mathfrak{s})=\begin{cases}\varphi(\mathfrak{s}), & \mathfrak{s}\in(-\infty,0],\\ S_q(\mathfrak{s})\varphi(0), & \mathfrak{s}\in J,\end{cases}\qquad \tilde{\psi}(\mathfrak{s})=\begin{cases}0, & \mathfrak{s}\le 0,\\ \psi(\mathfrak{s}), & \mathfrak{s}\in J.\end{cases}$$

Then $\tilde{\phi} \in \tilde{Z}_{\hat{a}}$. Set $x(\mathfrak{s}) = \tilde{\phi} + \psi, -\infty < \mathfrak{s} \leq \hat{a}$. Then x satisfies (1)–(3) if and only if $\psi_0 = 0$ and

$$\psi(\mathfrak{s}) = \begin{cases} -S_q(\mathfrak{s})[\varphi(0) - \mathcal{L}(0,\varphi)] + \mathcal{L}(\mathfrak{s}, \tilde{\phi}_{\tau(\mathfrak{s},\tilde{\phi}_\mathfrak{s}+\psi_\mathfrak{s})} + \psi_{\tau(\mathfrak{s},\tilde{\phi}_\mathfrak{s}+\psi_\mathfrak{s})}) \\ + \int_0^\mathfrak{s} S_q(\mathfrak{s}-\varsigma) f(\varsigma, \tilde{\phi}_{\tau(\varsigma,\tilde{\phi}_\varsigma+\psi_\varsigma)} + \psi_{\tau(\varsigma,\tilde{\phi}_\varsigma+\psi_\varsigma)}) d\varsigma \\ + \int_0^t (\mathfrak{s}-\varsigma)^{q-1} T_q(\mathfrak{s}-\varsigma) \sigma(\varsigma, \tilde{\phi}_{\tau(\varsigma,\tilde{\phi}_\varsigma+\psi_\varsigma)} + \psi_{\tau(\varsigma,\tilde{\phi}_\varsigma+\psi_\varsigma)}) dW(\varsigma) \\ + \int_0^t \int_Z (\mathfrak{s}-\varsigma)^{q-1} T_q(\mathfrak{s}-\varsigma) h(\varsigma, \tilde{\phi}_{\tau(\varsigma,\tilde{\phi}_\varsigma+\psi_\varsigma)} + \psi_{\tau(\varsigma,\tilde{\phi}_\varsigma+\psi_\varsigma)}, \eta) \lambda(d\eta) d\varsigma, \\ \qquad t \in [0, \mathfrak{s}_1], \\ \vdots \qquad \vdots \qquad \vdots \\ T_q(\mathfrak{s}-\mathfrak{s}_k)(\mathfrak{s}-\mathfrak{s}_k)^{q-1} \left[(\tilde{\phi}(\mathfrak{s}_k^-) + \psi(\mathfrak{s}_k^-) + I_k(\tilde{\phi}(\mathfrak{s}_k^-) + \psi(\mathfrak{s}_k^-)) \right] \\ + \mathcal{L}(\mathfrak{s}_k, \tilde{\phi}_{\tau(\mathfrak{s}_k,\tilde{\phi}_{\mathfrak{s}_k})} + \psi_{\tau(\mathfrak{s}_k,\psi_{\mathfrak{s}_k})}) \\ + \int_{\mathfrak{s}_k}^t S_q(\mathfrak{s}-\varsigma) f(\varsigma, \tilde{\phi}_{\tau(\varsigma,\tilde{\phi}_\varsigma+\psi_\varsigma)} + \psi_{\tau(\varsigma,\tilde{\phi}_\varsigma+\psi_\varsigma)}) d\varsigma \\ + \int_{\mathfrak{s}_k}^t (\mathfrak{s}-\varsigma)^{q-1} T_q(\mathfrak{s}-\varsigma) \sigma(\varsigma, \tilde{\phi}_{\tau(\varsigma,\tilde{\phi}_\varsigma+\psi_\varsigma)} + \psi_{\tau(\varsigma,\tilde{\phi}_\varsigma+\psi_\varsigma)}) dW(\varsigma) \\ + \int_{\mathfrak{s}_k}^\mathfrak{s} \int_Z (\mathfrak{s}-\varsigma)^{q-1} T_q(\mathfrak{s}-\varsigma) h(\varsigma, \tilde{\phi}_{\tau(\varsigma,\tilde{\phi}_\varsigma+\psi_\varsigma)} + \psi_{\tau(\varsigma,\tilde{\phi}_\varsigma+\psi_\varsigma)}, \eta) \lambda(d\eta) d\varsigma, \\ \qquad \mathfrak{s} \in (\mathfrak{s}_k, \mathfrak{s}_{k+1}]. \end{cases}$$

Let $\hat{\mathcal{B}}_{\hat{a}}^0 = \{\tilde{Z} \in \hat{\mathcal{B}}_{\hat{a}}; \tilde{Z}_0 = 0 \in \hat{\mathcal{B}}\}$. For any $\tilde{Z} \in \hat{\mathcal{B}}_{\hat{a}}^0$, one can have

$$\left\|\tilde{Z}\right\|_{\hat{a}} = \left\|\tilde{Z}_0\right\|_{\mathbb{B}} + \sup_{\varsigma\in[0,\hat{a}]} (E \left\|\tilde{Z}(\varsigma)\right\|^p)^{\frac{1}{p}} = \sup_{\varsigma\in[0,\hat{a}]} (E \left\|\tilde{Z}(\varsigma)\right\|^p)^{\frac{1}{p}}.$$

Thus, if $\tilde{Z}_{\hat{a}}^0 = \mathcal{P}C((-\infty,\hat{a}], L_p(\Omega, \mathbb{B}_{\hat{a}}^0))$, then $(\tilde{Z}_{\hat{a}}^0, \|\cdot\|_{\hat{a}})$ is a Banach space. For each $\vartheta \geq 0$ set $\hat{\mathcal{B}}_\vartheta = \{\tilde{Z} \in \tilde{Z}_{\hat{a}}^0 : \|Z\|_{\hat{a}}^p \leq \vartheta\}$. Then it is clear that $\hat{\mathcal{B}}_\vartheta$ is a bounded, closed, and convex set in $Z_{\hat{a}}^0$. From (4) and [3], one have $\|S_q(t)\| \leq M$, $\|T_q(\mathfrak{s})\| \leq \frac{M}{\Gamma(q)}$. For $\tilde{Z} \in \mathbb{B}_\vartheta$,

$$\begin{aligned} &\left\|\tilde{\phi}_{\tau(\mathfrak{s},\tilde{\phi}+\psi_\mathfrak{s})} + \psi_{\tau(\mathfrak{s},\tilde{\phi}+\psi_\mathfrak{s}))}\right\|_{\mathbb{B}}^p \\ &\leq 2^{p-1} \left\{ \left\|\tilde{\phi}_{\tau(t,\tilde{\phi}+\psi_\mathfrak{s})}\right\|_{\mathbb{B}}^p + \left\|\psi_{\tau(\mathfrak{s},\tilde{\phi}+\psi_\mathfrak{s}))}\right\|_{\mathbb{B}}^p \right\} \\ &\leq 4^{p-1} \Bigg\{ (\mathbb{H}_{\hat{a}} + J^\varphi)^p E \left\|\tilde{\phi}_0\right\|_{\mathbb{B}}^p + \tilde{N}_{\hat{a}}^p \sup_{0\leq\varsigma\leq\hat{a}} \left\|\tilde{\phi}(\varsigma)\right\|_{\hat{a}}^p \\ &\qquad + (\mathbb{H}_{\hat{a}} + J^\varphi)^p E \left\|\psi_0\right\|_{\mathbb{B}}^p + \tilde{N}_{\hat{a}}^p \sup_{0\leq\varsigma\leq\hat{a}} \left\|\psi(\varsigma)\right\|_{\hat{a}}^p \Bigg\} \end{aligned} \tag{7}$$

$$\leq 4^{p-1}\left\{\tilde{N}_{\hat{a}}^{p}\sup_{0\leq\varsigma\leq\hat{a}}\|\psi(\varsigma)\|_{\hat{a}}^{p}+\tilde{N}_{\hat{a}}^{p}\tilde{M}^{p}\left\|\tilde{\phi}(0)\right\|_{\hat{a}}^{p}+(\mathbb{H}_{\hat{a}}+J^{\varphi})^{p}E\left\|\tilde{\phi}\right\|_{\mathbb{B}}^{p}\right\}$$

$$\leq 4^{p-1}\left\{\tilde{N}_{\hat{a}}^{p}\vartheta+\tilde{N}^{p}\tilde{M}^{p}\mathbb{L}^{p}+(\mathbb{H}_{\hat{a}}+J^{\varphi})^{p}\left\|\tilde{\phi}\right\|_{\mathbb{B}}^{p}\right\}=\hat{\vartheta},\ \mathfrak{s}\in[0,\hat{a}]. \tag{8}$$

Let the operator $\Theta:\tilde{Z}_{\hat{a}}^{0}\to\tilde{Z}_{\hat{a}}^{0}$ be defined by

$$(\Theta\psi)(\mathfrak{s})=\begin{cases}-S_q(\mathfrak{s})[\varphi(0)-\mathcal{L}(0,\varphi)]+\mathcal{L}(\mathfrak{s},\tilde{\phi}_{\tau(\mathfrak{s},\tilde{\phi}_\varsigma+\psi_\mathfrak{s})}+\psi_{\tau(\mathfrak{s},\tilde{\phi}_\mathfrak{s}+\psi_\mathfrak{s})})\\ +\int_0^{\mathfrak{s}}S_q(\mathfrak{s}-\varsigma)f(\varsigma,\tilde{\phi}_{\tau(\varsigma,\tilde{\phi}_\varsigma+\psi_\varsigma)}+\psi_{\tau(\varsigma,\tilde{\phi}_\varsigma+\psi_\varsigma)})d\varsigma\\ +\int_0^{\mathfrak{s}}(\mathfrak{s}-\varsigma)^{q-1}T_q(\mathfrak{s}-\varsigma)\sigma(\varsigma,\tilde{\phi}_{\tau(\varsigma,\tilde{\phi}_\varsigma+\psi_\varsigma)}+\psi_{\tau(\varsigma,\tilde{\phi}_\varsigma+\psi_\varsigma)})dW(\varsigma)\\ +\int_0^{\mathfrak{s}}\int_Z(\mathfrak{s}-\varsigma)^{q}T_q(\mathfrak{s}-\varsigma)h(\varsigma,\tilde{\phi}_{\tau(\varsigma,\tilde{\phi}_\varsigma+\psi_\varsigma)}\\ +\psi_{\tau(\varsigma,\tilde{\phi}_\varsigma+\psi_\varsigma)},\eta)\lambda(d\eta)d\varsigma,\qquad \mathfrak{s}\in[0,\mathfrak{s}_1],\\ \vdots\qquad\qquad\vdots\qquad\qquad\vdots\\ T_q(\mathfrak{s}-\mathfrak{s}_k)(\mathfrak{s}-\mathfrak{s}_k)^{q-1}\left[(\tilde{\phi}(\mathfrak{s}_k^-)+\psi(t_k^-)+I_k(\tilde{\phi}(\mathfrak{s}_k^-)+\psi(\mathfrak{s}_k^-))\right]\\ +\mathcal{L}(\mathfrak{s}_k,\tilde{\phi}_{\tau(\mathfrak{s}_k,\tilde{\phi}_{\mathfrak{s}_k})}+\psi_{\tau(\mathfrak{s}_k,\psi_{\mathfrak{s}_k})})+\int_{\mathfrak{s}_k}^{\mathfrak{s}}S_q(\mathfrak{s}-\varsigma)f(\varsigma,\tilde{\phi}_{\tau(\varsigma,\tilde{\phi}_\varsigma+\psi_\varsigma)}\\ +\psi_{\tau(\varsigma,\tilde{\phi}_\varsigma+\psi_\varsigma)})d\varsigma\\ +\int_{\mathfrak{s}_k}^{\mathfrak{s}}(\mathfrak{s}-\varsigma)^{q-1}T_q(\mathfrak{s}-\varsigma)\sigma(\varsigma,\tilde{\phi}_{\tau(\varsigma,\tilde{\phi}_\varsigma+\psi_\varsigma)}+\psi_{\tau(\varsigma,\tilde{\phi}_\varsigma+\psi_\varsigma)})dW(\varsigma)\\ +\int_{\mathfrak{s}_k}^{t}\int_Z(\mathfrak{s}-\varsigma)^{q}T_q(\mathfrak{s}-\varsigma)h(\varsigma,\tilde{\phi}_{\tau(\varsigma,\tilde{\phi}_\varsigma+\psi_\varsigma)}\\ +\psi_{\tau(\varsigma,\tilde{\phi}_\varsigma+\psi_\varsigma)},\eta)\lambda(d\eta)d\varsigma,\qquad \mathfrak{s}\in(\mathfrak{s}_k,\mathfrak{s}_{k+1}].\end{cases}$$

Now, split the proof into three steps.

Step I: Θ is continuous. Choose ψ^n is a sequence satisfying $\psi^n\to\psi$ in $\mathcal{PC}(J,L_2(\Omega,\mathbb{H}))$ as $n\to\infty$. Then there exist $\hat{r}>0$ such that $\|\psi^n(\mathfrak{s})\|\leq\hat{r}$ for all ϑ and a.s. $\mathfrak{s}\in[0,\hat{a}]$. By using (7), one can have $\left\|\tilde{\phi}_{\tau(\varsigma,\tilde{\phi}_\varsigma+\psi_\varsigma)}+\psi_{\tau(\varsigma,\tilde{\phi}_\varsigma+\psi_\varsigma)}\right\|\leq\hat{\vartheta}$.

Using lemma, one can get

$$\begin{aligned}&\left\|\psi^n_{\tau(\mathfrak{s},\tilde{\phi}_\mathfrak{s}+\psi^n_\mathfrak{s})}-\psi_{\tau(\mathfrak{s},\ \tilde{\phi}_\mathfrak{s}+\psi_\mathfrak{s})}\right\|^p\\ &\leq 2^{p-1}\left[(\tilde{N}(\mathfrak{s}))^p\sup_{\varsigma\in[0,\mathfrak{s}]}\left\{\|\psi^n(\varsigma)-\psi(\varsigma)\|^p\right\}+(\mathbb{H}(\mathfrak{s}))^p\|\psi_0^n-\psi_0\|_{\mathbb{B}}^p\right]\\ &\leq 2^{p-1}\tilde{N}(\mathfrak{s})^p\|\psi^n(\varsigma)-\psi(\varsigma)\|^p\}\to 0,\ \text{as } n\to\infty.\end{aligned}$$

Through the Caratheodory continuity of functions f,σ, and h, one can have

$$\lim_{n\to\infty} f\Big(\varsigma, \tilde{\phi}_{\tau\left(\varsigma,\tilde{\phi}_\varsigma+\psi^n_\varsigma\right)} + {\psi^n}_{\tau\left(\varsigma,\tilde{\phi}_\varsigma+\psi^n_\varsigma\right)}\Big) = f\Big(\varsigma, \tilde{\phi}_{\tau\left(\varsigma,\tilde{\phi}_\varsigma+\psi_\varsigma\right)} + \psi_{\tau\left(\varsigma,\tilde{\phi}_\varsigma+\psi_\varsigma\right)}\Big),$$
$$\lim_{n\to\infty} \sigma\Big(\varsigma, \tilde{\phi}_{\tau\left(\varsigma,\tilde{\phi}_\varsigma+\psi^n_\varsigma\right)} + {\psi^n}_{\tau(\varsigma,\tilde{\phi}_\varsigma+\psi^n_\varsigma)}\Big) = \sigma\Big(\varsigma, \tilde{\phi}_{\tau\left(\varsigma,\tilde{\phi}_\varsigma+\psi_\varsigma\right)} + \psi_{\tau\left(\varsigma,\tilde{\phi}_\varsigma+\psi_\varsigma\right)}\Big),$$
$$\lim_{n\to\infty} h\Big(\varsigma, \tilde{\phi}_{\tau(\varsigma,\tilde{\phi}_\varsigma+\psi^n_\varsigma)} + {\psi^n}_{\tau\left(\varsigma,\tilde{\phi}_\varsigma+\psi^n_\varsigma\right)}, \eta\Big) = h\Big(\varsigma, \tilde{\phi}_{\tau\left(\varsigma,\tilde{\phi}_\varsigma+\psi_\varsigma\right)} + \psi_{\tau\left(\varsigma,\tilde{\phi}_\varsigma+\psi_\varsigma\right)}, \eta\Big),$$

