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Bayesian inference of reliability in the case of zero-failure data for the binomial distribution

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Abstract. This article explores Bayesian inference for the reliability of the binomial distribution in cases with zero-failure data. We propose Expected-Bayesian (E-Bayesian) and hierarchical Bayesian estimators of reliability, considering three different joint prior distributions of hyper-parameters in the prior distribution of reliability, which is assumed to follow a beta distribution. We derive closed-form expressions for the E-Bayesian estimators of reliability and propose hierarchical Bayesian estimators, which are subsequently evaluated using Monte Carlo simulations. Furthermore, we study the one-sided modified Bayesian (M-Bayesian) lower credible limits for reliability. The performance of the proposed methods is demonstrated through Monte Carlo simulations. Finally, a real data example is analyzed for illustrative purposes.

Keywords: E-Bayesian; Hierarchical Bayesian; M-Bayesian; Reliability; Zero-failure data.

1 Introduction

With the rapid advancement of manufacturing design techniques, many modern products are designed to operate without failure for several years, decades or more, such as high reliability and longevity products in aerospace, engineering and industry. In this situation, extensive discussions have taken place regarding reliability analysis for zero-failure data. Zero-failure data refers to situations where the test units exhibit no failure within the specified life testing time. Numerous statistical procedures have been proposed in the literature for analyzing reliability in such scenarios. For example, Martz and Waller (1979), Mao and Xia (1992), Han (2009), Chen et al. (1995) have contributed to this body of work. However, limited attention has been given to simultaneously addressing both point and interval estimators of reliability parameters for zero-failure data. Notable exceptions include the works of Fan and Chang (2009) and Zhang

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et al. (2019). In the paper, we will introduce and discuss the Expected-Bayesian (E-Bayesian) estimator, hierarchical Bayesian estimator, and one-sided modified Bayesian (M-Bayesian) lower credible limits of reliability for zero-failure data from the binomial distribution. These methods are explored under the assumption that the prior distribution of reliability follows a beta distribution.

For the zero-failure data, Martz and Waller (1979) first proposed a Bayesian zero-failure reliability demonstration testing procedure for an exponential failure-time model and a gamma prior distribution on the failure rate. Following Martz and Waller (1979), special attention has been paid to statistical analysis methods for zero-failure data. Mao and Luo (1989) proposed the matching distribution curve method, and Chen et al. (1993) developed a optimal confidence limit method. In addition, Fan and Chang (2009), Bailey (1997) and Fan et al. (2009) proposed Bayesian methods for estimating the reliability of explosive devices with binary data from destructive tests, where exponential lifetime distributions were used. Han (2007) first proposed an Expected-Bayesian (E-Bayesian) estimation method for estimating the failure probability, in the case of zero-failure data or very few zero-failure data. Jiang et al. (2010) put forward a modified maximum likelihood estimation (MMLE) and shrinkage estimation method to analyze zero-failure data from Weibull distribution. Subsequently, on the base of the thought of hierarchical prior distribution introduced by Lindley and Smith (1972). Han (2011) introduced E-Bayesian and hierarchical Bayesian estimation methods to estimate the reliability of the binomial distribution, however, the proposed methods only considered the case of only one hyper-parameter in the prior distribution of the reliability. Xia (2012) proposed a grey bootstrap method in the information poor theory for the reliability analysis of zero-failure data under the condition of a known or unknown lifetime distribution. Nam et al. (2013) presented a new accelerated zero-failure testing model for rolling bearings. Li et al. (2019) considered the classical Bayesian, E-Bayesian and hierarchical Bayesian estimates of reliability for zero-failure data from Weibull distribution, and confidence interval of reliability is also obtained by the parametric bootstrap method. Zhang et al. (2019) provided a Bayesian reliability evaluation method for very few failure data under the Weibull distribution. Zhang et al. (2021) proposed a Bayesian method for analyzing few or zero failure data using re-parametrization of the Weibull distribution. However, these researches only discussed the case of one of two hyper-parameters is random variable in the prior distribution of failure probability or reliability.

In recent years, the credible (confidence) intervals of reliability parameters based on zero failure data has also made new progress. For example, Han (2008) proposed a two-sided M-Bayesian credible limits method of reliability parameters, in the case of zero-failure data from exponential distribution; furthermore, the properties for two-sided M-Bayesian credible limits of failure rate were discussed. Subsequently, Han (2012) put forward a M-Bayesian credible limits method for estimating the reliability of binomial distribution, in the case of zero-failure data. Tian and Chen (2014) studied interval estimates of failure rate and reliability for the exponential distribution based on zero-failure data, using the method of two-sided M-Bayesian credible limits. Jiang et al. (2015) presented a interval estimation method of failure probability for Weibull distribution under zero-failure data situation. Zhang (2021) proposed a novel method that does not require prior information for estimating Weibull parameters in the absence of failure data, an unbiased estimator and confidence interval for the Weibull scale parameter are also obtained. However, these researches only considered the case of only one of two hyper-parameters is random in the prior distribution of reliability parameter. In the following, we aim to discuss the Bayesian inference for the reliability of binomial products, in the case of zero-failure data. This discussion encompasses the point estimation method and one-sided M-Bayesian lower credible limits for reliability, based on three different priors, when the prior distribution of reliability is assumed to be a beta distribution with hyper-parameters a and b.

The remainder of this article is organized as follows. In Section 2, we will first introduce the zero-failure data for type-I censored life testing and describe the prior distribution of reliability. Section 3 then presents point estimation of reliability for zero-failure data, including E-Bayesian estimator and hierarchical Bayesian estimator. In Section 4, we discuss one-sided M-Bayesian lower credible limits of reliability in the case of zero-failure data, considering three different joint prior distributions of hyper-parameters. In Section 5, numerical examples are provided, and Section 6 contains some conclusions and several remarks.

2 Zero-failure data for type-I censored life testing

In this section, we first introduce the zero-failure data for highly reliable products in type-I censored life testing. Furthermore, the prior distribution of reliability for zero-failure data is assumed based on engineering experience.

2.1 The zero-failure data

In some cases in reliability engineering, determining the life distribution of products becomes challenging when there is no failure information except for the number of failures. In such situations, reliability estimates can be obtained through the application of nonparametric methods designed for binomial distribution.

In this article, we assume that the distribution of the lifetime T of products is unknown, conduct type-I censored life testing k times, the censoring time of these independent trials are denoted as $t_1, t_2, ..., t_k$ $(0 < t_1 < t_2 < \cdots < t_k)$, respectively. There are n_i testing samples at the censoring time $t_i, i = 1, 2, ..., k$, if these testing units are all without failure at the censoring time t_i, it can be considered that the lifetimes of these testing units is greater than the censoring time $t_i, i = 1, 2, ..., k$. Therefore, the corresponding test data of the k times type-I censored life testing are called the zero-failure data, and denoted as $(t_i, n_i), i = 1, 2, ..., k$. According to the whole testing process of the type-I censored life testing, the following test information can be obtained.

(1) At the beginning time of the type-I censored life testing, the reliability of products can be expressed as $R_0 = Pr(T > 0) = 1$.

(2) At the censoring time t_i, the reliability of products denoted as $R_i = Pr(T > t_i), i = 1, 2, ..., k$, it is clear that $R_0 > R_1 > \cdots > R_k$.

(3) Let $s_i = n_i + n_{i+1} + \cdots + n_k$, it indicates that there are s_i testing units survival at the censoring time t_i, i.e., s_i is the number of products which lifetime is longer than $t_i, i = 1, 2, ..., k$.

Consequently, the above problem can be expressed as evaluating the reliability of products by using the zero-failure data $(t_i, n_i), i = 1, 2, ..., k$, in the type-I censored life testing.

2.2 The prior distribution of reliability

According to some engineering experience and the results of Han (2011), Beta distribution is often used as the conjugate prior distribution of reliability (i.e., the probability of success) in the reliability modeling of zero-failure data for binomial products. We thus take the beta distribution $Beta(a,b)$ as the conjugate prior distribution of the reliability R_i, and the probability density function (pdf) of R_i is

$$\pi(R_i|a,b) = \frac{1}{B(a,b)} R_i^{a-1}(1-R_i)^{b-1}, 0 < R_i < 1, i = 1,2,...,k, \tag{1}$$

where $B(a,b) = \int_0^1 t^{a-1}(1-t)^{b-1}\mathrm{d}t$ is the beta function, and hyper-parameters $a > 0, b > 0$.

With the continuous improvement of science and technology, the reliability of products is constantly improved, so the possibility of high reliability of products is large, and the possibility of low reliability is small. According to Han (2011), we choose a and b such that $\pi(R_i|a,b)$ is an increasing function of R_i. The derivative of $\pi(R_i|a,b)$ with respect to R_i is

$$\frac{\mathrm{d}\pi(R_i|a,b)}{\mathrm{d}R_i} = \frac{R_i^{a-1}(1-R_i)^{b-1}}{B(a,b)} \left[(a-1)(1-R_i) - (b-1)R_i\right], i = 1,2,...,k, \tag{2}$$

Since $a > 0, b > 0$ and $0 < R_i < 1$, from (2), we thus can easily see that $\mathrm{d}\pi(R_i|a,b)/\mathrm{d}R_i > 0$ on $a > 1, 0 < b < 1$, and then $\pi(R_i|a,b)$ is an increasing function of R_i.

According to the results of Berger (1985) and Han (2011) on the Bayesian estimation, from the perspective of the robustness of Bayesian estimation, the thinner the tail of a prior distribution, and the worse the robustness of Bayesian estimation. Therefore, the hyper-parameter a should not be too large, and there should be an upper bound c to be determined, where c is a constant greater than 1. Thereby, the hyper-parameters a and b can be selected in the range of $1 < a < c$ and $0 < b < 1$.

3 Point estimation of reliability

In this section, we investigate point estimation of reliability using the methods of E-Bayesian estimation and hierarchical Bayesian estimation, respectively.

3.1 E-Bayesian estimator of reliability

Referring to the concept of E-Bayesian estimator of failure probability proposed by Han (2007), this method is applicable in cases with either zero or few failure data. In this subsection, we present E-Bayesian estimators of R_i under three different joint prior distributions of hyper-parameters a and b in the prior distribution of R_i.

In the following, we first present the definition of the E-Bayesian estimator of R_i when the prior distribution of R_i is distributed as $Beta(a,b)$.

Definition 1. *Let $\hat{R}_{iB}(a,b)$ be continuous, if*

$$\iint_D \left|\hat{R}_{iB}(a,b)\right| \pi(a,b)\mathrm{d}a\mathrm{d}b < \infty, \;\; i = 1,2,...,k,$$

then

$$\hat{R}_{iEB} = \iint_D \hat{R}_{iB}(a,b)\pi(a,b)\mathrm{d}a\mathrm{d}b,\ \ i=1,2,...,k,$$

is called the Expected Bayesian (E-Bayesian) estimator of R_i*, where* $\hat{R}_{iB}(a,b)$ *is the Bayesian estimator of* R_i *with hyper-parameters* a *and* b*,* $D=\{(a,b)|1<a<c,0<b<1\}$*,* $\pi(a,b)$ *is the pdf of hyper-parameters* a *and* b *in the region* D*.*

From the Definition 1, it is easy to see that the E-Bayesian estimator of R_i is the mathematical expectation for the Bayesian estimator $\hat{R}_{iB}(a,b)$, that is,

$$\hat{R}_{iEB} = \iint_D \hat{R}_{iB}(a,b)\pi(a,b)\mathrm{d}a\mathrm{d}b = E[\hat{R}_{iB}(a,b)],\ \ i=1,2,...,k.$$

According to the Bayesian theorem and the Definition 1, we have the following theorem, the proof see Appendix A.

Theorem 1. *For the zero-failure data* (t_i,n_i) *from the* k *times type-I censored life testing, let* $s_i=\sum_{j=i}^{k} n_j$*,* $i=1,2,...,k$*. If the prior density function* $\pi(R_i|a,b)$ *of* R_i *is presented by the formula (1), then have the following conclusions.*

(1) Using the squared error loss function, the Bayesian estimator of R_i *is* $\hat{R}_{iB}(a,b)=\dfrac{s_i+a}{s_i+a+b}$*,*

(2) For the following three different joint prior distributions of hyper-parameters a *and* b*,*

$$\pi_1(a,b)=\frac{2(c-a)}{(c-1)^2},\ \ 1<a<c,0<b<1, \tag{3}$$

$$\pi_2(a,b)=\frac{1}{c-1},\ \ 1<a<c,0<b<1, \tag{4}$$

$$\pi_3(a,b)=\frac{2a}{c^2-1},\ \ 1<a<c,0<b<1, \tag{5}$$

the corresponding E-Bayesian estimators of R_i *are presented as follows, respectively,*

$$\hat{R}_{iEB1}=\frac{2}{(c-1)^2}\left[G_1(s_i,c)-G_2(s_i,c)\right],\ \ i=1,2,...,k,$$

$$\hat{R}_{iEB2}=\frac{1}{c-1}\left[G_3(s_i,c)-G_4(s_i,c)\right],\ \ i=1,2,...,k,$$

$$\hat{R}_{iEB3}=\frac{2}{c^2-1}\left[G_5(s_i,c)-G_6(s_i,c)\right],\ \ i=1,2,...,k,$$

where $G_1(s_i,c)$ *to* $G_6(s_i,c)$ *and* $q_1(s_i,c)$ *to* $q_5(s_i,c)$ *are provided in Appendix A.*

It is easy to show that the E-Bayesian estimators of R_i in the Theorem 1 have the following properties.