$$\begin{aligned}
&E\left\|(\Theta\psi^n)(\mathfrak{s}) - (\Theta\psi)(\mathfrak{s})\right\|^p \\
&\leq 4^{p-1}\Bigg\{E\left\|\mathcal{L}\Big(\mathfrak{s}, \tilde{\phi}_{\tau\left(\mathfrak{s},\tilde{\phi}_\mathfrak{s}+\psi^n_\mathfrak{s}\right)} + {\psi^n}_{\tau\left(\mathfrak{s},\tilde{\phi}_\mathfrak{s}+\psi^n_\mathfrak{s}\right)}\Big)\right. \\
&\left.-\mathcal{L}\Big(\mathfrak{s}, \tilde{\phi}_{\tau\left(\mathfrak{s},\tilde{\phi}_\mathfrak{s}+\psi_\mathfrak{s}\right)} + \psi_{\tau\left(t,\tilde{\phi}_\mathfrak{s}+\psi_\mathfrak{s}\right)}\Big)\right\|^p \\
&\quad + E\left\|\int_0^\mathfrak{s} S_q(\mathfrak{s}-\varsigma)\Big[f\Big(\varsigma, \tilde{\phi}_{\tau\left(\varsigma,\tilde{\phi}_\varsigma+\psi^n_\varsigma\right)} + {\psi^n}_{\tau\left(\varsigma,\tilde{\phi}_\varsigma+\psi^n_\varsigma\right)}\Big)\right. \\
&\quad \left.-f\Big(\varsigma, \tilde{\phi}_{\tau\left(\varsigma,\tilde{\phi}_\varsigma+\psi_\varsigma\right)} + \psi_{\tau\left(\varsigma,\tilde{\phi}_\varsigma+\psi_\varsigma\right)}\Big)\Big]d\varsigma\right\|^p \\
&E\left\|\int_0^\mathfrak{s} S_q(\mathfrak{s}-\varsigma)\Big[\sigma\Big(\varsigma, \tilde{\phi}_{\tau\left(\varsigma,\tilde{\phi}_\varsigma+\psi^n_\varsigma\right)} + {\psi^n}_{\tau\left(\varsigma,\tilde{\phi}_\varsigma+\psi^n_\varsigma\right)}\Big)\right. \\
&\quad \left.-\sigma\Big(\varsigma, \tilde{\phi}_{\tau(\varsigma,\tilde{\phi}_\varsigma+\psi_\varsigma)} + \psi_{\tau\left(\varsigma,\tilde{\phi}_\varsigma+\psi_\varsigma\right)}\Big)\Big]dW(\varsigma)\right\|^p \\
&\quad + E\left\|\int_0^t T_q(\mathfrak{s}-\varsigma)(\mathfrak{s}-\varsigma)^{q-1}\Big[h\Big(\varsigma, \tilde{\phi}_{\tau\left(\varsigma,\tilde{\phi}_\varsigma+\psi^n_\varsigma\right)} + {\psi^n}_{\tau\left(\varsigma,\tilde{\phi}_\varsigma+\psi^n_\varsigma\right)}, \eta\Big)\right. \\
&\qquad \left.-h\Big(\varsigma, \tilde{\phi}_{\tau\left(\varsigma,\tilde{\phi}_\varsigma+\psi_\varsigma\right)} + \psi_{\tau\left(\varsigma,\tilde{\phi}_\varsigma+\psi_\varsigma\right)}, \eta\Big)\Big]\lambda(d\eta)d\varsigma\right\|^p\Bigg\} \\
&\leq 4^{p-1}\Bigg\{C_4 E\left\|\Big(\tilde{\phi}_{\tau(\mathfrak{s},\tilde{\phi}_\mathfrak{s}+\psi^n_\mathfrak{s})} + {\psi^n}_{\tau(\mathfrak{s},\tilde{\phi}_\mathfrak{s}+\psi^n_\mathfrak{s})}\Big) - \Big(\tilde{\phi}_{\tau(\mathfrak{s},\tilde{\phi}_\mathfrak{s}+\psi_\mathfrak{s})} + \psi_{\tau(\mathfrak{s},\tilde{\phi}_\mathfrak{s}+\psi_\mathfrak{s})}\Big)\right\|^p \\
&\quad + \hat{M}^p\left(\int_0^\mathfrak{s} d\varsigma\right)^{p-1} E\left\|f(\tilde{\phi}_{\tau(\varsigma,\tilde{\phi}_\varsigma+\psi^n_\varsigma)} + \psi^n_{\tau(\varsigma,\tilde{\phi}_\varsigma+\psi^n_\varsigma)})\right. \\
&\left.-f(\tilde{\phi}_{\tau(\varsigma,\tilde{\phi}_\varsigma+\psi_\varsigma)} + \psi_{\tau(\varsigma,\tilde{\phi}_\varsigma+\psi_\varsigma)})\right\|^p d\varsigma + \frac{\hat{M}^p}{\Gamma^p(q)}\left(\int_0^\mathfrak{s}(\mathfrak{s}-\varsigma)^{2(q-1)}d\varsigma\right)^{\frac{p}{2}} \\
&\quad \times\Bigg[\Bigg(E\left\|\int_0^\mathfrak{s}\sigma(\tilde{\phi}_{\tau(\varsigma,\tilde{\phi}_\varsigma+\psi^n_\varsigma)} + \psi^n_{\tau(\varsigma,\tilde{\phi}_\varsigma+\psi^n_\varsigma)})\right. \\
&\quad \left.-\sigma(\tilde{\phi}_{\tau(\varsigma,\tilde{\phi}_\varsigma+\psi_\varsigma)} + \psi_{\tau(\varsigma,\tilde{\phi}_\varsigma+\psi_\varsigma)})\right\|^p\Bigg)^{\frac{2}{p}}d\varsigma\Bigg]^{\frac{p}{2}}
\end{aligned}$$

$$
\begin{aligned}
&+\frac{\hat{M}^p}{\Gamma^p(q)}\hat{C}_7\Bigg[\Big(\int_0^{\mathfrak{s}}(\mathfrak{s}-\varsigma)^{\frac{2p(q-1)}{p-2}}d\varsigma\Big)^{\frac{p-2}{2}}\\
&\times\int_0^{\mathfrak{s}}E\left\|\tilde{\phi}_{\tau(\varsigma,\tilde{\phi}_\varsigma+\psi^n_\varsigma)}+\psi^n_{\tau(\varsigma,\tilde{\phi}+\psi^n_\varsigma)}-(\tilde{\phi}_{\tau(\varsigma,\tilde{\phi}_\varsigma+\psi_\varsigma)}+\psi_{\tau(\varsigma,\tilde{\phi}_\varsigma+\psi_\varsigma)})\right\|^p d\varsigma\\
&+\int_0^{\mathfrak{s}}(\mathfrak{s}-\varsigma)^{2p(q-1)}E\left\|\tilde{\phi}_{\tau(\varsigma,\tilde{\phi}_\varsigma+\psi^n_\varsigma)}+\psi^n_{\tau(\varsigma,\psi^n_\varsigma)}-(\tilde{\phi}_{\tau(\varsigma,\tilde{\phi}_\varsigma+\psi_\varsigma)}\right.\\
&\left.+\psi_{\tau(\varsigma,\tilde{\phi}_\varsigma+\psi_\varsigma)})\right\|^p d\varsigma\Bigg]\Bigg\}\\
\le{}& 4^{p-1}\Bigg\{\hat{C}_4(E\left\|\tilde{\phi}^n_{\tau(\varsigma,\tilde{\phi}_\varsigma+\psi^n_\varsigma)}+\psi^n_{\tau(\varsigma,\tilde{\phi}_\varsigma+\psi^n_\varsigma)}-(\tilde{\phi}_{\tau(\varsigma,\tilde{\phi}_\varsigma+\psi_\varsigma)}+\psi_{\tau(\varsigma,\tilde{\phi}_\varsigma+\psi_\varsigma)})\right\|^p)\\
&+\hat{M}^p\,(\mathfrak{s})^{p-1}\,(\hat{C}_2+1)\int_0^{\mathfrak{s}}E\left\|\tilde{\phi}_{\tau(\varsigma,\tilde{\phi}_\varsigma+\psi^n_\varsigma)}+\psi^n_{\tau(\varsigma,\tilde{\phi}_\varsigma+\psi^n_\varsigma)}\right.\\
&\left.-(\tilde{\phi}_{\tau(\varsigma,\tilde{\phi}_\varsigma+\psi_\varsigma)}+\psi_{\tau(\varsigma,\tilde{\phi}_\varsigma+\psi_\varsigma)})\right\|^p d\varsigma\\
&+\frac{\hat{M}^p}{\Gamma^p(q)}\frac{\big((\mathfrak{s})^{2(q-1)+1}\big)^{\frac{p}{2}}}{(2q-2+1)^{\frac{p}{2}}}\int_0^{\mathfrak{s}}\hat{m}(\varsigma)\varrho_\sigma\\
&\times E\left\|\tilde{\phi}_{\tau(\varsigma,\tilde{\phi}_\varsigma+\psi^n_\varsigma)}+\psi^n_{\tau(\varsigma,\tilde{\phi}_\varsigma+\psi^n_\varsigma)}-(\tilde{\phi}_{\tau(\varsigma,\tilde{\phi}_\varsigma+\psi_\varsigma)}+\psi_{\tau(\varsigma,\tilde{\phi}_\varsigma+\psi_\varsigma)})\right\|^p d\varsigma\\
&+\frac{\hat{M}^p}{\Gamma^p(q)}\hat{C}_7\Bigg[\frac{(t)^{\frac{(2pq-p-2)}{2}}}{\big(\frac{2pq-p-2}{p-2}\big)^{\frac{p-2}{2}}}\int_0^t E\left\|\tilde{\phi}_{\tau(\varsigma,\tilde{\phi}_\varsigma+\psi^n_\varsigma)}+\psi^n_{\tau(\varsigma,\tilde{\phi}_\varsigma+\psi^n_\varsigma)}\right.\\
&\left.-(\tilde{\phi}_{\tau(\varsigma,\tilde{\phi}_\varsigma+\psi_\varsigma)}+\psi_{\tau(\varsigma,\tilde{\phi}_\varsigma+\psi_\varsigma})\right\|^p d\varsigma\\
&+\int_0^{\mathfrak{s}}(\mathfrak{s}-\varsigma)^{2p(q-1)}E\left\|\tilde{\phi}_{\tau(\varsigma,\tilde{\phi}_\varsigma+\psi^n_\varsigma)}+\psi^n_{\tau(\varsigma,\tilde{\phi}_\varsigma+\psi^n_\varsigma)}\right.\\
&\left.-(\tilde{\phi}_{\tau(\varsigma,\tilde{\phi}_\varsigma+\psi_\varsigma)}+\psi_{\tau(\varsigma,\tilde{\phi}_\varsigma+\psi_\varsigma)})\right\|^p d\varsigma\Bigg]\Bigg\}.
\end{aligned}
$$

By the Lebesgue dominated theorem, one can verify that
$E\left\|(\Theta\psi^n)(\mathfrak{s})-(\Theta\psi)(\mathfrak{s})\right\|^p\to 0$ as $n\to\infty$.

Thus, $\Theta\psi$ is continuous. For every $\mathfrak{s}\in(\mathfrak{s}_k,\mathfrak{s}_{k+1}]$, one can have

$$
\begin{aligned}
&E\left\|(\Theta\psi^n)(\mathfrak{s})-(\Theta\psi)(\mathfrak{s})\right\|^p\\
\le{}& 6^{p-1}\Bigg\{E\left\|T_q(\mathfrak{s}-\mathfrak{s}_k)(\mathfrak{s}-\mathfrak{s}_k)^{q-1}\Big[(\tilde{\phi}_{\mathfrak{s}_k^-}+\psi_{\mathfrak{s}_k^-})+I_k(\tilde{\phi}_{\mathfrak{s}_k^-}+\psi_{\mathfrak{s}_k^-})\Big]\right\|^p\\
&+E\left\|\mathcal{L}\Big(t,\tilde{\phi}_{\tau\left(\mathfrak{s},\tilde{\phi}_\varsigma+\psi^n_\varsigma\right)}+\psi^n{}_{\tau\left(\mathfrak{s},\tilde{\phi}_{\mathfrak{s}}+\psi^n_{\mathfrak{s}}\right)}\Big)-\mathcal{L}\Big(\mathfrak{s},\tilde{\phi}_{\tau\left(\mathfrak{s},\tilde{\phi}_{\mathfrak{s}}+\psi_{\mathfrak{s}}\right)}+\psi_{\tau\left(t,\tilde{\phi}_\varsigma+\psi_{\mathfrak{s}}\right)}\Big)\right\|^p
\end{aligned}
$$

$$
\begin{aligned}
&+E\left\|\int_{\mathfrak{s}_k}^{\mathfrak{s}} S_q(\mathfrak{s}-\varsigma)\left[f\Big(\varsigma,\tilde{\phi}_{\tau\left(\varsigma,\tilde{\phi}_\varsigma+\psi^n_\varsigma\right)}+\psi^n{}_{\tau\left(\varsigma,\tilde{\phi}_\varsigma+\psi^n_\varsigma\right)}\Big)\right.\right.\\
&\qquad\left.\left.-f\Big(\varsigma,\tilde{\phi}_{\tau\left(\varsigma,\tilde{\phi}_\varsigma+\psi_\varsigma\right)}+\psi_{\tau\left(\varsigma,\tilde{\phi}_\varsigma+\psi_\varsigma\right)}\Big)\right]d\varsigma\right\|^p\\
&+E\left\|\int_{\mathfrak{s}_k}^{t} T_q(\mathfrak{s}-\varsigma)(\mathfrak{s}-\varsigma)^{q-1}\left[\sigma\Big(\varsigma,\tilde{\phi}_{\tau\left(\varsigma,\tilde{\phi}_\varsigma+\psi^n_\varsigma\right)}+\psi^n{}_{\tau\left(\varsigma,\tilde{\phi}_\varsigma+\psi^n_\varsigma\right)}\Big)\right.\right.\\
&\qquad\left.\left.-\sigma\Big(\varsigma,\tilde{\phi}_{\tau\left(\varsigma,\tilde{\phi}_\varsigma+\psi_\varsigma\right)}+\psi_{\tau\left(\varsigma,\tilde{\phi}_\varsigma+\psi_\varsigma\right)}\Big)\right]dW(\varsigma)\right\|^p\\
&+E\left\|\int_{\mathfrak{s}_k}^{t} T_q(\mathfrak{s}-\varsigma)(\mathfrak{s}-\varsigma)^{q-1}\left[h\Big(\varsigma,\tilde{\phi}_{\tau\left(\varsigma,\tilde{\phi}_\varsigma+\psi^n_\varsigma\right)}+\psi^n{}_{\tau\left(\varsigma,\tilde{\phi}_\varsigma+\psi^n_\varsigma\right)},\eta\Big)\right.\right.\\
&\qquad\left.\left.-h\Big(\varsigma,\tilde{\phi}_{\tau\left(\varsigma,\tilde{\phi}_\varsigma+\psi_\varsigma\right)}+\psi_{\tau\left(\varsigma,\tilde{\phi}_\varsigma+\psi_\varsigma\right)},\eta\Big)\right]\lambda(d\eta)d\varsigma\right\|^p\Bigg\}\\
&\leq 6^{p-1}\Bigg\{\frac{\hat{M}^p}{\Gamma^p(q)}E\left\|(\mathfrak{s}-\mathfrak{s}_k)^{q-1}\left[(\tilde{\phi}_{\mathfrak{s}_k^-}+\psi_{\mathfrak{s}_k^-})+I_k(\tilde{\phi}_{\mathfrak{s}_k^-}+\psi_{\mathfrak{s}_k^-})\right]\right\|^p\\
&\quad+\hat{C}_4E\left\|\left[\Big(\tilde{\phi}_{\tau\left(\mathfrak{s},\tilde{\phi}_\varsigma+\psi^n_\varsigma\right)}+\psi^n{}_{\tau\left(\mathfrak{s},\tilde{\phi}_\varsigma+\psi^n_\varsigma\right)}\Big)-\Big(\tilde{\phi}_{\tau\left(\mathfrak{s},\tilde{\phi}_\varsigma+\psi_\varsigma\right)}+\psi_{\tau\left(\mathfrak{s},\tilde{\phi}_\varsigma+\psi_\varsigma\right)}\Big)\right]\right\|^p\\
&\quad+\hat{M}^p\left(\int_{\mathfrak{s}_k}^{\mathfrak{s}}d\varsigma\right)^{p-1}E\left\|\left[f\Big(\varsigma,\tilde{\phi}_{\tau\left(\varsigma,\tilde{\phi}_\varsigma+\psi^n_\varsigma\right)}+\psi^n{}_{\tau(\varsigma,\tilde{\phi}_\varsigma+\psi^n_\varsigma)}\Big)\right.\right.\\
&\qquad\left.\left.-f\Big(\varsigma,\tilde{\phi}_{\tau\left(\varsigma,\tilde{\phi}_\varsigma+\psi_\varsigma\right)}+\psi_{\tau\left(\varsigma,\tilde{\phi}_\varsigma+\psi_\varsigma\right)}\Big)\right]\right\|^p d\varsigma\\
&\quad+\frac{\hat{M}^p}{\Gamma^p(q)}\left(\int_{\mathfrak{s}_k}^{t}(\mathfrak{s}-\varsigma)^{2(q-1)}d\varsigma\right)^{\frac{p}{2}}\Bigg[\Bigg(E\Bigg\|\int_{\mathfrak{s}_k}^{\mathfrak{s}}\big[\sigma(\varsigma,\tilde{\phi}_{\tau(\varsigma,\tilde{\phi}_\varsigma+\psi^n_\varsigma)}+\psi^n{}_{\tau(\varsigma,\tilde{\phi}_\varsigma+\psi^n_\varsigma)})\\
&\qquad-\sigma(\varsigma,\tilde{\phi}_{\tau(\varsigma,\tilde{\phi}_\varsigma+\psi_\varsigma)}+\psi_{\tau(\varsigma,\tilde{\phi}_\varsigma+\psi_\varsigma)})\big]\Bigg\|^p\Bigg)^{\frac{2}{p}}d\varsigma\Bigg]^{\frac{p}{2}}\\
&\quad+\frac{\hat{M}^p}{\Gamma^p(q)}\hat{C}_7\Bigg[\Big(\int_0^t(\mathfrak{s}-\varsigma)^{\frac{2p(q-1)}{p-2}}d\varsigma\Big)^{\frac{p-2}{2}}\\
&\quad\times\int_0^t E\left\|\tilde{\phi}_{\tau\left(\varsigma,\tilde{\phi}_\varsigma+\psi^n_\varsigma\right)}+\psi^n_{\tau\left(\varsigma,\tilde{\phi}_\varsigma+\psi^n_\varsigma\right)}-\Big(\tilde{\phi}_{\tau\left(\varsigma,\tilde{\phi}_\varsigma+\psi_\varsigma\right)}+\psi_{\tau\left(\varsigma,\tilde{\phi}_\varsigma+\psi_\varsigma\right)}\Big)\right\|^p d\varsigma\\
&\quad+\int_{\mathfrak{s}_k}^{t}(\mathfrak{s}-\varsigma)^{2p(q-1)}E\left\|\tilde{\phi}_{\tau\left(\varsigma,\tilde{\phi}_\varsigma+\psi^n_\varsigma\right)}+\psi^n_{\tau\left(\varsigma,\tilde{\phi}_\varsigma+\psi^n_\varsigma\right)}\right.\\
&\quad\left.-\Big(\tilde{\phi}_{\tau\left(\varsigma,\tilde{\phi}_\varsigma+\psi_\varsigma\right)}+\psi_{\tau\left(\varsigma,\tilde{\phi}_\varsigma+\psi_\varsigma\right)}\Big)\right\|^p d\varsigma\Bigg]\Bigg\}
\end{aligned}
$$