Corollary 1. *In Theorem 1, for the E-Bayesian estimators* $\hat{R}_{iEBj}$ $(i=1,2,...,k,j=1,2,3)$*, the following two properties are hold almost surely,*

(1) For the fixed i*,* $\hat{R}_{iEB1}<\hat{R}_{iEB2}<\hat{R}_{iEB3}$*;*

(2) For the fixed i*,* $\lim_{s_i\to\infty}\hat{R}_{iEB1}=\lim_{s_i\to\infty}\hat{R}_{iEB2}=\lim_{s_i\to\infty}\hat{R}_{iEB3}$*.*

3.2 Hierarchical Bayesian estimator of reliability

In this subsection, we present the hierarchical Bayesian estimators of R_i $(i = 1, 2, ..., k)$, based on three different joint prior distributions of hyper-parameters a and b. In the following, we first introduce the hierarchical prior distribution of R_i $(i = 1, 2, ..., k)$.

According to the definition of hierarchical prior distribution proposed by Good (1983). If we select $\pi(R_i|a, b)$, which presented by the formula (1), as the prior density function of R_i, and take formulas (3) to (5) as the joint prior distributions of hyper-parameters a and b, respectively, then the hierarchical prior distributions of R_i $(i = 1, 2, ..., k)$ can be easily obtained. For instance,

$$\pi_4(R_i) = \int_0^1 \int_1^c \pi(R_i|a,b)\pi_1(a,b)\mathrm{d}a\mathrm{d}b = \frac{2}{(c-1)^2}\int_0^1 \int_1^c \frac{(c-a)}{B(a,b)} R_i^{a-1}(1-R_i)^{b-1}\mathrm{d}a\mathrm{d}b, \tag{6}$$

$$\pi_5(R_i) = \int_0^1 \int_1^c \pi(R_i|a,b)\pi_2(a,b)\mathrm{d}a\mathrm{d}b = \frac{1}{c-1}\int_0^1 \int_1^c \frac{1}{B(a,b)} R_i^{a-1}(1-R_i)^{b-1}\mathrm{d}a\mathrm{d}b, \tag{7}$$

$$\pi_6(R_i) = \int_0^1 \int_1^c \pi(R_i|a,b)\pi_3(a,b)\mathrm{d}a\mathrm{d}b = \frac{2}{c^2-1}\int_0^1 \int_1^c \frac{a}{B(a,b)} R_i^{a-1}(1-R_i)^{b-1}\mathrm{d}a\mathrm{d}b. \tag{8}$$

We can then use Bayes' theorem to derive the hierarchical Bayesian estimators of R_i, and the results are provided in the next theorem, the proof see Appendix B.

Theorem 2. *For the zero-failure data (t_i, n_i) from the k times type-I censored life testing, let $s_i = \sum_{j=i}^k n_j$, $i = 1, 2, ..., k$. If the hierarchical prior distributions of R_i are given by (6) to (8), then under the squared error loss function, the corresponding hierarchical Bayesian estimators of R_i are, respectively,*

$$\hat{R}_{iHB1} = \frac{\int_0^1 \int_1^c (c-a)\frac{B(a+s_i+1,b)}{B(a,b)}\mathrm{d}a\mathrm{d}b}{\int_0^1 \int_1^c (c-a)\frac{B(a+s_i,b)}{B(a,b)}\mathrm{d}a\mathrm{d}b}, \quad i = 1, 2, ..., k,$$

$$\hat{R}_{iHB2} = \frac{\int_0^1 \int_1^c \frac{B(a+s_i+1,b)}{B(a,b)}\mathrm{d}a\mathrm{d}b}{\int_0^1 \int_1^c \frac{B(a+s_i,b)}{B(a,b)}\mathrm{d}a\mathrm{d}b}, \quad i = 1, 2, ..., k,$$

$$\hat{R}_{iHB3} = \frac{\int_0^1 \int_1^c a\frac{B(a+s_i+1,b)}{B(a,b)}\mathrm{d}a\mathrm{d}b}{\int_0^1 \int_1^c a\frac{B(a+s_i,b)}{B(a,b)}\mathrm{d}a\mathrm{d}b}, \quad i = 1, 2, ..., k.$$

Taking use of the Monte Carlo simulation method, we can easily get the hierarchical Bayesian estimators of R_i in the Theorem 2.

4 One-sided credible limits of reliability

In this section, we investigate the one-sided modified Bayesian (M-Bayesian) lower credible limits of R_i $(i = 1, 2, ..., k)$, based on three different joint priors for the hyper-parameters a and b.

4.1 One-sided Bayesian lower credible limits

We first propose the following lemma, which aims to derive the one-sided Bayesian lower credible limits of R_i $(i=1,2,...,k)$ based on three different priors for the hyper-parameter a and b.

Lemma 1. *For the zero-failure data* (t_i,n_i) $(i=1,2,...,k)$, *if the prior distribution of* R_i *is given by* (1), *that is* R_i *is a beta random variable with parameters* a *and* b, *we then have* $\frac{a+s_i}{b}\frac{1-R_i}{R_i}$ *follows a* F *distribution with the degrees of freedom* $2b$ *and* $2(a+s_i)$, *i.e.,*

$$\frac{a+s_i}{b}\frac{1-R_i}{R_i}\sim F(2b,2(a+s_i)),\ \ i=1,2,..,k.$$

By means of the Lemma 1 and the theory of Bayesian credible limits, we can easily obtain the one-sided Bayesian lower credible limits of R_i. The result is given by the following corollary.

Corollary 2. *For the zero-failure data* (t_i,n_i) $(i=1,2,...,k)$, *if the prior distribution of* R_i *is given by* (1), *then the 100*$(1-\alpha)$% *one-sided Bayesian lower credible limit of* R_i *is*

$$\hat{R}_{iBL}(a,b)=\frac{a+s_i}{a+s_i+bF_{1-\alpha}(2b,2(a+s_i))},\ \ i=1,2,...,k. \tag{9}$$

where $F_{1-\alpha}(2b,2(a+s_i))$ *is the 1-*α *quantiles of* F *distribution with the degrees of freedom* $2b$ *and* $2(a+s_i)$.

4.2 One-sided M-Bayesian lower credible limits

In this subsection, we consider the one-sided M-Bayesian lower credible limits of R_i $(i=1,2,...,k)$. We thus first give the definition of one-sided M-Bayesian lower credible limit of R_i.

Definition 2. *Let* $\hat{R}_{iBL}(a,b)$ *be continuous, if*

$$\iint_D\left|\hat{R}_{iBL}(a,b)\right|\pi(a,b)\mathrm{d}a\mathrm{d}b<\infty,\ \ i=1,2,...,k,$$

then

$$\hat{R}_{iMBL}=\iint_D\hat{R}_{iBL}(a,b)\pi(a,b)\mathrm{d}a\mathrm{d}b,\ \ i=1,2,...,k,$$

is called the one-sided modified Bayesian (M-Bayesian) lower credible limit of the reliability R_i, *where* $D=\{(a,b)|1<a<c,0<b<1\}$ *is the domain of* a *and* b, $\hat{R}_{iBL}(a,b)$ *is the one-sided Bayesian lower credible limit of* R_i *with hyper-parameters* a *and* b, $\pi(a,b)$ *is the pdf of hyper-parameters* a *and* b *in the region* D.

We have the following theorem about one-sided M-Bayesian lower credible limits of R_i, for the proof see Appendix C.

Theorem 3. *For the zero-failure data* (t_i,n_i) $i=1,2,...,k$, *if the prior density function of* R_i *is given by* (1), *then under the credible level of* $1-\alpha$ $(0<\alpha<1)$, *for the three different joint prior* *distributions (3) to (5) of the hyper-parameters* a *and* b*, the corresponding one-sided M-Bayesian lower credible limits of* R_i *are, respectively,*

$$\hat{R}_{iMBL1} = \frac{2}{(c-1)^2}\int_0^1\int_1^c \frac{(c-a)(a+s_i)}{a+s_i+bF_{1-\alpha}(2b,2(a+s_i))}\mathrm{d}a\mathrm{d}b,\ \ i=1,2,...,k,$$
$$\hat{R}_{iMBL2} = \frac{1}{c-1}\int_0^1\int_1^c \frac{a+s_i}{a+s_i+bF_{1-\alpha}(2b,2(a+s_i))}\mathrm{d}a\mathrm{d}b,\ \ i=1,2,...,k,$$
$$\hat{R}_{iMBL3} = \frac{2}{c^2-1}\int_0^1\int_1^c \frac{a(a+s_i)}{a+s_i+bF_{1-\alpha}(2b,2(a+s_i))}\mathrm{d}a\mathrm{d}b,\ \ i=1,2,...,k.$$

The one-sided M-Bayesian lower credible limits in Theorem 3 can be evaluated by the Monte Carlo Integration method. Furthermore, the analysis results of the Monte Carlo simulation show that the one-sided M-Bayesian lower credible limits have the following properties.

Corollary 3. *In Theorem 3, for the one-sided M-Bayesian lower credible limits* $\hat{R}_{iMBLj}$ $(i=1,2,...,k, j=1,2,3)$*, the following two properties are hold almost surely,*

(1) For the specified i*,* $\hat{R}_{iMBL1} < \hat{R}_{iMBL2} < \hat{R}_{iMBL3}$*,*

(2) For the specified i*,* $\lim_{s_i\to\infty}\hat{R}_{iMBL1} = \lim_{s_i\to\infty}\hat{R}_{iMBL2} = \lim_{s_i\to\infty}\hat{R}_{iMBL3}$*.*

5 Numerical analysis

5.1 Simulation study

In this subsection, simulation study is conducted to investigate the performance of the point estimators of R_i (including E-Bayesian and hierarchical estimators). According to the Theorems 1 and 2 in Section 3, by simulating the value of s_i $(i=1,2,...,6)$, we can obtain the E-Bayesian estimators $\hat{R}_{iEBj}$ $(j=1,2,3)$, and the hierarchical Bayesian estimators $\hat{R}_{iHBj}$ $(j=1,2,3)$. The results of each $\hat{R}_{iEBj}, \hat{R}_{iHBj}$ are shown in Tables 1 and 2, respectively.

It is clearly that from Tables 1 and 2, for the fixed i $(i=1,2,...,k)$, the E-Bayesian estimators $\hat{R}_{iEBj}$ $(j=1,2,3)$ are very close to each other for different values of c (4, 5, 6, 7, 8). In addition, the same conclusion is suitable for the hierarchical Bayesian estimators $\hat{R}_{iHEBj}$ $(j=1,2,3)$. This indicates that $\hat{R}_{iEBj}$, $\hat{R}_{iHBj}$ $(j=1,2,3)$ are all robust for different values of c. Therefore, in practical application, we can take the midpoint of interval [4, 8] as the value of c, i.e., $c=6$. Besides, since $\hat{R}_{iEBj}$, $\hat{R}_{iHBj}$ $(j=1,2,3)$ are all robust for the specified c (4, 5, 6, 7, 8) under three different joint prior distributions of hyper-parameters a and b, in order to reduce the computational complexity, we thus suggest that the uniform distribution (**??**) can be used as the joint prior distribution of a and b. Finally, for the same values of c and i, the E-Bayesian estimators $\hat{R}_{iEBj}$ are very close to hierarchical Bayesian estimators $\hat{R}_{iHBj}$, thus the E-Bayesian method is preferred to estimate the reliability R_i.

5.2 An illustrative example

In this subsection, we present a real data set to assess the performances of the proposed methods for estimating reliability based on zero-failure data. We consider the zero-failure data of type-I censored life testing of engines from Han (2007), which is listed in Table 3 (time unit: hour).

Table 1: The Monte Carlo simulation results of $\hat{R}_{iEBj}$ $(j = 1,2,3)$ for different values of c.