$$
\begin{aligned}
&\le 6^{p-1}\Bigg\{\frac{\hat{M}^p}{\Gamma^p(q)}E\left\|(\mathfrak{s}-\mathfrak{s}_k)^{q-1}\left[(\tilde{\phi}_{\mathfrak{s}_k^-}+\psi_{\mathfrak{s}_k^-})+I_k(\tilde{\phi}_{\mathfrak{s}_k^-}+\psi_{\mathfrak{s}_k^-})\right]\right\|^p \\
&+\hat{C}_4\left\|\left[\left(\tilde{\phi^n}_{\tau\left(\mathfrak{s}_k,\tilde{\phi}_{\mathfrak{s}_k}+\psi^n_{\mathfrak{s}_k}\right)}+\psi^n{}_{\tau\left(\mathfrak{s}_k,\tilde{\phi}_{\mathfrak{s}_k}+\psi^n_{\mathfrak{s}_k}\right)}\right)\right.\right.\\
&\left.\left.-(\tilde{\phi}_{\tau\left(\mathfrak{s}_k,\tilde{\phi}_{\mathfrak{s}_k}+\psi_{\mathfrak{s}_k}\right)}+\psi_{\tau\left(\mathfrak{s}_k,\tilde{\phi}+\psi_{\mathfrak{s}_k}\right)}\Big)\right]\right\|^p \\
&+\hat{M}^p\left(\mathfrak{s}-\mathfrak{s}_k\right)^{p-1}(\hat{C}_2+1)\\
&\times\int_{\mathfrak{s}_k}^{\mathfrak{s}}\left\|\tilde{\phi}_{\tau\left(\mathfrak{s},\tilde{\phi}_\varsigma+\psi^n_\varsigma\right)}+\psi^n_{\tau\left(\mathfrak{s},\tilde{\phi}_\varsigma+\psi^n_\varsigma\right)}-\left(\tilde{\phi}_{\tau\left(\varsigma,\tilde{\phi}_\varsigma+\psi_\varsigma\right)}+\psi_{\tau\left(\varsigma,\tilde{\phi}_\varsigma+\psi_\varsigma\right)}\right)\right\|^p d\varsigma\\
&+\frac{\hat{M}^p}{\Gamma^p(q)}\frac{\left(\mathfrak{s}-\mathfrak{s}_k)^{2(q-1)+1}\right)^{\frac{p}{2}}}{(2q-2+1)^{\frac{p}{2}}}\\
&\times\left[\left(\int_{\mathfrak{s}_k}^{t}\hat{m}(\varsigma)\varrho_\sigma\left\|\tilde{\phi}_{\tau\left(\varsigma,\tilde{\phi}_\varsigma+\psi^n_\varsigma\right)}+\psi^n_{\tau\left(\varsigma,\tilde{\phi}_\varsigma+\psi^n_\varsigma\right)}\right.\right.\right.\\
&\left.\left.\left.-\left(\tilde{\phi}_{\tau\left(\varsigma,\tilde{\phi}_\varsigma+\psi_\varsigma\right)}+\psi_{\tau\left(\varsigma,\tilde{\phi}_\varsigma+\psi_\varsigma\right)}\right)\right\|^p\right)^{\frac{2}{p}}d\varsigma\right]^{\frac{p}{2}}\\
&+\frac{\hat{M}^p}{\Gamma^p(q)}\hat{C}_7\left[\frac{(\mathfrak{s}-\mathfrak{s}_k)^{\frac{(2pq-p-2)}{2}}}{\left(\frac{2pq-p-2}{p-2}\right)^{\frac{p-2}{2}}}\right.\\
&\times\int_{\mathfrak{s}_k}^{t}\left\|\tilde{\phi}_{\tau\left(\varsigma,\tilde{\phi}_\varsigma+\psi^n_\varsigma\right)}+\psi^n_{\tau\left(\varsigma,\tilde{\phi}_\varsigma+\psi^n_\varsigma\right)}-\left(\tilde{\phi}_{\tau\left(\varsigma,\tilde{\phi}_\varsigma+\psi_\varsigma\right)}+\psi_{\tau\left(\varsigma,\tilde{\phi}_\varsigma+\psi_\varsigma\right)}\right)\right\|^p d\varsigma\\
&+\int_{\mathfrak{s}_k}^{t}(\mathfrak{s}-\varsigma)^{2p(q-1)}\\
&\left.\times E\left\|\tilde{\phi}_{\tau\left(\varsigma,\tilde{\phi}_\varsigma+\psi^n_\varsigma\right)}+\psi^n_{\tau\left(\varsigma,\tilde{\phi}_\varsigma+\psi^n_\varsigma\right)}-\left(\tilde{\phi}_{\tau\left(\varsigma,\tilde{\phi}_\varsigma+\psi_\varsigma\right)}+\psi_{\tau\left(\varsigma,\tilde{\phi}_\varsigma+\psi_\varsigma\right)}\right)\right\|^p d\varsigma\right]\Bigg\}.
\end{aligned}
$$

By the Lebesgue dominated theorem, one can prove that

$$E\left\|(\Theta\psi^n)(\mathfrak{s})-(\Theta\psi)(\mathfrak{s})\right\|^p \to 0$$

as $n\to\infty$.

Hence Θ is continuous.

Step II: One can verify that $\Theta(\hat{\mathcal{B}}_\vartheta)\subset(\hat{\mathcal{B}}_\vartheta)$. There exists $\vartheta>0$ such that $E\left\|(\Theta\psi)(\mathfrak{s})\right\|^p\le\vartheta$. Let us contradictory assume that for each $\vartheta>0$, there exist $\psi\in\hat{\mathcal{B}}_\vartheta$ and $t\in[0,\hat{a}]$ such that $\vartheta<E\left\|(\Theta\psi)(\mathfrak{s})\right\|^p$. Now, for $\mathfrak{s}\in[0,\mathfrak{s}_1]$ and $\psi\in\hat{\mathcal{B}}_\vartheta$, one can have

$$
\begin{aligned}
\vartheta < E\left\|(\Theta\psi)(\mathfrak{s})\right\|^p \leq 5^{p-1}\Big\{ & -E\left\|S_q(\mathfrak{s})\mathcal{L}(0,\varphi(0))\right\|^p \\
& + E\left\|\mathcal{L}\Big(\mathfrak{s},\tilde{\phi}_{\tau\left(\mathfrak{s},\tilde{\phi}_{\mathfrak{s}}+\psi_{\mathfrak{s}}\right)}+\psi_{\tau\left(\mathfrak{s},\tilde{\phi}_{\mathfrak{s}}+\psi_{\mathfrak{s}}\right)}\Big)\right\|^p \\
& + E\left\|\int_0^{\mathfrak{s}} S_q(\mathfrak{s}-\varsigma)f\Big(\varsigma,\tilde{\phi}_{\tau\left(\varsigma,\tilde{\phi}_{\varsigma}+\psi_{\varsigma}\right)}+\psi_{\tau\left(\varsigma,\tilde{\phi}_{\varsigma}+\psi_{\varsigma}\right)}\Big)d\varsigma\right\|^p \\
& + E\left\|\int_0^{\mathfrak{s}} (\mathfrak{s}-\varsigma)^{q-1}T_q(\mathfrak{s}-\varsigma)\sigma\Big(\varsigma,\tilde{\phi}_{\tau\left(\varsigma,\tilde{\phi}_{\varsigma}+\psi_{\varsigma}\right)}+\psi_{\tau\left(\varsigma,\tilde{\phi}_{\varsigma}+\psi_{\varsigma}\right)}\Big)dW(\varsigma)\right\|^p \\
& + E\left\|\int_0^{\mathfrak{s}}\int_Z (\mathfrak{s}-\varsigma)^{q-1}T_q(\mathfrak{s}-\varsigma)h\Big(\varsigma,\tilde{\phi}_{\tau\left(\varsigma,\tilde{\phi}_{\varsigma}+\psi_{\varsigma}\right)}+\psi_{\tau\left(\varsigma,\tilde{\phi}_{\varsigma}+\psi_{\varsigma}\right)},\eta)(d\eta\Big)d\varsigma\right\|^p\Big\} \\
\leq 5^{p-1}\Big\{ & -\hat{M}^pC_4\left\|\varphi(0)\right\|^p + \hat{C}_4\left\|\tilde{\phi}_{\tau\left(\mathfrak{s},\tilde{\phi}_{\varsigma}+\psi_{\varsigma}\right)}+\psi_{\tau\left(t,\tilde{\phi}_{\varsigma}+\psi_{\varsigma}\right)}\right\|^p \\
& + \hat{M}^p\left(\int_0^{\mathfrak{s}} d\varsigma\right)^{p-1} E\left\|f\Big(\varsigma,\tilde{\phi}_{\tau\left(\varsigma,\tilde{\phi}_{\varsigma}+\psi_{\varsigma}\right)}+\psi_{\tau\left(\varsigma,\tilde{\phi}_{\varsigma}+\psi_{\varsigma}\right)}\Big)\right\|^p d\varsigma \\
& + \frac{\hat{M}^p}{\Gamma^p(q)}\left(\int_0^{\mathfrak{s}}(\mathfrak{s}-\varsigma)^{2(q-1)}d\varsigma\right)^{\frac{p}{2}} \\
& \times\Big[\Big(E\left\|\int_0^{\mathfrak{s}}\sigma\Big(\varsigma,\tilde{\phi}_{\tau\left(\varsigma,\tilde{\phi}_{\varsigma}+\psi_{\varsigma}\right)}+\psi_{\tau\left(\varsigma,\tilde{\phi}_{\varsigma}+\psi_{\varsigma}\right)}\Big)\right\|^p\Big)^{\frac{2}{p}}d\varsigma\Big]^{\frac{p}{2}} \\
& + \frac{\hat{M}^p}{\Gamma^p(q)}\hat{C}_7\Bigg[\Big(\int_0^{\mathfrak{s}}(\mathfrak{s}-\varsigma)^{\frac{2p(q-1)}{p-2}}d\varsigma\Big)^{\frac{p-2}{2}}\int_0^t E\left\|\tilde{\phi}_{\tau\left(\varsigma,\tilde{\phi}_{\varsigma}+\psi_{\varsigma}\right)}+\psi_{\tau\left(\varsigma,\tilde{\phi}_{\varsigma}+\psi_{\varsigma}\right)}\right\|^p d\varsigma \\
& + \int_0^t(\mathfrak{s}-\varsigma)^{2p(q-1)}E\left\|\tilde{\phi}_{\tau\left(\varsigma,\tilde{\phi}_{\varsigma}+\psi_{\varsigma}\right)}+\psi_{\tau\left(\varsigma,\tilde{\phi}_{\varsigma}+\psi_{\varsigma}\right)}\right\|^p d\varsigma\Bigg]\Big\} \\
\leq 5^{p-1}\Big\{ & -\hat{M}^pC_4(\left\|\varphi\right\|^p) + \hat{C}_4\left\|\tilde{\phi}_{\tau\left(t,\tilde{\phi}_{\varsigma}+\psi_{\varsigma}\right)}+\psi_{\tau\left(\mathfrak{s},\tilde{\phi}_{\varsigma}+\psi_{\varsigma}\right)}\right\|^p \\
& + \hat{M}^p(\mathfrak{s})^{p-1}(\hat{C}_2+1) \\
& \times\int_0^{\mathfrak{s}}\left\|\tilde{\phi}_{\tau\left(\varsigma,\tilde{\phi}_{\varsigma}+\psi_{\varsigma}\right)}+\psi_{\tau\left(\varsigma,\tilde{\phi}_{\varsigma}+\psi_{\varsigma}\right)}\right\|^p d\varsigma + \frac{\hat{M}^p}{\Gamma^p(q)}\frac{\left(\mathfrak{s}\right)^{2(q-1)+1)^{\frac{p}{2}}}}{(2q-2+1)^{\frac{p}{2}}} \\
& \times\int_0^{\mathfrak{s}}\hat{m}(\varsigma)\varrho_\sigma\left\|\tilde{\phi}_{\tau\left(\varsigma,\tilde{\phi}_{\varsigma}+\psi_{\varsigma}\right)}+\psi_{\tau\left(\varsigma,\tilde{\phi}_{\varsigma}+\psi_{\varsigma}\right)}\right\|^p d\varsigma \\
& + \frac{\hat{M}^p}{\Gamma^p(q)}\hat{C}_7\Bigg[\frac{(\mathfrak{s})^{\frac{(2pq-p-2)}{2}}}{\left(\frac{2pq-p-2}{p-2}\right)^{\frac{p-2}{2}}}\int_0^{\mathfrak{s}}\left\|\tilde{\phi}_{\tau\left(\varsigma,\tilde{\phi}_{\varsigma}+\psi_{\varsigma}\right)}+\psi_{\tau\left(\varsigma,\tilde{\phi}_{\varsigma}+\psi_{\varsigma}\right)}\right\|^p d\varsigma \\
& + \int_0^{\mathfrak{s}}(\mathfrak{s}-\varsigma)^{2p(q-1)}E\left\|\tilde{\phi}_{\tau\left(\varsigma,\tilde{\phi}_{\varsigma}+\psi_{\varsigma}\right)}+\psi_{\tau\left(\varsigma,\tilde{\phi}_{\varsigma}+\psi_{\varsigma}\right)}\right\|^p d\varsigma\Bigg]\Big\}.
\end{aligned}
$$

Furthermore, for $\mathfrak{s} \in (\mathfrak{s}_k, \mathfrak{s}_{k+1}]$,

$$
\begin{aligned}
&\vartheta < E\left\|(\Theta\psi)(t)\right\|^p \\
&\leq 6^{p-1}\Bigg\{E\left\|T_q(\mathfrak{s}-\mathfrak{s}_k)(\mathfrak{s}-\mathfrak{s}_k)^{q-1}\left(\tilde{\phi}_{\tau\left(\mathfrak{s}_k,\tilde{\phi}_{\mathfrak{s}_k}+\psi_{\mathfrak{s}_k}\right)}+\psi_{\tau\left(\mathfrak{s}_k,\tilde{\phi}_{\mathfrak{s}_k}+\psi_{\mathfrak{s}_k}\right)}\right)\right\|^P \\
&\quad + E\left\|T_q(\mathfrak{s}-\mathfrak{s}_k)(\mathfrak{s}-\mathfrak{s}_k)^{q-1}I_k\left(\tilde{\phi}_{\tau\left(\mathfrak{s}_k,\tilde{\phi}_{\mathfrak{s}_k}+\psi_{\mathfrak{s}_k}\right)}+\psi_{\tau\left(\mathfrak{s}_k,\tilde{\phi}_{\mathfrak{s}_k}+\psi_{\mathfrak{s}_k}\right)}\right)\right\|^p \\
&\quad + E\left\|\mathcal{L}\Big(t,\tilde{\phi}_{\tau\left(\mathfrak{s},\tilde{\phi}_{\mathfrak{s}}+\psi_{\mathfrak{s}}\right)}+\psi_{\tau\left(\mathfrak{s},\tilde{\phi}_{\mathfrak{s}}+\psi_{\mathfrak{s}}\right)}\Big)\right\|^p \\
&\quad + E\left\|\int_{\mathfrak{s}_k}^{\mathfrak{s}} S_q(\mathfrak{s}-\varsigma)\left[f\Big(\varsigma,\tilde{\phi}_{\tau\left(\varsigma,\tilde{\phi}_{\varsigma}+\psi_{\varsigma}\right)}+\psi_{\tau\left(\varsigma,\tilde{\phi}_{\varsigma}+\psi_{\varsigma}\right)}\Big)\right]d\varsigma\right\|^p \\
&\quad + E\left\|\int_{\mathfrak{s}_k}^{\mathfrak{s}}(\mathfrak{s}-\varsigma)^{q-1}T_q(\mathfrak{s}-\varsigma)\sigma\Big(\varsigma,\tilde{\phi}_{\tau\left(\varsigma,\tilde{\phi}_{\varsigma}+\psi_{\varsigma}\right)}+\psi_{\tau\left(\varsigma,\tilde{\phi}_{\varsigma}+\psi_{\varsigma}\right)}\Big)dW(\varsigma)\right\|^p \\
&\quad + E\left\|\int_{\mathfrak{s}_k}^{\mathfrak{s}}\int_Z(\mathfrak{s}-\varsigma)^{q-1}T_q(\mathfrak{s}-\varsigma)h\Big(\varsigma,\tilde{\phi}_{\tau\left(\varsigma,\tilde{\phi}_{\varsigma}+\psi_{\varsigma}\right)}+\psi_{\tau\left(\varsigma,\tilde{\phi}_{\varsigma}+\psi_{\varsigma}\right)},\eta\Big)(d\eta)d\varsigma\right\|^p\Bigg\} \\
&\leq 6^{p-1}\Bigg\{-\hat{M}^pC_4(\|\varphi(0)\|^p)+\hat{C}_4\left\|\tilde{\phi}_{\tau\left(\mathfrak{s},\tilde{\phi}_{\mathfrak{s}}+\psi_{\mathfrak{s}}\right)}+\psi_{\tau\left(\mathfrak{s},\tilde{\phi}_{\mathfrak{s}}+\psi_{\mathfrak{s}}\right)}\right\|^p \\
&\quad + \hat{M}^p\left(\int_{\mathfrak{s}_k}^t d\varsigma\right)^{p-1}E\left\|f\Big(\varsigma,\tilde{\phi}_{\tau\left(\varsigma,\tilde{\phi}_{\varsigma}+\psi_{\varsigma}\right)}+\psi_{\tau\left(\varsigma,\tilde{\phi}_{\varsigma}+\psi_{\varsigma}\right)}\Big)\right\|^p d\varsigma \\
&\quad + \frac{\hat{M}^p}{\Gamma^p(q)}\left(\int_{\mathfrak{s}_k}^t(\mathfrak{s}-\varsigma)^{2(q-1)}d\varsigma\right)^{\frac{p}{2}}\left[\left(E\left\|\int_{\mathfrak{s}_k}\sigma\Big(\varsigma,\tilde{\phi}_{\tau\left(\varsigma,\tilde{\phi}_{\varsigma}+\psi_{\varsigma}\right)}+\psi_{\tau\left(\varsigma,\tilde{\phi}_{\varsigma}+\psi_{\varsigma}\right)}\Big)\right\|^p\right)^{\frac{2}{p}}d\varsigma\right]^{\frac{p}{2}} \\
&\quad + \frac{\hat{M}^p}{\Gamma^p(q)}(\hat{C}_7+1)\Bigg[\left(\int_0^{\mathfrak{s}}(\mathfrak{s}-\varsigma)^{\frac{2p(q-1)}{p-2}}d\varsigma\right)^{\frac{p-2}{2}}\int_0^t E\left\|\tilde{\phi}_{\tau\left(\varsigma,\tilde{\phi}_{\varsigma}+\psi_{\varsigma}\right.}+\psi_{\tau(\varsigma,\tilde{\phi}_{\varsigma}+\psi_{\varsigma})}\right\|^p\Bigg)^{\frac{2}{p}}d\varsigma\Bigg]^{\frac{p}{2}} \\
&\quad + \int_{\mathfrak{s}_k}^{\mathfrak{s}}(\mathfrak{s}-\varsigma)^{2p(q-1)}E\left\|\tilde{\phi}_{\tau\left(\varsigma,\tilde{\phi}_{\varsigma}+\psi_{\varsigma}\right)}+\psi_{\tau\left(\varsigma,\tilde{\phi}_{\varsigma}+\psi_{\varsigma}\right)}\right\|^p d\varsigma\Bigg]\Bigg\} \\
&\leq 6^{p-1}\Bigg\{\frac{\hat{M}^p}{\Gamma^p(q)}E\Bigg\|(\mathfrak{s}-\mathfrak{s}_k)^{q-1}\Bigg[\left(\tilde{\phi}_{\tau\left(\mathfrak{s}_k,\tilde{\phi}_{\mathfrak{s}_k}+\psi_{\mathfrak{s}_k}\right)}+\psi_{\tau\left(\mathfrak{s}_k,\tilde{\phi}_{\mathfrak{s}_k}+\psi_{\mathfrak{s}_k}\right)}\right) \\
&\quad +I_k\left(\tilde{\phi}_{\tau(\mathfrak{s}_k,\tilde{\phi}_{\mathfrak{s}_k}+\psi_{\mathfrak{s}_k})}+\psi_{\tau(\mathfrak{s}_k,\tilde{\phi}_{\mathfrak{s}_k}+\psi_{\mathfrak{s}_k})}\right)\Bigg]\Bigg\|^p \\
&\quad + (\hat{C}_4+1)\left\|\tilde{\phi}_{\tau(\mathfrak{s},\tilde{\phi}_{\mathfrak{s}})}+\psi_{\tau(\mathfrak{s},\psi_{\mathfrak{s}})}\right\|^p+\hat{M}^p(\mathfrak{s}-\mathfrak{s}_k)^{p-1}(\hat{C}_2+1) \\
&\quad \times\int_{\mathfrak{s}_k}^{\mathfrak{s}}\left\|\tilde{\phi}_{\tau(\mathfrak{s},\tilde{\phi}_{\mathfrak{s}})}+\psi_{\tau(\mathfrak{s},\psi_{\mathfrak{s}})}\right\|^p d\varsigma+\frac{\hat{M}^p}{\Gamma^p(q)}\frac{\left((\mathfrak{s}-\mathfrak{s}_k)^{2(q-1)+1}\right)^{\frac{p}{2}}}{(2q-2+1)^{\frac{p}{2}}} \\
&\quad \times\left[\left(\int_{\mathfrak{s}_k}^{\mathfrak{s}}\hat{m}(\varsigma)\varrho_\sigma\left\|\tilde{\phi}_{\tau(\varsigma,\tilde{\phi}_{\varsigma})}+\psi_{\tau(\varsigma,\psi_{\varsigma})}\right\|^p\right)^{\frac{2}{p}}d\varsigma\right]^{\frac{p}{2}} \\
&\quad + \frac{\hat{M}^p}{\Gamma^p(q)}(\hat{C}_7+1)\left[\frac{(\mathfrak{s}-\mathfrak{s}_k)^{\frac{(2pq-p-2)}{2}}}{\left(\frac{2pq-p-2}{p-2}\right)^{\frac{p-2}{2}}}\int_{\mathfrak{s}_k}^{\mathfrak{s}}\left\|\tilde{\phi}_{\tau(\varsigma,\tilde{\phi}_{\varsigma}+\psi_{\varsigma}}+\psi_{\tau(\varsigma,\tilde{\phi}_{\varsigma}+\psi_{\varsigma})}\right\|^p d\varsigma\right.
\end{aligned}
$$

$$+\int_{\mathfrak{s}_k}^{\mathfrak{s}}(\mathfrak{s}-\varsigma)^{2p(q-1)}E\left\|\tilde{\phi}_{\tau(\varsigma,\tilde{\phi}_\varsigma}+\psi_{\tau(\varsigma,\psi_\varsigma)}\right\|^p d\varsigma\Bigg]\Bigg\}.$$

Hence for every $t\in[0,\hat{a}]$, one can have

$$\begin{aligned}
&\vartheta < E\left\|(\Theta\psi)(\mathfrak{s})\right\|^p(\mathbb{H}+J^{\varphi})^p\\
&\leq \tilde{C}^*+6^{p-1}\Bigg\{\frac{\hat{M}^p(\mathfrak{s}-\mathfrak{s}_k)^{p(q-1)}}{\Gamma^p(q)}E\Bigg\|4^{p-1}\Bigg[\tilde{N}^p_{\hat{a}}\vartheta+\tilde{N}^p\tilde{M}^p\mathbb{L}^p+(\mathbb{H}_{\hat{a}}+J^{\varphi})^p\left\|\tilde{\phi}\right\|^p_{\mathbb{B}}\Bigg]\\
&\quad+\overline{q}_k4^{p-1}\Bigg[\tilde{N}^p_{\hat{a}}\vartheta+\tilde{N}^p\tilde{M}^p\mathbb{L}^p+(\mathbb{H}_{\hat{a}}+J^{\varphi})^p\left\|\tilde{\phi}\right\|^p_{\mathbb{B}}\Bigg]\Bigg\|^p\\
&\quad+\hat{C}_44^{p-1}\Bigg[\tilde{N}^p_{\hat{a}}\vartheta+\tilde{N}^p\tilde{M}^p\mathbb{L}^p+(\mathbb{H}_{\hat{a}}+J^{\varphi})^p\left\|\tilde{\phi}\right\|^p_{\mathbb{B}}\Bigg]+\hat{M}^p\left(\mathfrak{s}-\mathfrak{s}_k\right)^{p-1}\hat{C}_2\\
&\quad\times\int_{\mathfrak{s}_k}^{t}4^{p-1}\Bigg[\tilde{N}^p_{\hat{a}}\vartheta+\tilde{N}^p\tilde{M}^p\mathbb{L}^p+(\mathbb{H}_{\hat{a}}+J^{\varphi})^p\left\|\tilde{\phi}\right\|^p_{\mathbb{B}}\Bigg]d\varsigma+\frac{\hat{M}^p}{\Gamma^p(q)}\frac{\left((\mathfrak{s}-\mathfrak{s}_k)^{2(q-1)+1}\right)^{\frac{p}{2}}}{(2q-2+1)^{\frac{p}{2}}}\\
&\quad\times\Bigg[\Bigg(\int_{\mathfrak{s}_k}^{\mathfrak{s}}\hat{m}(\varsigma)\varrho_\sigma4^{p-1}\Bigg[\tilde{N}^p_{\hat{a}}\vartheta+\tilde{N}^p\tilde{M}^p\mathbb{L}^p+(\mathbb{H}_{\hat{a}}+J^{\varphi})^p\left\|\tilde{\phi}\right\|^p_{\mathbb{B}}\Bigg]\Bigg)^{\frac{2}{p}}d\varsigma\Bigg]^{\frac{p}{2}}\\
&\quad+\frac{\hat{M}^p}{\Gamma^p(q)}\hat{C}_7\Bigg[\frac{(\mathfrak{s}-\mathfrak{s}_k)^{\frac{(2pq-p-2)}{2}}}{\left(\frac{2pq-p-2}{p-2}\right)^{\frac{p-2}{2}}}\int_{\mathfrak{s}_k}^{t}4^{p-1}\Bigg[\tilde{N}^p_{\hat{a}}\vartheta+\tilde{N}^p\tilde{M}^p\mathbb{L}^p+(\mathbb{H}_{\hat{a}}+J^{\varphi})^p\left\|\tilde{\phi}\right\|^p_{\mathbb{B}}\Bigg]d\varsigma\\
&\quad+\int_{\mathfrak{s}_k}^{t}(\mathfrak{s}-\varsigma)^{2p(q-1)}4^{p-1}\Bigg[\tilde{N}^p_{\hat{a}}\vartheta+\tilde{N}^p\tilde{M}^p\mathbb{L}^p+(\mathbb{H}_{\hat{a}}+J^{\varphi})^p\left\|\tilde{\phi}\right\|^p_{\mathbb{B}}\Bigg]d\varsigma\Bigg]\Bigg\},
\end{aligned}$$

$$\begin{aligned}
&\text{where } \tilde{C}^*=6^{p-1}\Bigg\{\frac{\hat{M}^p(\mathfrak{s}-\mathfrak{s}_k)^{p(q-1)}}{\Gamma^p(q)}\overline{q}_k+\hat{C}_4+\hat{M}^p\left(\mathfrak{s}-\mathfrak{s}_k\right)^{p-1}\hat{C}_2\\
&\quad+\frac{\hat{M}^p}{\Gamma^p(q)}\frac{\left((\mathfrak{s}-\mathfrak{s}_k)^{2(q-1)+1}\right)^{\frac{p}{2}}}{(2q-2+1)^{\frac{p}{2}}}\Bigg[\Bigg(\int_{\mathfrak{s}_k}^{\mathfrak{s}}\hat{m}(\varsigma)\varrho_\sigma\Bigg)^{\frac{2}{p}}d\varsigma\Bigg]^{\frac{p}{2}}\\
&\quad+\frac{\hat{M}^p}{\Gamma^p(q)}\hat{C}_7\Bigg[\frac{(\mathfrak{s}-\mathfrak{s}_k)^{\frac{(2pq-p-2)}{2}}}{\left(\frac{2pq-p-2}{p-2}\right)^{\frac{p-2}{2}}}\int_{\mathfrak{s}_k}^{t}d\varsigma+\int_{\mathfrak{s}_k}^{\mathfrak{s}}(\mathfrak{s}-\varsigma)^{2p(q-1)}\Bigg]\Bigg\}.
\end{aligned}$$

Next, dividing both sides of (8) through ϑ and $\vartheta\to\infty$, one can obtained that

$$\begin{aligned}
1<4^{p-1}\tilde{N}_{\hat{a}}\max_{1\leq k\leq\tilde{n}}&\Bigg\{(\hat{C}_4+1)+\hat{M}^p\left(\mathfrak{s}-\mathfrak{s}_k\right)^{p-1}(\hat{C}_2+1)\\
&+\frac{\hat{M}^p}{\Gamma^p(q)}\frac{\left((\mathfrak{s}-\mathfrak{s}_k)^{2(q-1)+1}\right)^{\frac{p}{2}}}{(2q-2+1)^{\frac{p}{2}}}\hat{m}(t)\varrho_\sigma
\end{aligned}$$

$$+\frac{\hat{M}^p}{\Gamma^p(q)}(\hat{C}_7+1)\left[\frac{(\mathfrak{s}-\mathfrak{s}_k)^{\frac{(2pq-p-2)}{2}}}{\left(\frac{2pq-p-2}{p-2}\right)^{\frac{p-2}{2}}}+\frac{(\mathfrak{s}-\mathfrak{s}_k)^{2pq-2p+1}}{2pq-2p+1}\right]\Bigg\},$$

That contradicts our assumption (A_7). Thus there exists a positive constant $\vartheta > 0$ such that $\Theta(\hat{\mathcal{B}}_\vartheta) \subset (\hat{\mathcal{B}}_\vartheta)$. Then one can conclude that the system (1)–(3) has a mild solution.

Step III: To prove the uniqueness. For $t \in [0, \mathfrak{s}_1]$,

$$\begin{aligned}
&E\left\|(\Theta\psi)(\mathfrak{s})-(\Theta\psi^*)(\mathfrak{s})\right\|^p\\
&\leq 4^{p-1}\Bigg\{E\left\|\mathcal{L}\Big(\mathfrak{s},\tilde{\phi}_{\tau\left(\mathfrak{s},\tilde{\phi}_\mathfrak{s}+\psi_\mathfrak{s}\right)}+\psi_{\tau\left(\mathfrak{s},\tilde{\phi}_\mathfrak{s}+\psi_\mathfrak{s}\right)}\Big)-\mathcal{L}\Big(\mathfrak{s},\tilde{\phi}_{\tau(\mathfrak{s},\tilde{\phi}_\mathfrak{s}+\psi^*_\mathfrak{s})}+\psi^*_{\tau\left(\mathfrak{s},\tilde{\phi}_\mathfrak{s}+\psi^*_\mathfrak{s}\right)}\Big)\right\|^p\\
&\quad+E\Bigg\|\int_0^\mathfrak{s} S_q(\mathfrak{s}-\varsigma)\\
&\quad\times\left[f\Big(\varsigma,\tilde{\phi}_{\tau\left(\varsigma,\tilde{\phi}_\varsigma+\psi_\varsigma\right)}+\psi_{\tau\left(\varsigma,\tilde{\phi}_\varsigma+\psi_\varsigma\right)}\Big)-f\Big(\varsigma,\tilde{\phi}_{\tau\left(\varsigma,\tilde{\phi}_\varsigma+\psi^*_\varsigma\right)}+\psi^*_{\tau\left(\varsigma,\tilde{\phi}_\varsigma+\psi^*_\varsigma\right)}\Big)\right]d\varsigma\Bigg\|^p\\
&\quad+E\Bigg\|\int_0^t(\mathfrak{s}-\varsigma)^{q-1}T_q(\mathfrak{s}-\varsigma)\\
&\quad\times\left[\sigma\Big(\varsigma,\tilde{\phi}_{\tau\left(\varsigma,\tilde{\phi}_\varsigma+\psi_\varsigma\right)}+\psi_{\tau\left(\varsigma,\tilde{\phi}_\varsigma+\psi_\varsigma\right)}\Big)-\sigma\Big(\varsigma,\tilde{\phi}_{\tau\left(\varsigma,\tilde{\phi}_\varsigma+\psi^*_\varsigma\right)}+\psi^*_{\tau\left(\varsigma,\tilde{\phi}_\varsigma+\psi^*_\varsigma\right)}\Big)\right]dW(\varsigma)\Bigg\|^p\\
&\quad+E\Bigg\|\int_0^\mathfrak{s}\int_Z(\mathfrak{s}-\varsigma)^{q-1}T_q(\mathfrak{s}-\varsigma)\\
&\quad\times\Bigg[h\Big(\varsigma,\tilde{\phi}_{\tau\left(\varsigma,\tilde{\phi}_\varsigma+\psi_\varsigma\right)}+\psi_{\tau\left(\varsigma,\tilde{\phi}_\varsigma+\psi_\varsigma\right)},\eta\Big)\\
&\quad-h\Big(\varsigma,\tilde{\phi}_{\tau\left(\varsigma,\tilde{\phi}_\varsigma+\psi^*_\varsigma\right)}+\psi^*_{\tau\left(\varsigma,\tilde{\phi}_\varsigma+\psi^*_\varsigma\right)},\eta\Big)\Bigg]\lambda(d\eta)d\varsigma\Bigg\|^p\Bigg\}\\
&\leq 4^{p-1}\Bigg\{\hat{C}_4\left\|\Big(\tilde{\phi}_{\tau(\mathfrak{s},\tilde{\phi}_\mathfrak{s}+\psi_\mathfrak{s})}+\psi_{\tau\left(\mathfrak{s},\tilde{\phi}_\mathfrak{s}+\psi_\mathfrak{s}\right)}\Big)-\Big(\tilde{\phi}_{\tau\left(\mathfrak{s},\tilde{\phi}_\mathfrak{s}+\psi^*_\mathfrak{s}\right)}+\psi^*_{\tau\left(\mathfrak{s},\tilde{\phi}_\mathfrak{s}+\psi^*_\mathfrak{s}\right)}\Big)\right\|^p\\
&\quad+\hat{M}^p\left(\int_0^t d\varsigma\right)^{p-1}\\
&\quad\times E\left\|f\Big(\varsigma,\tilde{\phi}_{\tau\left(\varsigma,\tilde{\phi}_\varsigma+\psi_\varsigma\right)}+\psi_{\tau\left(\varsigma,\tilde{\phi}_\varsigma+\psi_\varsigma\right)}\Big)-f\Big(\varsigma,\tilde{\phi}_{\tau(\varsigma,\tilde{\phi}_\varsigma+\psi^*_\varsigma)}+\psi^*_{\tau\left(\varsigma,\tilde{\phi}_\varsigma+\psi^*_\varsigma\right)}\Big)\right\|^p d\varsigma\\
&\quad+\frac{\hat{M}^p}{\Gamma^p(q)}\left(\int_0^\mathfrak{s}(\mathfrak{s}-\varsigma)^{2(q-1)}d\varsigma\right)^{\frac{p}{2}}\\
&\quad\times\Bigg[\Bigg(E\Bigg\|\int_0^\mathfrak{s}\sigma\Big(\varsigma,\tilde{\phi}_{\tau\left(\varsigma,\tilde{\phi}_\varsigma+\psi_\varsigma\right)}+\psi_{\tau\left(\varsigma,\tilde{\phi}+\psi_\varsigma\right)}\Big)
\end{aligned}$$