s_i	$\hat{R}_{iEBj}$	c = 4	5	6	7	8
1000	$\hat{R}_{iEB1}$	0.999501	0.999501	0.999502	0.999502	0.999502
	$\hat{R}_{iEB2}$	0.999502	0.999502	0.999502	0.999502	0.999503
	$\hat{R}_{iEB3}$	0.999502	0.999502	0.999502	0.999503	0.999503
500	$\hat{R}_{iEB1}$	0.999005	0.999006	0.999007	0.999007	0.999008
	$\hat{R}_{iEB2}$	0.999006	0.999007	0.999008	0.999009	0.999010
	$\hat{R}_{iEB3}$	0.999007	0.999008	0.999009	0.999011	0.999012
100	$\hat{R}_{iEB1}$	0.995130	0.995145	0.995161	0.995176	0.995191
	$\hat{R}_{iEB2}$	0.995153	0.995176	0.995199	0.995222	0.995244
	$\hat{R}_{iEB3}$	0.995167	0.995197	0.995226	0.995256	0.995285
50	$\hat{R}_{iEB1}$	0.990504	0.990563	0.990620	0.990677	0.990732
	$\hat{R}_{iEB2}$	0.990593	0.990679	0.990762	0.990844	0.990924
	$\hat{R}_{iEB3}$	0.990646	0.990756	0.990864	0.990970	0.991074
20	$\hat{R}_{iEB1}$	0.977918	0.978223	0.978516	0.978798	0.979069
	$\hat{R}_{iEB2}$	0.978385	0.978821	0.979234	0.979627	0.980001
	$\hat{R}_{iEB3}$	0.978665	0.979219	0.979747	0.980249	0.980726
10	$\hat{R}_{iEB1}$	0.960392	0.961329	0.962204	0.963025	0.963797
	$\hat{R}_{iEB2}$	0.961847	0.963139	0.964322	0.965412	0.966419
	$\hat{R}_{iEB3}$	0.962720	0.964345	0.965835	0.967202	0.968459

According to the results of simulation study, we select $c = 6$ as a supper limit of the hyper-parameter a. Therefore, according to the Theorems 1 and 2, we can easily obtain the E-Bayesian estimators $\hat{R}_{iEBj}$ $(j = 1,2,3)$, and the hierarchical Bayesian estimators $\hat{R}_{iHBj}$ $(j = 1,2,3)$ of R_i at each censoring time t_i $(i = 1,2,...,9)$. The results are provided in Table 4 and Figure 1, respectively.

By Theorem 3, with different credible levels $1-\alpha$=0.99, 0.95, 0.9, 0.85, 0.8, 0.75, 0.7, and taking c =6 as a upper limit of hyper-parameter a, we can then obtain the one-sided M-Bayesian lower credible limits $\hat{R}_{iMBLj}$ $(j = 1,2,3)$ of R_i at each censoring time t_i $(i = 1,2,...,9)$. The corresponding results are presented in Table 5 and Figures 2 and 3.

According to the Table 5 and Figures 2 and 3, we can see that the one-sided M-Bayesian lower credible limits of R_i satisfy the Corollary 3 under the same credible level 1-α. In addition, Table 5 and Figures 2 and 3 show that there ia a very small difference among the three kinds of one-sided M-Bayesian lower credible limits of R_i.

6 Conclusion

This paper investigates point estimation and credible limits for the reliability of the binomial distribution in the case of zero-failure data from a Bayesian perspective.

Table 2: The Monte Carlo simulation results of $\hat{R}_{iHBj}$ $(j = 1,2,3)$ for different values of c.

		c				
s_i	$\hat{R}_{iHBj}$	4	5	6	7	8
1000	$\hat{R}_{iHB1}$	0.999845	0.999841	0.999838	0.999835	0.999832
	$\hat{R}_{iHB2}$	0.999839	0.999834	0.999830	0.999827	0.999823
	$\hat{R}_{iHB3}$	0.999836	0.999830	0.999825	0.999821	0.999817
500	$\hat{R}_{iHB1}$	0.999655	0.999645	0.999638	0.999631	0.999624
	$\hat{R}_{iHB2}$	0.999641	0.999630	0.999620	0.999612	0.999604
	$\hat{R}_{iHB3}$	0.999633	0.999620	0.999608	0.999598	0.999590
100	$\hat{R}_{iHB1}$	0.997749	0.997690	0.997640	0.997596	0.997556
	$\hat{R}_{iHB2}$	0.997660	0.997590	0.997530	0.997480	0.997438
	$\hat{R}_{iHB3}$	0.997612	0.997528	0.997460	0.997406	0.997357
50	$\hat{R}_{iHB1}$	0.994988	0.994876	0.994785	0.994704	0.994732
	$\hat{R}_{iHB2}$	0.994813	0.994685	0.994589	0.994506	0.994639
	$\hat{R}_{iHB3}$	0.994717	0.994572	0.994464	0.994377	0.994444
20	$\hat{R}_{iHB1}$	0.986136	0.985793	0.985880	0.985824	0.985802
	$\hat{R}_{iHB2}$	0.985863	0.985751	0.985707	0.985708	0.985757
	$\hat{R}_{iHB3}$	0.985725	0.985623	0.985605	0.985642	0.985728
10	$\hat{R}_{iHB1}$	0.971898	0.972036	0.972229	0.972512	0.972796
	$\hat{R}_{iHB2}$	0.971990	0.972361	0.972809	0.973292	0.973778
	$\hat{R}_{iHB3}$	0.972052	0.972548	0.973145	0.973780	0.974458

Table 3: The zero-failure data of some engine.

i	1	2	3	4	5	6	7	8	9
t_i	250	450	650	850	1050	1250	1450	1650	1850
n_i	3	3	3	3	4	4	4	4	4
s_i	32	29	26	23	20	16	12	8	4

Table 4: The estimators of $\hat{R}_{iEBj}, \hat{R}_{iHBj}$ $(j = 1,2,3)$ of reliability of some engine at t_i $(i = 1,2,...,9)$ $(c = 6)$.

t_i	250	450	650	850	1050	1250	1450	1650	1850
$\hat{R}_{iEB1}$	0.985833	0.984515	0.982926	0.980974	0.978516	0.974040	0.967197	0.955402	0.930041
$\hat{R}_{iHB1}$	0.991436	0.990469	0.989283	0.987794	0.985878	0.982282	0.976559	0.966194	0.942310
$\hat{R}_{iEB2}$	0.986152	0.984895	0.983386	0.981541	0.979234	0.975075	0.968815	0.958294	0.936684
$\hat{R}_{iHB2}$	0.991199	0.990233	0.989053	0.987583	0.985706	0.982230	0.976807	0.967228	0.946544
$\hat{R}_{iEB3}$	0.986381	0.985167	0.983715	0.981974	0.979747	0.975814	0.969971	0.960359	0.941430
$\hat{R}_{iHB3}$	0.991053	0.990087	0.988911	0.987453	0.985601	0.982198	0.976961	0.967968	0.949221