$$
\begin{aligned}
&-\sigma\Big(\varsigma, \tilde{\phi}_{\tau(\varsigma,\tilde{\phi}_\varsigma+\psi^*_\varsigma)} + \psi^*_{\tau\left(\varsigma,\tilde{\phi}_\varsigma+\psi^*_\varsigma\right)}\Big)\Big\|^p\Big)^{\frac{2}{p}} d\varsigma\Big]^{\frac{p}{2}} \\
&+ \frac{\hat{M}^p}{\Gamma^p(q)}\hat{C}_7\Bigg[\Big(\int_0^{\mathfrak{s}} (\mathfrak{s}-\varsigma)^{\frac{2p(q-1)}{p-2}} d\varsigma\Big)^{\frac{p-2}{2}} \\
&\times \int_0^{\mathfrak{s}} E\left\|\Big(\tilde{\phi}_{\tau\left(\varsigma,\tilde{\phi}_\varsigma+\psi_\varsigma\right)} + \psi_{\tau\left(\varsigma,\tilde{\phi}_\varsigma+\psi_\varsigma\right)}\Big) - \Big(\tilde{\phi}_{\tau\left(\varsigma,\tilde{\phi}_\varsigma+\psi^*_\varsigma\right)} + \psi^*_{\tau\left(\varsigma,\tilde{\phi}_\varsigma+\psi^*_\varsigma\right)}\Big)\right\|^p d\varsigma \\
&+ \int_0^{\mathfrak{s}} (\mathfrak{s}-\varsigma)^{2p(q-1)} \\
&\times E\left\|\Big(\tilde{\phi}_{\tau\left(\varsigma,\tilde{\phi}_\varsigma+\psi_\varsigma\right)} + \psi_{\tau\left(\varsigma,\tilde{\phi}_\varsigma+\psi_\varsigma\right)}\Big) - \Big(\tilde{\phi}_{\tau\left(\varsigma,\tilde{\phi}_\varsigma+\psi^*_\varsigma\right)} + \psi^*_{\tau\left(\varsigma,\tilde{\phi}_\varsigma+\psi^*_\varsigma\right)}\Big)\right\|^p d\varsigma\Bigg]\Bigg\} \\
\le{}& 4^{p-1}\Bigg\{\hat{C}_4\left\|\Big(\tilde{\phi}_{\tau\left(\mathfrak{s},\tilde{\phi}_{\mathfrak{s}}+\psi_{\mathfrak{s}}\right)} + \psi_{\tau\left(\mathfrak{s},\tilde{\phi}_{\mathfrak{s}}+\psi_{\mathfrak{s}}\right)}\Big) - \Big(\tilde{\phi}_{\tau\left(\mathfrak{s},\tilde{\phi}_{\mathfrak{s}}+\psi^*_{\mathfrak{s}}\right)} + \psi^*_{\tau\left(\mathfrak{s},\tilde{\phi}_{\mathfrak{s}}+\psi^*_{\mathfrak{s}}\right)}\Big)\right\|^p \\
&+ \hat{M}^p(\mathfrak{s})^{p-1}\hat{C}_2\int_0^{\mathfrak{s}}\left\|\tilde{\phi}_{\tau\left(\varsigma,\tilde{\phi}_\varsigma+\psi_\varsigma\right)} + \psi_{\tau\left(\varsigma,\tilde{\phi}_\varsigma+\psi_\varsigma\right)} - \Big(\tilde{\phi}_{\tau\left(\varsigma,\tilde{\phi}_\varsigma+\psi^*_\varsigma\right)} + \psi^*_{\tau\left(\varsigma,\tilde{\phi}_\varsigma+\psi^*_\varsigma\right)}\Big)\right\|^p d\varsigma \\
&+ \frac{\hat{M}^p}{\Gamma^p(q)}\frac{\big(\mathfrak{s})^{2(q-1)+1}\big)^{\frac{p}{2}}}{(2q-2+1)^{\frac{p}{2}}} \\
&\times \int_0^{\mathfrak{s}} \hat{m}(\varsigma)\varrho_\sigma\left\|\tilde{\phi}_{\tau\left(\varsigma,\tilde{\phi}_\varsigma+\psi_\varsigma\right)} + \psi_{\tau\left(\varsigma,\tilde{\phi}_\varsigma+\psi_\varsigma\right)} - \Big(\tilde{\phi}_{\tau\left(\varsigma,\tilde{\phi}_\varsigma+\psi^*_\varsigma\right)} + \psi^*_{\tau\left(\varsigma,\tilde{\phi}_\varsigma+\psi^*_\varsigma\right)}\Big)\right\|^p d\varsigma \\
&+ \frac{\hat{M}^p}{\Gamma^p(q)}\hat{C}_7\Bigg[\frac{(\mathfrak{s})^{\frac{(2pq-p-2)}{2}}}{\big(\frac{2pq-p-2}{p-2}\big)^{\frac{p-2}{2}}} \\
&\times \int_0^{\mathfrak{s}}\left\|\tilde{\phi}_{\tau\left(\varsigma,\tilde{\phi}_\varsigma+\psi_\varsigma\right)} + \psi_{\tau\left(\varsigma,\tilde{\phi}_\varsigma+\psi_\varsigma\right)} - (\tilde{\phi}_{\tau\left(\varsigma,\tilde{\phi}_\varsigma+\psi^*_\varsigma\right)} + \psi^*_{\tau\left(\varsigma,\tilde{\phi}_\varsigma+\psi^*_\varsigma\right)})\right\|^p d\varsigma \\
&+ \int_0^{\mathfrak{s}} (\mathfrak{s}-\varsigma)^{2p(q-1)} E\Bigg\|\tilde{\phi}_{\tau\left(\varsigma,\tilde{\phi}_\varsigma+\psi_\varsigma\right)} + \psi_{\tau\left(\varsigma,\tilde{\phi}_\varsigma+\psi_\varsigma\right)} \\
&-\Big(\tilde{\phi}_{\tau\left(\varsigma,\tilde{\phi}_\varsigma+\psi^*_\varsigma\right)} + \psi^*_{\tau\left(\varsigma,\tilde{\phi}_\varsigma+\psi^*_\varsigma\right)}\Big)\Bigg\|^p d\varsigma\Bigg]\Bigg\} \\
\le{}& 4^{p-1}\Bigg\{\hat{C}_4 + \hat{M}^p(\mathfrak{s})^{p-1}\hat{C}_2 + \frac{\hat{M}^p}{\Gamma^p(q)}\frac{\big(\mathfrak{s})^{2(q-1)+1}\big)^{\frac{p}{2}}}{(2q-2+1)^{\frac{p}{2}}}\hat{m}(\mathfrak{s})\varrho_\sigma \\
&+ \frac{\hat{M}^p}{\Gamma^p(q)}\hat{C}_7\Bigg[\frac{(\mathfrak{s})^{\frac{(2pq-p-2)}{2}}}{\big(\frac{2pq-p-2}{p-2}\big)^{\frac{p-2}{2}}} + \frac{\mathfrak{s}^{2pq-1}}{2pq-1}\Bigg]\Bigg\}\tilde{N}^p_{\hat{a}}E\left\|\tilde{\phi}(\mathfrak{s}) - \psi(\mathfrak{s})\right\|^p \\
&+ (\mathbb{H}_{\hat{a}} + J^{\varphi})^p E\left\|\tilde{\phi}_0\right\|^p_{\mathbb{B}} + (\mathbb{H}_{\hat{a}} + J^{\varphi})^p E\left\|\psi_0\right\|^p_{\mathbb{B}} \\
\le{}& 4^{p-1}\Bigg\{\hat{C}_4 + \hat{M}^p(\mathfrak{s})^{p-1}\hat{C}_2 + \frac{\hat{M}^p}{\Gamma^p(q)}\frac{\big(\mathfrak{s})^{2(q-1)+1}\big)^{\frac{p}{2}}}{(2q-2+1)^{\frac{p}{2}}}\hat{m}(\mathfrak{s})\varrho_\sigma\mathfrak{s}^{\frac{p}{2}}
\end{aligned}
$$

$$+\frac{\hat{M}^p}{\Gamma^p(q)}\hat{C}_7\left[\frac{(\mathfrak{s})^{\frac{(2pq-p-2)}{2}}}{\left(\frac{2pq-p-2}{p-2}\right)^{\frac{p-2}{2}}}+\frac{\mathfrak{s}^{2pq-1}}{2pq-1}\right]\Bigg\}\tilde{N}^p_{\hat{a}}E\left\|\tilde{\phi}(\mathfrak{s})-\psi(\mathfrak{s})\right\|^p.$$

For $\mathfrak{s}\in(\mathfrak{s}_k,\mathfrak{s}_{k+1}]$,

$$\begin{aligned}
&E\left\|(\Theta\psi)(\mathfrak{s})-(\Theta\psi^*)(\mathfrak{s})\right\|^p\\
&\leq 6^{p-1}\Bigg\{E\Bigg\|T_q(\mathfrak{s}-\mathfrak{s}_k)(\mathfrak{s}-\mathfrak{s}_k)^{q-1}\Big[\Big(\tilde{\phi}_{\tau\left(\mathfrak{s}_k,\tilde{\phi}_{\mathfrak{s}_k}+\psi_{\mathfrak{s}_k}\right)}+\psi_{\tau\left(\mathfrak{s}_k,\tilde{\phi}_{\mathfrak{s}_k}+\psi_{\mathfrak{s}_k}\right)}\Big)\\
&\quad-\Big(\tilde{\phi}_{\tau\left(\mathfrak{s}_k,\tilde{\phi}_{\mathfrak{s}_k}+\psi^*_{\mathfrak{s}_k}\right)}+\psi^*_{\tau(\mathfrak{s}_k,\tilde{\phi}_{\mathfrak{s}_k}+\psi^*_{\mathfrak{s}_k})}\Big)\Big]\Bigg\|^p\\
&\quad+E\Bigg\|\Big[T_q(\mathfrak{s}-\mathfrak{s}_k)(\mathfrak{s}-\mathfrak{s}_k)^{q-1}I_k\Big(\tilde{\phi}_{\tau\left(\mathfrak{s}_k,\tilde{\phi}_{\mathfrak{s}_k}+\psi_{\mathfrak{s}_k}\right)}+\psi_{\tau\left(\mathfrak{s}_k,\tilde{\phi}_{\mathfrak{s}_k}+\psi_{\mathfrak{s}_k}\right)}\Big)\\
&\quad-\Big(\tilde{\phi}_{\tau\left(\mathfrak{s}_k,\tilde{\phi}_{\mathfrak{s}_k}+\psi^*_{\mathfrak{s}_k}\right)}+\psi^*_{\tau\left(\mathfrak{s}_k,\tilde{\phi}_{\mathfrak{s}_k}+\psi^*_{\mathfrak{s}_k}\right)}\Big)\Big]\Bigg\|^p\\
&\quad+E\Bigg\|\mathcal{L}\Big(\mathfrak{s},\tilde{\phi}_{\tau\left(\mathfrak{s},\tilde{\phi}_{\mathfrak{s}}+\psi_{\mathfrak{s}}\right)}+\psi_{\tau\left(t,\tilde{\phi}_{\mathfrak{s}}+\psi_{\mathfrak{s}}\right)}\Big)-\mathcal{L}\Big(\mathfrak{s},\tilde{\phi}_{\tau\left(t,\tilde{\phi}_{\mathfrak{s}}+\psi^*_{\mathfrak{s}}\right)}+\psi^*_{\tau\left(t,\tilde{\phi}_{\mathfrak{s}}+\psi^*_{\mathfrak{s}}\right)}\Big)\Bigg\|^p\\
&\quad+E\Bigg\|\int_{\mathfrak{s}_k}^{\mathfrak{s}}S_q(\mathfrak{s}-\varsigma)f\Big(\varsigma,\tilde{\phi}_{\tau\left(\varsigma,\tilde{\phi}_\varsigma+\psi_\varsigma\right)}+\psi_{\tau\left(\varsigma,\tilde{\phi}_\varsigma+\psi_\varsigma\right)}\Big)\\
&\quad-f\Big(\varsigma,\tilde{\phi}_{\tau\left(\varsigma,\tilde{\phi}_\varsigma+\psi^*_\varsigma\right)}+\psi^*_{\tau\left(\varsigma,\tilde{\phi}_\varsigma+\psi^*_\varsigma\right)}\Big)d\varsigma\Bigg\|^p\\
&\quad+E\Bigg\|\int_{\mathfrak{s}_k}^{\mathfrak{s}}(\mathfrak{s}-\varsigma)^{q-1}T_q(\mathfrak{s}-\varsigma)\sigma\Big(\varsigma,\tilde{\phi}_{\tau\left(\varsigma,\tilde{\phi}_\varsigma+\psi_\varsigma\right)}+\psi_{\tau\left(\varsigma,\tilde{\phi}_\varsigma+\psi_\varsigma\right)}\Big)\\
&\quad-\sigma\Big(\varsigma,\tilde{\phi}_{\tau\left(\varsigma,\tilde{\phi}_\varsigma+\psi^*_\varsigma\right)}+\psi^*_{\tau\left(\varsigma,\tilde{\phi}_\varsigma+\psi^*_\varsigma\right)}\Big)dW(\varsigma)\Bigg\|^p\\
&\quad+E\Bigg\|\int_{\mathfrak{s}_k}^{\mathfrak{s}}\int_Z(\mathfrak{s}-\varsigma)^{q-1}T_q(\mathfrak{s}-\varsigma)\\
&\quad\times\Big[\Big(h\Big(\varsigma,\tilde{\phi}_{\tau\left(\varsigma,\tilde{\phi}_\varsigma+\psi_\varsigma\right)}+\psi_{\tau\left(\varsigma,\tilde{\phi}_\varsigma+\psi_\varsigma\right)},\eta\Big)\\
&-h\Big(\varsigma,\tilde{\phi}_{\tau\left(\varsigma,\tilde{\phi}_\varsigma+\psi^*_\varsigma\right)}+\psi^*_{\tau\left(\varsigma,\tilde{\phi}_\varsigma+\psi^*_\varsigma\right)},\eta\Big)\lambda(d\eta)d\varsigma\Big]\Bigg\|^p\Bigg\}\\
&\leq 6^{p-1}\Bigg\{\frac{\hat{M}^p}{\Gamma^p(q)}E\Bigg\|(\mathfrak{s}-\mathfrak{s}_k)^{q-1}\Big[\Big(\tilde{\phi}_{\tau(\mathfrak{s}_k,\tilde{\phi}_{\mathfrak{s}_k}+\psi_{\mathfrak{s}_k})}+\psi_{\tau\left(\mathfrak{s}_k,\tilde{\phi}_{\mathfrak{s}_k}+\psi_{\mathfrak{s}_k}\right)}\Big)\\
&\quad-\Big(\tilde{\phi}_{\tau\left(\mathfrak{s}_k,\tilde{\phi}_{\mathfrak{s}_k}+\psi^*_{\mathfrak{s}_k}\right)}+\psi^*_{\tau\left(\mathfrak{s}_k,\tilde{\phi_{\mathfrak{s}_{\mathfrak{k}}}}\psi^*_{\mathfrak{s}_k}\right)}\Big)\Big]\Bigg\|^p\\
&\quad+E\Bigg\|\frac{\hat{M}^p}{\Gamma^p(q)}\Big[I_k(\mathfrak{s}-\mathfrak{s}_k)^{q-1}\Big(\tilde{\phi}_{\tau\left(\mathfrak{s}_k,\tilde{\phi}_{\mathfrak{s}_k}+\psi_{\mathfrak{s}_k}\right)}+\psi_{\tau\left(\mathfrak{s}_k,\tilde{\phi}_{\mathfrak{s}_k}+\psi_{\mathfrak{s}_k}\right)}\Big)
\end{aligned}$$