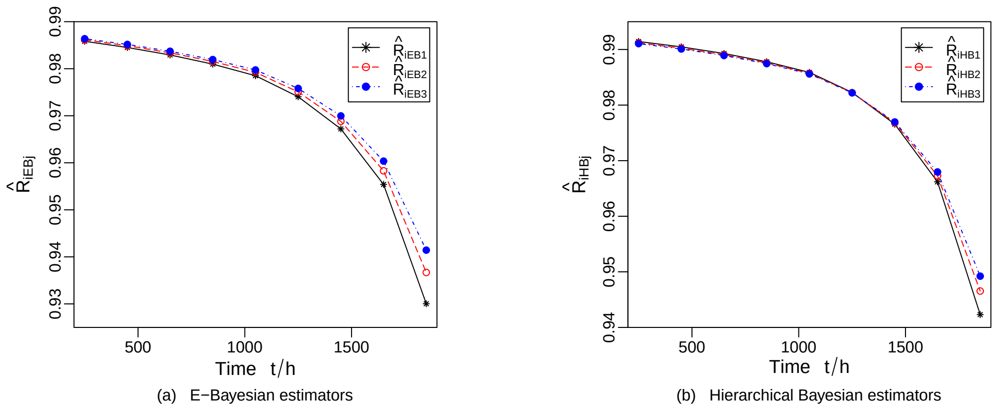

Figure 1: E-Bayesian estimators $\hat{R}_{iEBj}$ and hierarchical Bayesian estimators $\hat{R}_{iHBj}$ of reliability R_i.

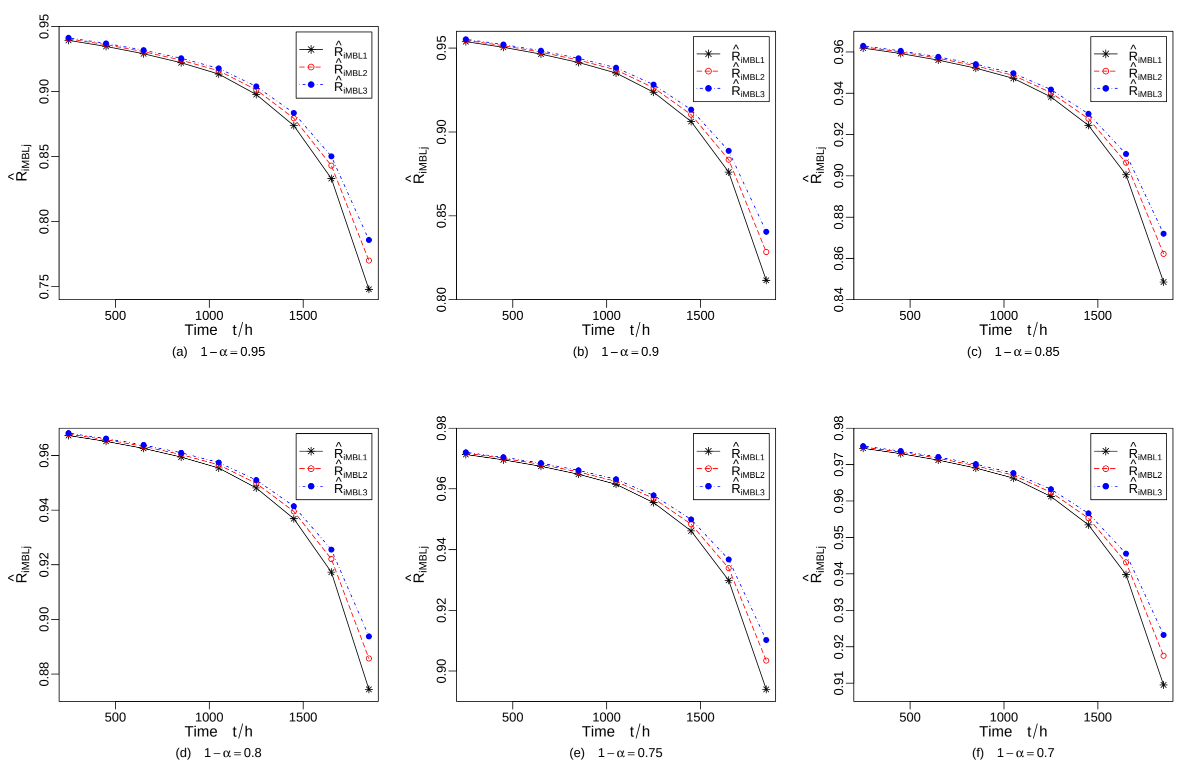

Figure 2: One-sided M-Bayesian lower credible limits of reliability R_i at different credible level 1-α.

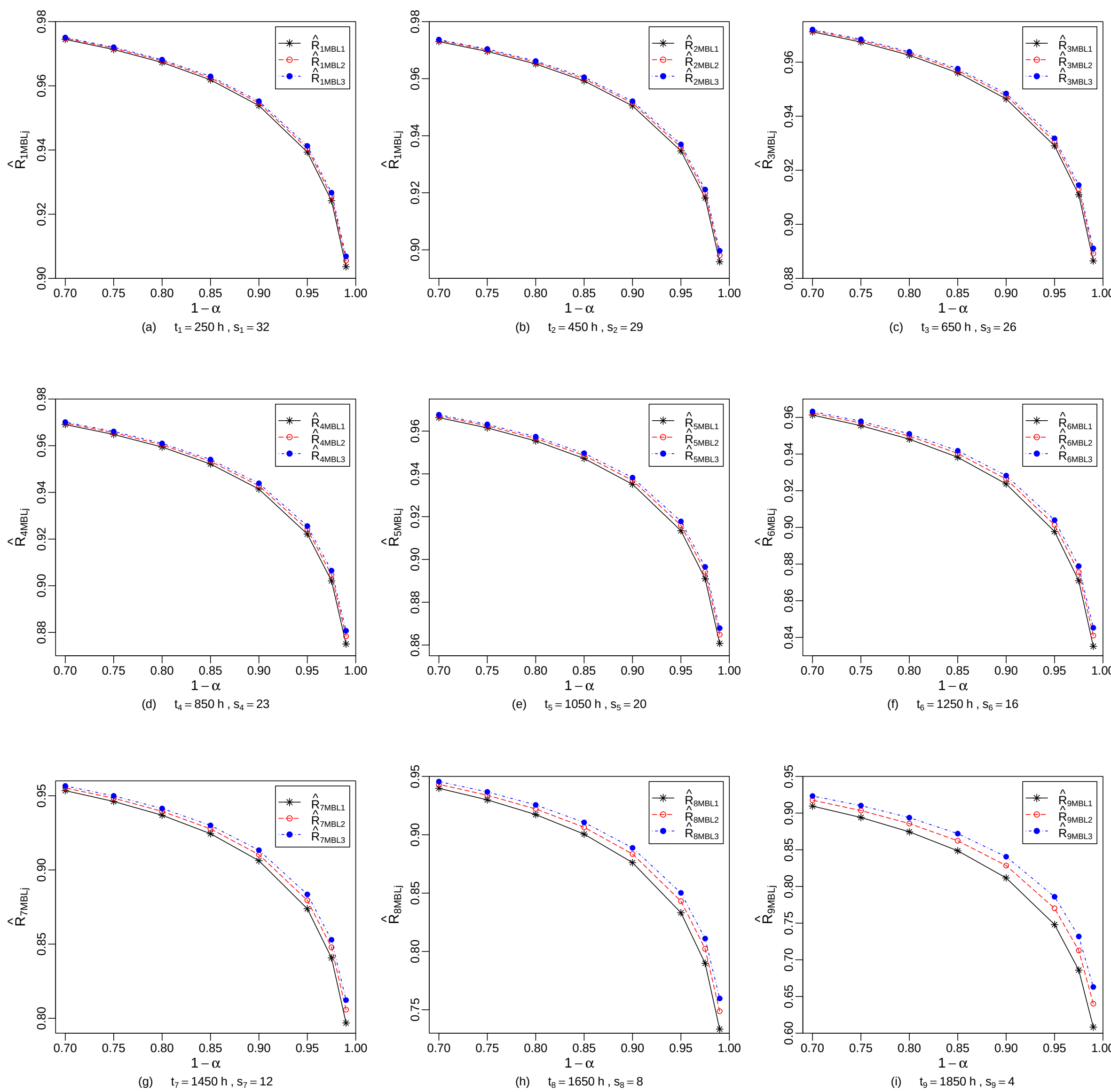

Figure 3: One-sided M-Bayesian lower credible limits of reliability R_i for different credible level at each censoring time.

Table 5: One-sided M-Bayesian credible lower limits of R_i for different credible levels 1-α (c =6).

			$1-\alpha$						
t_i	s_i	$\hat{R}_{iHBj}$	0.99	0.95	0.90	0.85	0.80	0.75	0.7
250	32	$\hat{R}_{iMBL1}$	0.903628	0.939340	0.953827	0.961854	0.967281	0.971307	0.974455
		$\hat{R}_{iMBL2}$	0.905528	0.940475	0.954643	0.962491	0.967797	0.971732	0.974809
		$\hat{R}_{iMBL3}$	0.906885	0.941285	0.955226	0.962947	0.968166	0.972036	0.975062
450	29	$\hat{R}_{iMBL1}$	0.895819	0.934663	0.950460	0.959222	0.965150	0.969549	0.972990
		$\hat{R}_{iMBL2}$	0.898069	0.936012	0.951431	0.959981	0.965765	0.970056	0.973413
		$\hat{R}_{iMBL3}$	0.899677	0.936976	0.952125	0.960523	0.966204	0.970419	0.973715
650	26	$\hat{R}_{iMBL1}$	0.886458	0.929035	0.946401	0.956046	0.962577	0.967424	0.971220
		$\hat{R}_{iMBL2}$	0.889164	0.930665	0.947576	0.956965	0.963322	0.968041	0.971733
		$\hat{R}_{iMBL3}$	0.891098	0.931828	0.948415	0.957622	0.963854	0.968480	0.972099
850	23	$\hat{R}_{iMBL1}$	0.875030	0.922131	0.941412	0.952138	0.959408	0.964807	0.969039
		$\hat{R}_{iMBL2}$	0.878347	0.924138	0.942863	0.953275	0.960330	0.965571	0.969673
		$\hat{R}_{iMBL3}$	0.880716	0.925571	0.943898	0.954086	0.960988	0.966144	0.970127
1050	20	$\hat{R}_{iMBL1}$	0.860764	0.913460	0.935132	0.947212	0.955410	0.961503	0.966282
		$\hat{R}_{iMBL2}$	0.864925	0.915993	0.936967	0.948652	0.956579	0.962472	0.967088
		$\hat{R}_{iMBL3}$	0.867897	0.917803	0.938278	0.949681	0.957414	0.963162	0.967662
1250	16	$\hat{R}_{iMBL1}$	0.835142	0.897735	0.923696	0.938223	0.948103	0.955459	0.961235
		$\hat{R}_{iMBL2}$	0.841038	0.901365	0.926340	0.940303	0.949794	0.956858	0.962404
		$\hat{R}_{iMBL3}$	0.845249	0.903960	0.928228	0.941788	0.951002	0.957861	0.963236
1450	12	$\hat{R}_{iMBL1}$	0.796842	0.873848	0.906212	0.924428	0.936862	0.946142	0.953443
		$\hat{R}_{iMBL2}$	0.805837	0.879485	0.910348	0.927695	0.939525	0.948351	0.955288
		$\hat{R}_{iMBL3}$	0.812262	0.883514	0.913301	0.930027	0.941428	0.949932	0.956608
1650	8	$\hat{R}_{iMBL1}$	0.733382	0.833161	0.876101	0.900522	0.917297	0.929874	0.939803
		$\hat{R}_{iMBL2}$	0.748753	0.843100	0.883482	0.906394	0.922107	0.933878	0.943158
		$\hat{R}_{iMBL3}$	0.759733	0.850202	0.888754	0.910587	0.925544	0.936744	0.945559
1850	4	$\hat{R}_{iMBL1}$	0.608222	0.748032	0.811571	0.848596	0.874404	0.893959	0.909514
		$\hat{R}_{iMBL2}$	0.640133	0.770141	0.828453	0.862237	0.885703	0.903440	0.917528
		$\hat{R}_{iMBL3}$	0.662926	0.785934	0.840513	0.871982	0.893774	0.910220	0.923242

For the point estimation of reliability based on zero-failure data, the authors consider the E-Bayesian and hierarchical Bayesian estimators, utilizing three different joint prior distributions for the two hyper-parameters in the prior distribution of reliability. Besides, closed-form expressions for the E-Bayesian estimators of reliability are obtained, and the hierarchical Bayesian estimators are evaluated using the Monte Carlo simulation method. The results of numerical examples and a real example indicate that both the E-Bayesian estimators and hierarchical Bayesian estimators are robust for different values of the upper bound c. Therefore, we suggest selecting the midpoint of integer interval [4, 8] as the value of the upper bound c in applications, namely, c=6. Besides, compared to the hierarchical Bayesian estimation method, the E-Bayesian estimation method is much easier to implement in practical engineering applications.

To obtain the credible limits of reliability based on zero-failure data, we study the one-sided M-Bayesian lower credible limits of reliability using three different joint prior distributions for two hyper-parameters. We propose estimation expressions for these one-sided M-Bayesian lower credible limits and discuss the properties of these estimators. The results of an illustrative example show that the proposed one-sided M-Bayesian lower credible limits perform excellently, and the properties of these estimators are stable.

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Declaration of interest statement

The authors declare that there are no conflicts of interest related to this study.