$$
\begin{aligned}
&-(\tilde{\phi}_{\tau\left(\mathfrak{s}_k,\tilde{\phi}_{\mathfrak{s}_k}+\psi^*_{\mathfrak{s}_k}\right)}+\psi^*_{\tau\left(\mathfrak{s}_k,\tilde{\phi_{\mathfrak{s}_\mathfrak{k}}}\psi^*_{\mathfrak{s}_k}\right)}\Big)\Big]\Big\|^p \\
&+E\left\|\mathcal{L}\Big(\varsigma,\tilde{\phi}_{\tau\left(\varsigma,\tilde{\phi}_\varsigma+\psi_\varsigma\right)}+\psi_{\tau\left(\varsigma,\tilde{\phi}+\psi_\varsigma\right)}\Big)-\mathcal{L}\Big(\varsigma,\tilde{\phi}_{\tau\left(\varsigma,\tilde{\phi}_\varsigma+\psi^*_\varsigma\right)}+\psi^*_{\tau\left(\varsigma,\tilde{\phi}_\varsigma\psi^*_\varsigma\right)}\Big)\right\|^p \\
+&\hat{M}^p\left(\int_{\mathfrak{s}_k}^{\mathfrak{s}}d\varsigma\right)^{p-1}E\Big\|f\Big(\varsigma,\tilde{\phi}_{\tau\left(\varsigma,\tilde{\phi}_\varsigma+\psi_\varsigma\right)}+\psi_{\tau\left(\varsigma,\tilde{\phi}_\varsigma+\psi_\varsigma\right)}\Big) \\
&-f\Big(\varsigma,\tilde{\phi}_{\tau\left(\varsigma,\tilde{\phi}_\varsigma+\psi^*_\varsigma\right)}+\psi^*_{\tau\left(\varsigma,\tilde{\phi}_\varsigma+\psi^*_\varsigma\right)}\Big)\Big\|^p d\varsigma \\
&+\frac{\hat{M}^p}{\Gamma^p(q)}\left(\int_{\mathfrak{s}_k}^{\mathfrak{s}}(\mathfrak{s}-\varsigma)^{2(q-1)}d\varsigma\right)^{\frac{p}{2}} \\
&\times\Big[\Big(E\Big\|\int_{\mathfrak{s}_k}^{\mathfrak{s}}\sigma\Big(\varsigma,\tilde{\phi}_{\tau\left(\varsigma,\tilde{\phi}_\varsigma+\psi_\varsigma\right)}+\psi_{\tau\left(\varsigma,\tilde{\phi}_\varsigma+\psi_\varsigma\right)}\Big) \\
&-\sigma\Big(\varsigma,\tilde{\phi}_{\tau\left(\varsigma,\tilde{\phi}_\varsigma+\psi^*_\varsigma\right)}+\psi^*_{\tau\left(\varsigma,\tilde{\phi}_\varsigma+\psi^*_\varsigma\right)}\Big)\Big\|^p\Big)^{\frac{2}{p}}d\varsigma\Big]^{\frac{p}{2}} \\
&+\frac{\hat{M}^p}{\Gamma^p(q)}\hat{C}_7 \\
&\times\left[\Big(\int_0^{\mathfrak{s}}(\mathfrak{s}-\varsigma)^{\frac{2p(q-1)}{p-2}}d\varsigma\Big)^{\frac{p-2}{2}}\int_0^{\mathfrak{s}}E\Big\|\tilde{\phi}_{\tau\left(\varsigma,\tilde{\phi}_\varsigma+\psi_{\mathfrak{s}_k}\varsigma\right)}+\psi_{\tau\left(\varsigma,\tilde{\phi}_\varsigma+\psi_\varsigma\right)}\right. \\
&-\Big(\tilde{\phi}_{\tau\left(\varsigma,\tilde{\phi}_\varsigma+\psi^*_\varsigma\right)}+\psi^*_{\tau\left(\varsigma,\tilde{\phi}_\varsigma+\psi^*_\varsigma\right)}\Big)\Big\|^p d\varsigma\Big] \\
&+\int_{\mathfrak{s}_k}^{\mathfrak{s}}(\mathfrak{s}-\varsigma)^{2p(q-1)}E\Big\|\tilde{\phi}_{\tau\left(\varsigma,\tilde{\phi}_\varsigma+\psi_{\mathfrak{s}_k}\varsigma\right)}+\psi_{\tau\left(\varsigma,\tilde{\phi}_\varsigma+\psi_\varsigma\right)} \\
&\left.-\Big(\tilde{\phi}_{\tau\left(\varsigma,\tilde{\phi}_\varsigma+\psi^*_\varsigma\right)}+\psi^*_{\tau\left(\varsigma,\tilde{\phi}_\varsigma+\psi^*_\varsigma\right)}\Big)\Big\|^p d\varsigma\right]\Bigg\} \\
\leq\ &6^{p-1}\Bigg\{\frac{\hat{M}^p}{\Gamma^p(q)}E\Big\|(\mathfrak{s}-\mathfrak{s}_k)^{q-1}\Big[(\tilde{\phi}_{\tau\left(\mathfrak{s}_k,\tilde{\phi}_{\mathfrak{s}_k}+\psi_{\mathfrak{s}_k}\right)}+\psi_{\tau\left(\mathfrak{s}_k,\tilde{\phi}_{\mathfrak{s}_k}+\psi_{\mathfrak{s}_k}\right)}\Big) \\
&-\Big(\tilde{\phi}_{\tau\left(\mathfrak{s}_k,\tilde{\phi}_{\mathfrak{s}_k}+\psi^*_{\mathfrak{s}_k}\right)}+\psi^*_{\tau\left(\mathfrak{s}_k,\tilde{\phi_{\mathfrak{s}_\mathfrak{k}}}\psi^*_{\mathfrak{s}_k}\right)}\Big)\Big]\Big\|^p \\
&+E\Big\|\frac{\hat{M}^p}{\Gamma^p(q)}\Big[I_k(\mathfrak{s}-\mathfrak{s}_k)^{q-1}\Big(\tilde{\phi}_{\tau\left(\mathfrak{s}_k,\tilde{\phi}_{\mathfrak{s}_k}+\psi_{\mathfrak{s}_k}\right)}+\psi_{\tau\left(\mathfrak{s}_k,\tilde{\phi}_{\mathfrak{s}_k}+\psi_{\mathfrak{s}_k}\right)}\Big) \\
&-\Big(\tilde{\phi}_{\tau\left(\mathfrak{s}_k,\tilde{\phi}_{\mathfrak{s}_k}+\psi^*_{\mathfrak{s}_k}\right)}+\psi^*_{\tau(\mathfrak{s}_k,\tilde{\phi_{\mathfrak{s}_\mathfrak{k}}}\psi^*_{\mathfrak{s}_k})}\Big)\Big]\Big\|^p \\
&+\hat{C}_4E\Big\|\tilde{\phi}_{\tau(t,\tilde{\phi}_\mathfrak{s}+\psi_\mathfrak{s})}+\psi_{\tau\left(\mathfrak{s},\tilde{\phi}_\mathfrak{s}\psi_\mathfrak{s}\right)}-\Big(\tilde{\phi}_{\tau\left(t,\tilde{\phi}_\mathfrak{s}+\psi^*_\mathfrak{s}\right)}+\psi^*_{\tau\left(\mathfrak{s},\tilde{\phi}_\mathfrak{s}\psi^*_\mathfrak{s}\right)}\Big)\Big\|^p \\
&+\hat{M}^p\,(t-\mathfrak{s}_k)^{p-1}\,(\hat{C}_2+1)
\end{aligned}
$$

$$\times \int_{\mathfrak{s}_k}^{\mathfrak{s}} E\left\|\tilde{\phi}_{\tau\left(\varsigma,\tilde{\phi}_\varsigma+\psi_\varsigma\right)}+\psi_{\tau\left(\varsigma,\tilde{\phi}_\varsigma\psi_\varsigma\right)}-\left(\tilde{\phi}_{\tau\left(\varsigma,\tilde{\phi}_\varsigma+\psi^*_\varsigma\right)}+\psi^*_{\tau\left(\varsigma,\tilde{\phi}_\varsigma\psi^*_\varsigma\right)}\right)\right\|^p d\varsigma$$

$$+\frac{\hat{M}^p}{\Gamma^p(q)}\frac{\left((\mathfrak{s}-\mathfrak{s}_k)^{2(q-1)+1}\right)^{\frac{p}{2}}}{(2q-2+1)^{\frac{p}{2}}}$$

$$\times\left[\left(\int_{\mathfrak{s}_k}^{t}\hat{m}(\varsigma)\varrho_\sigma\left\|\tilde{\phi}_{\tau\left(\varsigma,\tilde{\phi}_\varsigma+\psi_\varsigma\right)}+\psi_{\tau\left(\varsigma,\tilde{\phi}_\varsigma\psi_\varsigma\right)}-\left(\tilde{\phi}_{\tau\left(\varsigma,\tilde{\phi}_\varsigma+\psi^*_\varsigma\right)}+\psi^*_{\tau\left(\varsigma,\tilde{\phi}_\varsigma\psi^*_\varsigma\right)}\right)\right\|^p\right)^{\frac{2}{p}}d\varsigma\right]^{\frac{p}{2}}$$

$$+\frac{\hat{M}^p}{\Gamma^p(q)}\hat{C}_7$$

$$\times\left[\frac{(\mathfrak{s}-\mathfrak{s}_k)^{\frac{(2pq-p-2)}{2}}}{\left(\frac{2pq-p-2}{p-2}\right)^{\frac{p-2}{2}}}\int_{\mathfrak{s}_k}^{t}E\left\|\tilde{\phi}_{\tau\left(\varsigma,\tilde{\phi}_\varsigma+\psi_\varsigma\right)}+\psi_{\tau\left(\varsigma,\tilde{\phi}_\varsigma\psi_\varsigma\right)}\right.\right.$$

$$\left.-\left(\tilde{\phi}_{\tau\left(\varsigma,\tilde{\phi}_\varsigma+\psi^*_\varsigma\right)}+\psi^*_{\tau\left(\varsigma,\tilde{\phi}_\varsigma\psi^*_\varsigma\right)}\right)\right\|^p d\varsigma$$

$$+\int_{\mathfrak{s}_k}^{\mathfrak{s}}(\mathfrak{s}-\varsigma)^{2p(q-1)}E\left\|\tilde{\phi}_{\tau(\varsigma,\tilde{\phi}_\varsigma+\psi_\varsigma)}+\psi_{\tau(\varsigma,\tilde{\phi}_\varsigma\psi_\varsigma)}\right.$$

$$\left.\left.\left.-\left(\tilde{\phi}_{\tau\left(\varsigma,\tilde{\phi}_\varsigma+\psi^*_\varsigma\right)}+\psi^*_{\tau\left(\varsigma,\tilde{\phi}_\varsigma\psi^*_\varsigma\right)}\right)\right\|^p d\varsigma\right]\right\}$$

$$\le 6^{p-1}\left\{(1+q_k)(\mathfrak{s}-\mathfrak{s}_k)^{q-1}+\hat{C}_4+\hat{M}^p\,(\mathfrak{s}-\mathfrak{s}_k)^{p-1}\,\hat{C}_2\right.$$

$$+\frac{\hat{M}^p}{\Gamma^p(q)}\frac{\left((\mathfrak{s}-\mathfrak{s}_k)^{2(q-1)+1}\right)^{\frac{p}{2}}}{(2q-2+1)^{\frac{p}{2}}}\hat{m}(\mathfrak{s})\varrho_\sigma\mathfrak{s}^{\frac{p}{2}}$$

$$\left.+\frac{\hat{M}^p}{\Gamma^p(q)}\hat{C}_7\left[\frac{(\mathfrak{s}-\mathfrak{s}_k)^{\frac{(2pq-p-2)}{2}}}{\left(\frac{2pq-p-2}{p-2}\right)^{\frac{p-2}{2}}}+\frac{(\mathfrak{s}-\mathfrak{s}_k)^{2pq-1}}{2pq-1}\right]\right\}\tilde{N}^p_{\hat{a}}E\left\|\tilde{\phi}(\mathfrak{s})-\psi(\mathfrak{s})\right\|^p.$$

By the Holder inequality, along with the Lipschitz property of $\mathcal{L}, f, h, \sigma$, and $I_k, k=1,2,\dots,$ one can have

$$E\left\|(\Theta\psi)(\mathfrak{s})-(\Theta\psi^*)(\mathfrak{s})\right\|^p$$

$$\le 6^{p-1}\left\{\frac{\hat{M}^p}{\Gamma^p(q)}E\left\|(\mathfrak{s}-\mathfrak{s}_k)^{q-1}\left[\left(\tilde{\phi}_{\tau\left(\mathfrak{s}_k,\tilde{\phi}_{\mathfrak{s}_k}+\psi_{\mathfrak{s}_k}\right)}+\psi_{\tau\left(\mathfrak{s}_k,\tilde{\phi}_{\mathfrak{s}_k}+\psi_{\mathfrak{s}_k}\right)}\right)\right.\right.\right.$$

$$\left.\left.-\left(\tilde{\phi}_{\tau\left(\mathfrak{s}_k,\tilde{\phi}_{\mathfrak{s}_k}+\psi^*_{\mathfrak{s}_k}\right)}+\psi^*_{\tau\left(\mathfrak{s}_k,\tilde{\phi_{\mathfrak{s}_\mathfrak{k}}}+\psi^*_{\mathfrak{s}_k}\right)}\right)\right]\right\|^p$$

$$+E\left\|\frac{\hat{M}^p}{\Gamma^p(q)}\left[I_k(\mathfrak{s}-\mathfrak{s}_k)^{q-1}\left(\tilde{\phi}_{\tau\left(\mathfrak{s}_k,\tilde{\phi}_{\mathfrak{s}_k}+\psi_{\mathfrak{s}_k}\right)}+\psi_{\tau\left(\mathfrak{s}_k,\tilde{\phi}_{\mathfrak{s}_k}+\psi_{\mathfrak{s}_k}\right)}\right)\right.\right.$$

$$\left.\left.-\left(\tilde{\phi}_{\tau\left(\mathfrak{s}_k,\tilde{\phi}_{\mathfrak{s}_k}+\psi^*_{\mathfrak{s}_k}\right)}+\psi^*_{\tau\left(\mathfrak{s}_k,\tilde{\phi_{\mathfrak{s}_\mathfrak{k}}}+\psi^*_{\mathfrak{s}_k}\right)}\right)\right]\right\|^p$$

$$
\begin{aligned}
&+\hat{C}_4 E\left\|\tilde{\phi}_{\tau\left(\mathfrak{s},\tilde{\phi}_{\mathfrak{s}}+\psi_{\mathfrak{s}}\right)}+\psi_{\tau\left(\mathfrak{s},\tilde{\phi}_{\mathfrak{s}}\psi_{\mathfrak{s}}\right)}-\left(\tilde{\phi}_{\tau\left(\mathfrak{s},\tilde{\phi}_{\mathfrak{s}}+\psi^*_{\mathfrak{s}}\right)}+\psi^*_{\tau\left(\mathfrak{s},\tilde{\phi}_{\mathfrak{s}}\psi^*_{\mathfrak{s}}\right)}\right)\right\|^p \\
&+\hat{M}^p\left(t-\mathfrak{s}_k\right)^{p-1}\left(\hat{C}_2+1\right) \\
&\times\int_{\mathfrak{s}_k}^t E\left\|\tilde{\phi}_{\tau\left(\varsigma,\tilde{\phi}_\varsigma+\psi_\varsigma\right)}+\psi_{\tau\left(\varsigma,\tilde{\phi}_\varsigma\psi_\varsigma\right)}-\left(\tilde{\phi}_{\tau\left(\varsigma,\tilde{\phi}_\varsigma+\psi^*_\varsigma\right)}+\psi^*_{\tau\left(\varsigma,\tilde{\phi}_\varsigma\psi^*_\varsigma\right)}\right)\right\|^p d\varsigma \\
&+\frac{\hat{M}^p}{\Gamma^p(q)}\frac{\left(\mathfrak{s}-\mathfrak{s}_k\right)^{2(q-1)+1}\Big)^{\frac{p}{2}}}{(2q-2+1)^{\frac{p}{2}}}\Big[\Big(\int_{\mathfrak{s}_k}^t\hat{m}(\varsigma)\varrho_\sigma\left\|\tilde{\phi}_{\tau\left(\varsigma,\tilde{\phi}_\varsigma+\psi_\varsigma\right)}+\psi_{\tau\left(\varsigma,\tilde{\phi}_\varsigma\psi_\varsigma\right)}\right. \\
&\left.-\left(\tilde{\phi}_{\tau\left(\varsigma,\tilde{\phi}_\varsigma+\psi^*_\varsigma\right)}+\psi^*_{\tau\left(\varsigma,\tilde{\phi}_\varsigma\psi^*_\varsigma\right)}\right)\right\|^p\Big)^{\frac{2}{p}}d\varsigma\Big]^{\frac{p}{2}} \\
&+\frac{\hat{M}^p}{\Gamma^p(q)}\hat{C}_7\left[\frac{\left(\mathfrak{s}-\mathfrak{s}_k\right)^{\frac{(2pq-p-2)}{2}}}{\left(\frac{2pq-p-2}{p-2}\right)^{\frac{p-2}{2}}}\int_{\mathfrak{s}_k}^t E\left\|\tilde{\phi}_{\tau\left(\varsigma,\tilde{\phi}_\varsigma+\psi_\varsigma\right)}+\psi_{\tau\left(\varsigma,\tilde{\phi}_\varsigma\psi_\varsigma\right)}\right.\right. \\
&\left.-\left(\tilde{\phi}_{\tau\left(\varsigma,\tilde{\phi}_\varsigma+\psi^*_\varsigma\right)}+\psi^*_{\tau\left(\varsigma,\tilde{\phi}_\varsigma\psi^*_\varsigma\right)}\right)\right\|^p d\varsigma \\
&+\int_{\mathfrak{s}_k}^{\mathfrak{s}}(\mathfrak{s}-\varsigma)^{2p(q-1)}E\left\|\tilde{\phi}_{\tau\left(\varsigma,\tilde{\phi}_\varsigma+\psi_\varsigma\right)}+\psi_{\tau\left(\varsigma,\tilde{\phi}_\varsigma\psi_\varsigma\right)}\right. \\
&\left.\left.\left.-\left(\tilde{\phi}_{\tau\left(\varsigma,\tilde{\phi}_\varsigma+\psi^*_\varsigma\right)}+\psi^*_{\tau\left(\varsigma,\tilde{\phi}_\varsigma\psi^*_\varsigma\right)}\right)\right\|^p d\varsigma\right]\right\} \\
&\leq 6^{p-1}\Big\{(1+q_k)(\mathfrak{s}-\mathfrak{s}_k)^{q-1}+\hat{C}_4+\hat{M}^p\left(\mathfrak{s}-\mathfrak{s}_k\right)^{p-1}\hat{C}_2 \\
&+\frac{\hat{M}^p}{\Gamma^p(q)}\frac{\left(\mathfrak{s}-\mathfrak{s}_k\right)^{2(q-1)+1}\Big)^{\frac{p}{2}}}{(2q-2+1)^{\frac{p}{2}}}\hat{m}(\mathfrak{s})\varrho_\sigma t^{\frac{p}{2}} \\
&\quad+\frac{\hat{M}^p}{\Gamma^p(q)}\hat{C}_7\left[\frac{\left(\mathfrak{s}-\mathfrak{s}_k\right)^{\frac{(2pq-p-2)}{2}}}{\left(\frac{2pq-p-2}{p-2}\right)^{\frac{p-2}{2}}}+\frac{\left(\mathfrak{s}-\mathfrak{s}_k\right)^{2pq-1}}{2pq-1}\right]\Big\}\tilde{N}^p_{\hat{a}}E\left\|\tilde{\phi}(\mathfrak{s})-\psi(\mathfrak{s})\right\|^p.
\end{aligned}
$$

Hence, one can get

$$
\sup_{\varsigma\in[-s,\mathfrak{s}]}E\left\|(\Theta\psi)(\mathfrak{s})-(\Theta\psi^*)(\mathfrak{s})\right\|^p\leq\sup_{\varsigma\in[-s,\mathfrak{s}]}\hat{\rho}E\left\|\tilde{\phi}(\mathfrak{s})-\psi(\mathfrak{s})\right\|^p,
$$

where

$$
\begin{aligned}
\hat{\rho}(\mathfrak{s})=6^{p-1}\Big\{&(1+q_k)(\mathfrak{s}-\mathfrak{s}_k)^{q-1}+\hat{C}_4+\hat{M}^p\left(\mathfrak{s}-\mathfrak{s}_k\right)^{p-1}\hat{C}_2 \\
&+\frac{\hat{M}^p}{\Gamma^p(q)}\frac{\left(\mathfrak{s}-\mathfrak{s}_k\right)^{2(q-1)+1}\Big)^{\frac{p}{2}}}{(2q-2+1)^{\frac{p}{2}}}
\end{aligned}
$$

$$\times\,\hat{m}(\mathfrak{s})\varrho_\sigma \mathfrak{s}^{\frac{p}{2}} + \frac{\hat{M}^p}{\Gamma^p(q)}\hat{C}_7\left[\frac{(\mathfrak{s}-\mathfrak{s}_k)^{\frac{(2pq-p-2)}{2}}}{\left(\frac{2pq-p-2}{p-2}\right)^{\frac{p-2}{2}}} + \frac{(\mathfrak{s}-\mathfrak{s}_k)^{2pq-1}}{2pq-1}\right]\Bigg\}\tilde{N}^p_{\hat{a}} \quad (9)$$

Hence, according to the contraction mapping principle theory, Θ has a unique fixed point $\psi(\mathfrak{s})$. □

Lemma 3. Consider the constants $\hat{C}_i (i = 1, 2, 3)$, $I_k(k = 1, 2, \ldots, m), \hat{\gamma} > 0$ and a function $\psi(\mathfrak{s}) : [-s, \infty) \to [0, \infty)$ such that

$$\psi(\mathfrak{s}) \le \hat{C}_1 e^{-\hat{\gamma}\mathfrak{s}}, \qquad \text{for } \mathfrak{s} \in [-s, 0],$$

and

$$\psi(\mathfrak{s}) = \hat{C}_1 e^{-\hat{\gamma}\mathfrak{s}} + \hat{C}_2 \sup_{\theta\in[-s,0]} \psi(\mathfrak{s}+\theta) + \hat{C}_3 \int_0^{\mathfrak{s}} e^{(\mathfrak{s}-\varsigma)} \sup_{\theta\in[-s,0]} \psi(\mathfrak{s}+\theta) d\varsigma$$
$$+ \sum_{\mathfrak{s}_k<\mathfrak{s}} \beta_k e^{-\varsigma(\mathfrak{s}-\mathfrak{s}_k)}\psi(\mathfrak{s}_k^-), \qquad \text{for every } \ \mathfrak{s} \ge 0.$$

If $C_2 + \frac{C_3}{\hat{\gamma}} + \sum_{k=1}^{m} \beta_k < 1$, then $\psi(\mathfrak{s}) \le \mathcal{N}_0 e^{-\mu\mathfrak{s}}$ for $\mathfrak{s} \ge -s$, where $\mu > 0$ is the unique solution to the equation $\hat{C}_2 + \frac{C_3}{(\mu-\gamma)} e^{\mu s} + \sum_{k=1}^{m} \beta_k = 1$ and $\mathcal{N}_0 = \max\{C_1, \frac{C_1(\gamma-\mu)}{C_3 e^{\varsigma} s}\} > 0$.

Exponential stability: To ensure the exponential stability of the system (1)–(3), we need to state the following additional assumptions.

(A_7) From (7) and (8), we have

$$\|S_q(\mathfrak{s})\| \le M e^{-q\alpha\mathfrak{s}}, \qquad \mathfrak{s} \ge 0, \alpha > 0, \quad (10)$$
$$\|T_q(\mathfrak{s})\| \le \frac{M}{\Gamma(q)} e^{-q\alpha\mathfrak{s}}, \qquad \mathfrak{s} \ge 0, \alpha > 0. \quad (11)$$

We assume that $\|S_q(\mathfrak{s})\|$ and $\|T_q(\mathfrak{s})\|$ for $\mathfrak{s}$ are strongly continuous semigroup and exponentially stable.

(A_8) For all $\mathfrak{s} \ge 0$ and $\psi \in \mathbb{B}$, there exist nonnegative real numbers $\mathcal{R}_1, \mathcal{R}_2, \mathcal{R}_3 \ge 0$ and continuous function $\pi_i : [0, \infty) \to \mathbb{B}\ R_+, i = 1, 2, 3,$ such that

$$\|\mathcal{L}(\mathfrak{s}, x_{\mathfrak{s}})\|^p \le \mathcal{R}_1[\|x_{\mathfrak{s}}\|^p] + \pi_1(\mathfrak{s}),$$

$$\|f(\mathfrak{s}, x_{\mathfrak{s}})\|^p \leq \mathcal{R}_2[\|x_{\mathfrak{s}}\|^p] + \pi_2(\varsigma),$$
$$\|h(\mathfrak{s}, \varsigma, x_{\mathfrak{s}})\|^p \leq \mathcal{R}_3[\|x_{\mathfrak{s}}\|^p] + \pi_3(t\mathfrak{s}).$$

(A_9) There exist nonnegative real numbers $r_1, r_2 r_3 \geq 0$ such that

$$\pi_i \leq r_i e^{-\alpha \mathfrak{s}}, \qquad \mathfrak{s} \geq 0, i = 1, 2, 3.$$

(A_{10}) The function $\sigma : J \to L_2(\Omega, \mathbb{B}, \mathcal{B})$ satisfies the following conditions:

$$\int_0^\infty e^{\alpha\varsigma} \|\sigma(\varsigma)\|^p < \infty.$$

Theorem 2. Suppose that the assumptions $(A_2) - (A_{10})$ are satisfied. Then the given system (1)–(3) is exponentially stable.

Proof. Now, choose $\Theta(\psi(\mathfrak{s}))$ as the mild solution of the system (1)–(3), and take $\hat{\mu} = \gamma\epsilon$. Then, one can obtain

$$(\Theta\psi)(\mathfrak{s}) = \begin{cases} -S_q(\mathfrak{s})[\varphi(0) - \mathcal{L}(0, \varphi)] + \mathcal{L}\Big(\mathfrak{s}, \tilde{\phi}_{\tau\left(\mathfrak{s}, \tilde{\phi}_{\mathfrak{s}} + \psi_{\mathfrak{s}}\right)} + \psi_{\tau\left(\mathfrak{s}, \tilde{\phi}_{\mathfrak{s}} + \psi_{\mathfrak{s}}\right)}\Big) \\ + \int_0^{\mathfrak{s}} S_q(\mathfrak{s} - \varsigma) f\Big(\varsigma, \tilde{\phi}_{\tau\left(\varsigma, \tilde{\phi}_\varsigma + \psi_\varsigma\right)} + \psi_{\tau\left(\varsigma, \tilde{\phi}_\varsigma + \psi_\varsigma\right)}\Big) d\varsigma \\ + \int_0^{\mathfrak{s}} (\mathfrak{s} - \varsigma)^{q-1} T_q(\mathfrak{s} - \varsigma) \sigma\Big(\varsigma, \tilde{\phi}_{\tau\left(\varsigma, \tilde{\phi}_\varsigma + \psi_\varsigma\right)} + \psi_{\tau\left(\varsigma, \tilde{\phi}_\varsigma + \psi_\varsigma\right)}\Big) dW(\varsigma) \\ + \int_0^{\mathfrak{s}} \int_Z (\mathfrak{s} - \varsigma)^q T_q(\mathfrak{s} - \varsigma) h\Big(\varsigma, \tilde{\phi}_{\tau\left(\varsigma, \tilde{\phi}_\varsigma + \psi_\varsigma\right)} + \psi_{\tau\left(\varsigma, \tilde{\phi}_\varsigma + \psi_\varsigma\right)}, \eta\Big) \\ \times \lambda(d\eta) d\varsigma, \qquad \mathfrak{s} \in [0, \mathfrak{s}_1], \end{cases}$$

$$\begin{cases} \vdots \qquad\qquad\qquad \vdots \qquad\qquad\qquad \vdots \\ (\mathfrak{s} - \mathfrak{s}_k)^{q-1} T_q(\mathfrak{s} - \mathfrak{s}_k) \Big[\tilde{\phi}_{\tau\left(\tilde{\phi}_{\mathfrak{s}_k} + \psi_{\mathfrak{s}_k}\right)} + \psi_{\tau\left(\tilde{\phi}_{\mathfrak{s}_k} + \psi_{\mathfrak{s}_k}\right)} \\ + I_k(\tilde{\phi}_{\tau\left(\tilde{\phi}_{\mathfrak{s}_k} + \psi_{\mathfrak{s}_k}\right)} + \psi_{\tau\left(\tilde{\phi}_{\mathfrak{s}_k} + \psi_{\mathfrak{s}_k}\right)}\Big)\Big] \\ + \mathcal{L}\Big(\mathfrak{s}_k, \tilde{\phi}_{\tau\left(\mathfrak{s}_k, \tilde{\phi}_{\mathfrak{s}_k} + \psi_{\mathfrak{s}_k}\right)} + \psi_{\tau\left(\mathfrak{s}_k, \tilde{\phi}_{\mathfrak{s}_k} + \psi_{\mathfrak{s}_k}\right)}\Big) \\ + \int_{\mathfrak{s}_k}^{\mathfrak{s}} S_q(\mathfrak{s} - \varsigma) f\Big(\varsigma, \tilde{\phi}_{\tau\left(\varsigma, \tilde{\phi}_\varsigma + \psi_\varsigma\right)} + \psi_{\tau\left(\varsigma, \tilde{\phi}_\varsigma + \psi_\varsigma\right)}\Big) d\varsigma \\ + \int_{\mathfrak{s}_k}^{\mathfrak{s}} (\mathfrak{s} - \varsigma)^{q-1} T_q(\mathfrak{s} - \varsigma) \int_0^\varsigma \sigma\Big(\varsigma, \tilde{\phi}_{\tau\left(\varsigma, \tilde{\phi}_\varsigma + \psi_\varsigma\right)} + \psi_{\tau\left(\varsigma, \tilde{\phi}_\varsigma + \psi_\varsigma\right)}\Big) dW(\varsigma) \\ + \int_{\mathfrak{s}_k}^{\mathfrak{s}} \int_Z (\mathfrak{s} - \varsigma)^q T_q(\mathfrak{s} - \varsigma) h\Big(\varsigma, \tilde{\phi}_{\tau\left(\varsigma, \tilde{\phi}_\varsigma + \psi_\varsigma\right)} + \psi_{\tau\left(\varsigma, \tilde{\phi}_\varsigma + \psi_\varsigma\right)}, \eta\Big) \\ \times \lambda(d\eta) d\varsigma, \qquad \mathfrak{s} \in (\mathfrak{s}_k, \mathfrak{s}_{k+1}]. \end{cases}$$

For $\mathfrak{s} \in [0, \mathfrak{s}_1]$, it follows from (8), (9) , (A_8) and (A_9) that

$$\begin{aligned} & E \left\|(\Theta\psi)(\mathfrak{s})\right\|_{\mathbb{B}}^p && (12)\\ & \leq 5^{p-1}\Big\{ S_q(\mathfrak{s}) E \left\|[\varphi(0) - \mathcal{L}(0,\varphi)]\right\|_{\mathbb{B}}^p + E\left\|\mathcal{L}\Big(\mathfrak{s}, \tilde{\phi}_{\tau\left(\mathfrak{s},\tilde{\phi}_{\mathfrak{s}}\right)} + \psi_{\tau\left(\mathfrak{s},\psi_{\mathfrak{s}}\right)}\Big)\right\|_{\mathbb{B}}^p \\ & \quad + E\left\| S_q(\mathfrak{s}-\varsigma)\int_0^{\mathfrak{s}} f\Big(\varsigma, \tilde{\phi}_{\tau\left(\varsigma,\tilde{\phi}_{\varsigma}\right)} + \psi_{\tau\left(\varsigma,\psi_{\varsigma}\right)}\Big) d\varsigma\right\|_{\mathbb{B}}^p \\ & \quad + E\left\|\int_0^{\mathfrak{s}} (\mathfrak{s}-\varsigma)^{q-1} T_q(\mathfrak{s}-\varsigma)\sigma\Big(\varsigma, \tilde{\phi}_{\tau\left(\varsigma,\tilde{\phi}_{\varsigma}\right)} + \psi_{\tau\left(\varsigma,\psi_{\varsigma}\right)}\Big) dW(\varsigma)\right\|_{\mathbb{B}}^p \\ & \quad + E\left\|\int_0^{\mathfrak{s}}\int_Z (\mathfrak{s}-\varsigma)^{q-1} T_q(\mathfrak{s}-\varsigma) h\Big(\varsigma, \tilde{\phi}_{\tau\left(\varsigma,\tilde{\phi}_{\varsigma}\right)} + \psi_{\tau\left(\varsigma,\psi_{\varsigma}\right)}, \eta\Big)\lambda(d\eta) d\varsigma\right\|_{\mathbb{B}}^p \Big\} \\ & \leq \sum_{i=1}^{5} \mathcal{G}_i. && (13) \end{aligned}$$

From the assumption A_8, one can have

$$\begin{aligned} \mathcal{G}_1 &= S_q(\mathfrak{s}) E \left\|[\varphi(0) - \mathcal{L}(0,\varphi)]\right\|_{\mathbb{B}}^p \\ &\leq M^p e^{-p\alpha\mathfrak{s}} E \left\|\varphi(0)\right\|^p. && (14) \end{aligned}$$

From (A_8) and (A_9), one can obtain

$$\begin{aligned} \mathcal{G}_2 &= E\left\|\mathcal{L}\Big(\mathfrak{s}, \tilde{\phi}_{\tau\left(\mathfrak{s},\tilde{\phi}_{\mathfrak{s}}+\psi_{\mathfrak{s}}\right)} + \psi_{\tau\left(\mathfrak{s},\tilde{\phi}_{\mathfrak{s}}+\psi_{\mathfrak{s}}\right)}\Big)\right\|_{\mathbb{B}}^p \\ &\leq M^p(\hat{C}_4+1)\Big\{ E\left\|\mathcal{R}_1\tilde{\phi}_{\tau\left(\mathfrak{s},\tilde{\phi}_{\mathfrak{s}}+\psi_{\mathfrak{s}}\right)} + \psi_{\tau(\mathfrak{s},\tilde{\phi}_{\mathfrak{s}}+\psi_{\mathfrak{s}})}\right\|^p + \pi_1(\mathfrak{s})\Big\} \\ &\leq M^p(\hat{C}_4+1)\Big\{ E\left\|\mathcal{R}_1\tilde{\phi}_{\tau\left(\mathfrak{s},\tilde{\phi}_{\mathfrak{s}}+\psi_{\mathfrak{s}}\right)} + \psi_{\tau\left(\mathfrak{s},\tilde{\phi}_{\mathfrak{s}}+\psi_{\mathfrak{s}}\right)}\right\|^p\Big\} + \hat{Q}_2 e^{-\alpha\mathfrak{s}}, && (15) \end{aligned}$$

where $\hat{Q}_2 = M^p(\hat{C}_4+1)r_1$.

From (A_8) and (A_9), one can get

$$\begin{aligned} \mathcal{G}_3 &= E\left\|\int_0^{\mathfrak{s}} S_q(\mathfrak{s}-\varsigma) f(\varsigma, \tilde{\phi}_{\tau(\varsigma,\tilde{\phi}_{\varsigma}+\psi_{\varsigma})} + \psi_{\tau(\varsigma,\tilde{\phi}_{\varsigma}+\psi_{\varsigma})}) d\varsigma\right\|_{\mathbb{B}}^p \\ &\leq M^p(\hat{C}_2+1)\Big\{ E\left\|\int_0^{\mathfrak{s}} e^{-q\alpha}\mathcal{R}_2\tilde{\phi}_{\tau(\varsigma,\tilde{\phi}_{\varsigma}+\psi_{\varsigma})} + \psi_{\tau(\varsigma,\tilde{\phi}_{\varsigma}+\psi_{\varsigma})}\right\|^p + \pi_2(\mathfrak{s})\Big\} \\ &\leq M^p(\hat{C}_2+1)\Big\{ E\left\|\int_0^{\mathfrak{s}} e^{-q\alpha}\mathcal{R}_2\tilde{\phi}_{\tau(\varsigma,\tilde{\phi}_{\varsigma}+\psi_{\varsigma})} + \psi_{\tau(\varsigma,\tilde{\phi}_{\varsigma}+\psi_{\varsigma})}\right\|^p\Big\} + \hat{Q}_3 e^{-\alpha\mathfrak{s}}, \\ & && (16) \end{aligned}$$

where $\hat{Q}_3 = M^p(\hat{C}_2 + 1)r_2$.
By the hypotheses (A_4) and (A_{10}), one can have

$$\begin{aligned}
\mathcal{G}_4 &= E\left\|\int_0^{\mathfrak{s}}(\mathfrak{s}-\varsigma)^{q-1}T_q(\mathfrak{s}-\varsigma)\sigma\Big(\varsigma,\tilde{\phi}_{\tau\left(\varsigma,\tilde{\phi}_\varsigma+\psi_\varsigma\right)}+\psi_{\tau\left(\varsigma,\tilde{\phi}_\varsigma+\psi_\varsigma\right)}\Big)dW(\varsigma)\right\|_{\mathbb{B}}^p \\
&\leq \frac{M^p}{\Gamma^p(q)}E\left\|\int_0^{\mathfrak{s}}(\mathfrak{s}-\varsigma)^{q-1}e^{-q\alpha\varsigma}\sigma\Big(\varsigma,\tilde{\phi}_{\tau\left(\varsigma,\tilde{\phi}_\varsigma+\psi_\varsigma\right)}+\psi_{\tau\left(\varsigma,\tilde{\phi}_\varsigma+\psi_\varsigma\right)}\Big)\right\|_{L_Q^0}^p d\varsigma \\
&\leq \hat{Q}_4, \qquad \hat{Q}_4 > 0.
\end{aligned}$$

Hence, one can get $\mathcal{G}_4 \leq \hat{Q}_4 e^{-\mu\mathfrak{s}}$.
Furthermore, from (A_8) and (A_9), one can get

$$\begin{aligned}
\mathcal{G}_5 &= E\left\|\int_0^{\mathfrak{s}}(\mathfrak{s}-\varsigma)^{q-1}T_q(\mathfrak{s}-\varsigma)\Big(\varsigma,\tilde{\phi}_{\tau\left(\varsigma,\tilde{\phi}_\varsigma+\psi_\varsigma\right)}+\psi_{\tau\left(\varsigma,\tilde{\phi}_\varsigma+\psi_\varsigma\right)},\eta\Big)\lambda(d\eta)d\varsigma\right\|_{\mathbb{B}}^p \\
&\leq \frac{M^p}{\Gamma^p(q)}(\hat{C}_6+1)\left\{E\left\|\int_0^{\mathfrak{s}}e^{-q\alpha\varsigma}\Big(\mathcal{R}_3\tilde{\phi}_{\tau\left(\varsigma,\tilde{\phi}_\varsigma+\psi_\varsigma\right)}+\psi_{\tau\left(\varsigma,\tilde{\phi}_\varsigma+\psi_\varsigma\right)}\Big)d\varsigma\right\|^p+\pi_3(t)\right\} \\
&\leq \frac{M^p}{\Gamma^p(q)}(\hat{C}_6+1)\left\{E\left\|\int_0^{\mathfrak{s}}e^{-q\alpha\varsigma}\mathcal{R}_3\Big(\tilde{\phi}_{\tau\left(\varsigma,\tilde{\phi}_\varsigma+\psi_\varsigma\right)}+\psi_{\tau\left(\varsigma,\tilde{\phi}_\varsigma+\psi_\varsigma\right)}\Big)d\varsigma\right\|^p\right\}+\hat{Q}_5e^{-\alpha\mathfrak{s}},
\end{aligned} \tag{17}$$

where $\hat{Q}_5 = \frac{M^p}{\Gamma^p(q)}(\hat{C}_6 + 1)r_3$.

Now, from (A_5)&(A_8), one can have

$$\begin{aligned}
\mathcal{G}_6 &= (\mathfrak{s}-\varsigma)^{q-1}T_q(\mathfrak{s}-\varsigma)\big[E\left\|\tilde{\phi}_{\tau(\mathfrak{s},\tilde{\phi}_\mathfrak{s})}+\psi_{\tau(\mathfrak{s},\psi_\mathfrak{s})}\right\|^p+I_kE\left\|(\tilde{\phi}_{\tau(\varsigma,\tilde{\phi}_\varsigma)}+\psi_{\tau(\varsigma,\psi_\varsigma)})\right\|^p\big] \\
&\leq \frac{M^p}{\Gamma^p(q)}e^{-pq\alpha(\mathfrak{s}-\mathfrak{s}_k)}E\left\|\tilde{\phi}_{\tau(\mathfrak{s},\tilde{\phi}_\mathfrak{s}+\psi_\mathfrak{s})}+\psi_{\tau(\mathfrak{s},\tilde{\phi}+\psi_\mathfrak{s})}\right\|^p+\overline{q}_k\frac{M^p}{\Gamma^p(q)}e^{-pq\alpha(\mathfrak{s}-\mathfrak{s}_k)} \\
&\quad\times E\left\|\tilde{\phi}_{\tau(\mathfrak{s},\tilde{\phi}_\mathfrak{s}+\psi_\mathfrak{s})}+\psi_{\tau(\mathfrak{s},\tilde{\phi}_\mathfrak{s}\psi_\mathfrak{s})}\right\|^p \\
&\leq (\frac{M^p}{\Gamma^p(q)}e^{-pq\alpha(\mathfrak{s}-\mathfrak{s}_k)}+\overline{q}_k\frac{M^p}{\Gamma^p(q)}e^{-pq\alpha(\mathfrak{s}-\mathfrak{s}_k)})\tilde{N}_{\hat{a}}^p.
\end{aligned} \tag{18}$$

From the above inequalities (13)–(16) it follows that $E\left\|(\Theta\psi)(\mathfrak{s})\right\|^p \leq \hat{\rho}e^{-\mu\mathfrak{s}}$ for $\mathfrak{s} \in [-s, 0]$, and for every $\mathfrak{s} \geq 0$, we have
$E\left\|(\Theta\psi)(\mathfrak{s})\right\|^p \leq \hat{\rho}e^{-\mu\mathfrak{s}}+\overline{k}\sup_{-s\leq\leq 0}E\left\|(\Theta\psi)(\mathfrak{s}+\theta)\right\|^p+k\int_0^{\mathfrak{s}}e^{-\mu(\mathfrak{s}-\varsigma)}E\left\|(\Theta\psi)(\mathfrak{s})\right\|^p$
$\leq \hat{\rho}e^{-\mu\mathfrak{s}}+\overline{k}\sup_{-s\leq\leq 0}E\left\|(\Theta\psi)(\mathfrak{s}+\theta)\right\|^p d\varsigma+T_q(\mathfrak{s}_k)e^{-pq\alpha(\mathfrak{s}-\mathfrak{s}_k)})E\left\|(\Theta\psi)(\mathfrak{s})\right\|^p$
$+e^{-pq\alpha(\mathfrak{s}-\mathfrak{s}_k)})I_kE\left\|(\Theta\psi)(\mathfrak{s})\right\|^p$,
where $\overline{k} = (\hat{C}_4+1)\mathcal{R}_1+M^p(\hat{C}_2+1)\mathcal{R}_2+\frac{M^p}{\Gamma^p(q)}(\hat{C}_6+1)\mathcal{R}_3$ and $\hat{\rho}$

$$= \max\left(\sum_{i=1^6} \mathcal{G}_i \sup_{-s\leq\mathfrak{s}\leq 0} E\left\|(\Theta\psi)(\mathfrak{s}+\theta)\right\|^p\right).$$

The mild solution of system (1)–(3) is exponentially stable in $p|$th moment, since $\overline{k} + \frac{k}{\mu} + I_k < 1$. □

4 Applications

Consider the following INFSIDE driven by the Poisson jump, illustrating the obtained theory as follows:

$$\frac{d}{d\mathfrak{s}}\left[J^q x(\mathfrak{s}) - \int_{-3}^{0} e^{-2(\varsigma-\mathfrak{s})} x\big(\frac{\varsigma-\tau(\|x(\mathfrak{s})\|)}{25} d\varsigma\big) - \phi(0) + \mathcal{N}(0,\phi)\right] \tag{19}$$

$$= 0.2\left[x(t) - \int_{-3}^{0} e^{-2(\varsigma-\mathfrak{s})} x\big(\frac{\varsigma-\tau(\|x(\mathfrak{s})\|)}{25} d\varsigma\big)\right]$$

$$+ J_{\mathfrak{s}}^{1-q} \int_{-3}^{0} e^{-2(\varsigma-\mathfrak{s})} x\big(\frac{\varsigma-\tau(\|x(\mathfrak{s})\|)}{16}, z d\varsigma\big)$$

$$+ \int_{-3}^{0} e^{-2(\varsigma-\mathfrak{s})} x\big(\frac{\varsigma-\tau(\|x(\mathfrak{s})\|)}{36}, z\big) dW(\mathfrak{s})$$

$$+ \int_Z \int_{-3}^{0} e^{-4(\varsigma-\mathfrak{s})} \sin 3x\big(\frac{\varsigma-\tau(\|x(\mathfrak{s})\|)}{49}, z, \eta\big) \tilde{N}(d\mathfrak{s}, d\eta), \quad \mathfrak{s}\in[0,10],\ x\in(0,\pi),$$

$$x_0(\mathfrak{s}) = \varphi(\mathfrak{s}) \in \mathbb{B}, \tag{20}$$

$$\Delta^{1-q} x(t = \frac{1^-}{2}) = \sin(\frac{1}{7}\left\|\tau(\frac{1^-}{2}, x)\right\|), \tag{21}$$

where $o < q < 1$ and $\theta \in \mathbb{B}$ with impulsive moments. Choose $\tilde{\mathbb{V}} = L_2[0,\pi], \omega(t)$ is standard cylindrical wiener process in $\tilde{\mathbb{V}}$ described on a basis of stochastic $(\Omega, \{\mathcal{F}_{\mathfrak{s}}\}_{\mathfrak{s}\geq 0}, \mathbb{P})$. The operator $A : D(A) \subset \tilde{\mathbb{V}} \to \tilde{\mathbb{V}}$ is defined as $Ax = x$" with domain

$$D(A) = \{x \in \tilde{\mathbb{V}} : x, x' \text{are absolutely continuous } x\text{"} \in \tilde{\mathbb{V}}, x(0) = x(\pi) = 0\}.$$

Also, $T(\mathfrak{s})$ is defined as

$$T(t)x = \sum_{n=1}^{\infty} e^{-n^2\mathfrak{s}}(x, e_n)e_n.$$

Choose $\hat{P} \in (1, \infty), \tilde{s} \in [0, \infty)$, and $\Phi(-\infty, -\tilde{s}) \to \mathcal{R}$ is a nonnegative measurable function such that $\mathcal{L}$ is a locally integrable function, and there exists a nonnegative locally bounded function ϱ on $(-\infty, 0]$ such that $\Phi(\rho + \hat{\beta}) \leq \varrho(\rho)\Phi(\hat{\beta})$ for every $\tilde{s} \leq 0$ and $\hat{\beta} \in (-\infty, -\tilde{s})/M_\gamma$, where $M_\gamma \subseteq (-\infty, -\tilde{s})$ is a set whose Lebesgue measure is zero. Moreover, $\mathcal{P}C_{\tilde{s}} \times L^\alpha(\Phi, \tilde{\mathbb{V}})$ the set consists of all classes of function $\varphi : (-\infty, 0] \to \tilde{\mathbb{V}}$ such that $\varphi/(-\tilde{s}, 0) \in \mathcal{P}C([-\tilde{s}, 0], \tilde{\mathbb{V}}), \varphi(\cdot)$ is Lebesgue measurable in $(-\infty, -\tilde{s})$, and $\Phi \left\|\varphi\right\|^\alpha$ is Lebesgue integrable on $(-\infty, \tilde{s})$. The norm is denoted by

$$\|\varphi\|_{\mathbb{B}} = \sup_{\tilde{s} \leq \hat{\beta} \leq 0} \left\|\varphi(\hat{\beta})\right\| + \Big(\int_{-\infty}^{-\tilde{s}} \Phi(\hat{\beta}) \left\|\varphi\right\|^\alpha d\hat{\beta} \Big)^{\frac{1}{\alpha}}.$$

The space $\mathbb{B} = \mathcal{P}C_{-\tilde{s}} \times L^\alpha(\Phi, \tilde{\mathbb{V}})$ satisfies axioms. Furthermore, when $\tilde{s} = 0$ and $\alpha = 2$, one can assume $\tilde{H} = 1, \tilde{M}(\mathfrak{s}) = \sigma(-\mathfrak{s})^{\frac{1}{2}}$, and $K(\mathfrak{s}) = 1 + \Big(\int_{-\mathfrak{s}}^0 \Phi(\hat{\beta}) d\hat{\beta} \Big)^{\frac{1}{2}}$ for $\mathfrak{s} \geq 0$.

As one can have to rewrite system (17)–(19) in abstract form of (1)–(3), one can introduce the following notations:

$$\begin{aligned} x(\mathfrak{s}) &= \mu(\mathfrak{s}, \xi) \quad \text{for } \mathfrak{s} \geq 0 \text{ and } \xi \in [0, \pi] \\ \varphi(\mathfrak{s})(\tau) &= \mu_0(\mathfrak{s}, \xi) \quad \text{for } \mathfrak{s} \in (-\infty, 0] \text{ and } \xi \in [0, \pi]. \end{aligned}$$

Now, one can define the maps $\mathcal{L} : [0, \hat{a}] \times \mathbb{B} \to \tilde{\mathbb{V}}, f : [0, \hat{a}] \times \mathbb{B} \to \tilde{\mathbb{V}}, \sigma : [0, \hat{a}] \times L(\tilde{\mathbb{V}}) \to \tilde{\mathbb{V}}, I_k : [0, \hat{a}] \times \mathbb{B} \to \tilde{\mathbb{V}}$, and $\tau : [0, \hat{a}] \times \mathbb{B} \to \mathcal{R}$, denoted by

$$\begin{aligned} \mathcal{L}(\mathfrak{s}, x)(\xi) &= \frac{1}{25} \int_{-\infty}^0 e^{2\varsigma} \varphi d\varsigma, \\ f(\mathfrak{s}, x)(\xi) &= \frac{1}{16} \int_{-\infty}^0 e^{2\varsigma} \varphi d\varsigma, \\ \sigma(\mathfrak{s}, x)(\xi) &= \frac{1}{36} \int_{-\infty}^0 e^{2\varsigma} \varphi dW(\varsigma), \\ h(\mathfrak{s}, x, \eta)(\xi) &= \frac{1}{49} \int_{-\infty}^0 e^{2\varsigma} \varphi \lambda(d\eta) d\varsigma, \\ \tau(\mathfrak{s}, x) &= t \frac{1 + 2\left\|\varsigma(0)\right\|}{1 + 4\left\|\varsigma(0)\right\|}. \end{aligned}$$

Next, for $\varphi \in \mathbb{B}$, then the abstract form of (1)–(3), as in [5] if $\hat{\gamma}$ is a bounded and $\tilde{C}$ function such that

$$
\begin{aligned}
E\left\|\mathcal{L}(\mathfrak{s},\tilde{\phi})-\mathcal{L}(\mathfrak{s},\psi)\right\|_{\mathbb{B}}^{2} &\leq E\left[\int_{0}^{\pi}\left(\int_{-3}^{0}\frac{e^{\varsigma}}{25}\left\|\tilde{\phi}-\psi\right\|d\varsigma\right)^{2}d\xi\right]\\
&\leq \frac{\pi}{625}\left[\left(\int_{-3}^{0}\frac{e^{4x}}{x^{2}+1}dx\right)^{\frac{1}{2}}\left(\int_{-3}^{0}x^{2}+1dx\right)^{\frac{1}{2}}E\left\|\tilde{\phi}-\psi\right\|^{2}\right]\\
&=\frac{9.8778}{625}\left[0.4787\times 3.4641i\right]^{2}E\left\|\tilde{\phi}-\psi\right\|^{2}\\
&=-0.020804E\left\|\tilde{\phi}-\psi\right\|^{2},
\end{aligned}
$$

$$
\begin{aligned}
E\left\|f(\mathfrak{s},\tilde{\phi})-f(\mathfrak{s},\psi)\right\|_{\mathbb{B}}^{2} &\leq E\left[\int_{0}^{\pi}\left(\int_{-3}^{0}\frac{e^{\varsigma}}{16}\left\|\tilde{\phi}-\psi\right\|d\varsigma\right)^{2}d\xi\right]\\
&\leq \frac{\pi}{256}\left[\left(\frac{e^{4x}}{x^{3}+1}dx\right)^{\frac{1}{2}}\left(\int_{-3}^{0}x^{3}+1dx\right)^{\frac{1}{2}}E\left\|\tilde{\phi}-\psi\right\|^{2}\right]\\
&=\frac{9.8778}{256}\left[0.8881\times 4.15331i\right]^{2}E\left\|\tilde{\phi}-\psi\right\|^{2}\\
&=-0.524967E\left\|\tilde{\phi}-\psi\right\|_{\mathbb{B}}^{2},
\end{aligned}
$$

$$
\begin{aligned}
E\left\|\sigma(\mathfrak{s},\tilde{\phi})-\sigma(\mathfrak{s},\psi)\right\|_{\mathbb{B}}^{2} &\leq E\left[\int_{0}^{\pi}\left(\int_{-3}^{0}\frac{e^{\varsigma}}{36}\left\|\tilde{\phi}-\psi\right\|d\varsigma\right)^{2}d\xi\right]\\
&\leq \frac{\pi}{1296}\left[\left(\frac{e^{4x}}{x^{5}+1}dx\right)^{\frac{1}{2}}\left(\in\mathfrak{s}_{-3}^{0}x^{5}+1dx\right)^{\frac{1}{2}}E\left\|\tilde{\phi}-\psi\right\|^{2}\right]\\
&=\frac{9.8778}{1296}\left[(0.25237)^{\frac{1}{2}}\times(-118.5)^{\frac{1}{2}}\right]^{2}E\left\|\tilde{\phi}-\psi\right\|^{2}\\
&=-0.903178E\left\|\tilde{\phi}-\psi\right\|_{\mathbb{B}}^{2},
\end{aligned}
$$

$$
\begin{aligned}
E\left\|h(\mathfrak{s},\tilde{\phi})-h(\mathfrak{s},\psi)\right\|_{\mathbb{B}}^{2} &\leq E\left[\int_{0}^{\pi}\left(\int_{-3}^{0}\frac{e^{\varsigma}}{49}\left\|\tilde{\phi}-\psi\right\|d\varsigma\right)^{2}d\xi\right]\\
&\leq \frac{\pi}{2401}\left[\left(\frac{e^{4x}}{x^{7}+1}dx\right)^{\frac{1}{2}}\left(\int_{-3}^{0}x^{7}+1dx\right)^{\frac{1}{2}}E\left\|\tilde{\phi}-\psi\right\|^{2}\right]\\
&=\frac{9.8778}{2401}\left[(0.249033)^{\frac{1}{2}}\times(-817.125)^{\frac{1}{2}}\right]^{2}E\left\|\tilde{\phi}-\psi\right\|^{2}\\
&=-0.83715E\left\|\tilde{\phi}-\psi\right\|_{\mathbb{B}}^{2}.
\end{aligned}
$$

By substituting these values in (8), we got $-0.910264510<1$. Hence, the system (1)–(3) has a mild solution and is exponentially stable. One can

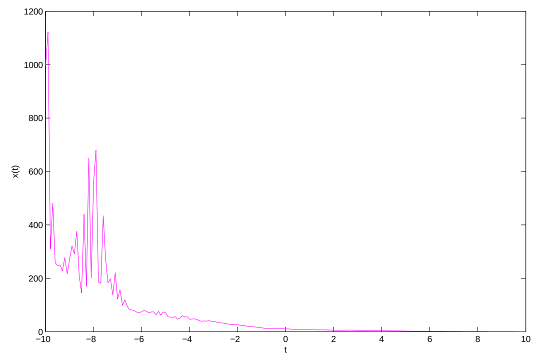

Figure 1: Stability trajectory of the power system of fractional power $q = 0.6$

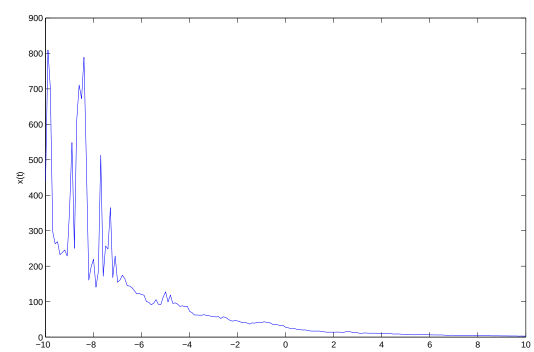

Figure 2: Stability trajectory of the power system of fractional power $q = 0.8$

observe the graphical representation of the exponential stability trajectory of INFSIDE and the synchronization of electrical signals between the generator through fractional power at $q = 0.6$ in Figure 1. The exponential stability trajectory of INFSIDE and the synchronization of electrical signals between the generators through fractional power at $q = 0.8$ are shown in Figure 2.

Remark 1. In the numerical example, simulations have been demonstrated in the mean square norm.

Remark 2. The proposed theoretical result is an effective tool to reach the goal subject to unpredictable Bm.
• Stability of BM and Poisson jumps are supported to synchronize the electrical signals in the power system.

5 Conclusion

Sufficient conditions have been obtained for the mild solutions of exponential stability of INFSIDE driven by Poisson jump with state-dependent delay. Fractional calculus, analytic resolvent operators, Mainardi function, and contraction mapping fixed point theorem are utilized to attain the main result. Numerical examples have been used to validate the theoretical results, and a real-world power system model using a fractional differential system has been supplied. In the future, the authors will extend the results to controllability, which supports controlling other abrupt changes like earthquakes and sudden fires in wires.

Disclosure statement
No potential conflict of interest was reported by the authors.

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