Appendix A

The proof of Theorem 1. (i) For the zero-failure data (t_i, n_i) $(i = 1,2,...,k)$ from the k times type-I censored life testing. The likelihood function of R_i is

$$L(0|R_i) = R_i^{s_i}, \ \ i = 1,2,...,k.$$

We thus according to the Bayesian theorem, the posterior density function of R_i can be given by,

$$\pi(R_i|s_i) = \frac{L(0|R_i)\pi(R_i|a,b)}{\int_0^1 L(0|R_i)\pi(R_i|a,b)\mathrm{d}R_i} = \frac{R_i^{s_i+a-1}(1-R_i)^{b-1}}{B(s_i+a,b)}, \ \ i = 1,2,...,k,$$

this indicates that the posterior distribution of R_i is the $Beta(s_i + a, b)$ distribution. Therefore, using the squared error loss function, the Bayesian estimator of R_i is

$$\hat{R}_{iB}(a,b) = \int_0^1 R_i \pi(R_i|s_i)\mathrm{d}R_i = \frac{1}{B(s_i+a,b)}\int_0^1 R_i^{s_i+a}(1-R_i)^{b-1}\mathrm{d}R_i = \frac{s_i+a}{s_i+a+b}, \ \ i = 1,2,...,k.$$

(ii) If the prior distributions of hyper-parameters a and b is presented by (3) to (5), then according to the Definition 1, the E-Bayesian estimators of R_i can be provided as, respectively,

$$\begin{aligned}\hat{R}_{iEB1} &= \int_1^c \int_0^1 \frac{s_i+a}{s_i+a+b} \cdot \frac{2(c-a)}{(c-1)^2} \mathrm{d}b\mathrm{d}a \\ &= \frac{2}{(c-1)^2}\left[\int_1^c (s_i+a)(c-a)\ln(s_i+a+1)\mathrm{d}a - \int_1^c (s_i+a)(c-a)\ln(s_i+a)\mathrm{d}a\right] \\ &= \frac{2}{(c-1)^2}[G_1(s_i,c)-G_2(s_i,c)], \\ \hat{R}_{iEB2} &= \int_1^c \int_0^1 \frac{s_i+a}{s_i+a+b} \cdot \frac{1}{c-1} \mathrm{d}b\mathrm{d}a \\ &= \frac{1}{c-1}\left[\int_1^c (s_i+a)\ln(s_i+a+1)\mathrm{d}a - \int_1^c (s_i+a)\ln(s_i+a)\mathrm{d}a\right] \\ &= \frac{1}{c-1}[G_3(s_i,c)-G_4(s_i,c)],\end{aligned}$$

and

$$\begin{aligned}\hat{R}_{iEB3} &= \int_1^c \int_0^1 \frac{s_i+a}{s_i+a+b} \cdot \frac{2a}{c^2-1} \mathrm{d}b\mathrm{d}a \\ &= \frac{2}{c^2-1}\left[\int_1^c a(s_i+a)\ln(s_i+a+1)\mathrm{d}a - \int_1^c a(s_i+a)\ln(s_i+a)\mathrm{d}a\right] \\ &= \frac{2}{c^2-1}[G_5(s_i,c)-G_6(s_i,c)],\end{aligned}$$

where

$$\begin{aligned}G_1(s_i,c) &= \frac{1}{2}(s_i+c+2)q_1(s_i,c) - (s_i+c+1)q_2(s_i,c) - \frac{1}{3}q_3(s_i,c), \\ G_2(s_i,c) &= \frac{1}{2}(s_i+c)q_4(s_i,c) - \frac{1}{3}q_5(s_i,c), \\ G_3(s_i,c) &= \frac{1}{2}q_1(s_i,c) - q_2(s_i,c), \\ G_4(s_i,c) &= \frac{1}{2}q_4(s_i,c), \\ G_5(s_i,c) &= -\frac{1}{2}(s_i+2)q_1(s_i,c) + (s_i+1)q_2(s_i,c) + \frac{1}{3}q_3(s_i,c), \\ G_6(s_i,c) &= \frac{1}{3}q_5(s_i,c) - \frac{s_i}{2}q_4(s_i,c),\end{aligned}$$

$$q_1(s_i,c) = (s_i+c+1)^2\left[\ln(s_i+c+1)-\frac{1}{2}\right]-(s_i+2)^2\left[\ln(s_i+2)-\frac{1}{2}\right],$$
$$q_2(s_i,c) = (s_i+c+1)\left[\ln(s_i+c+1)-1\right]-(s_i+2)\left[\ln(s_i+2)-1\right],$$
$$q_3(s_i,c) = (s_i+c+1)^3\left[\ln(s_i+c+1)-\frac{1}{3}\right]-(s_i+2)^3\left[\ln(s_i+2)-\frac{1}{3}\right],$$
$$q_4(s_i,c) = (s_i+c)^2\left[\ln(s_i+c)-\frac{1}{2}\right]-(s_i+1)^2\left[\ln(s_i+1)-\frac{1}{2}\right],$$
$$q_5(s_i,c) = (s_i+c)^3\left[\ln(s_i+c)-\frac{1}{3}\right]-(s_i+1)^3\left[\ln(s_i+1)-\frac{1}{3}\right].$$

Appendix B

The proof of Theorem 2. Since the hierarchical prior densities function of R_i are given by (6) to (8), respectively, we then according to the Bayesian theorem, the posterior densities function of R_i can be expressed as follows, respectively,

$$h_1(R_i|s_i) = \frac{L(0|R_i)\pi_4(R_i)}{\int_0^1 L(0|R_i)\pi_4(R_i)\mathrm{d}R_i} = \frac{\displaystyle\int_0^1\int_1^c (c-a)\frac{R_i^{s_i+a-1}(1-R_i)^{b-1}}{B(a,b)}\mathrm{d}a\mathrm{d}b}{\displaystyle\int_0^1\int_1^c (c-a)\frac{B(a+s_i,b)}{B(a,b)}\mathrm{d}a\mathrm{d}b}, \quad i=1,2,...,k,$$
$$h_2(R_i|s_i) = \frac{L(0|R_i)\pi_5(R_i)}{\int_0^1 L(0|R_i)\pi_5(R_i)\mathrm{d}R_i} = \frac{\displaystyle\int_0^1\int_1^c \frac{R_i^{s_i+a-1}(1-R_i)^{b-1}}{B(a,b)}\mathrm{d}a\mathrm{d}b}{\displaystyle\int_0^1\int_1^c \frac{B(a+s_i,b)}{B(a,b)}\mathrm{d}a\mathrm{d}b}, \quad i=1,2,...,k,$$
$$h_3(R_i|s_i) = \frac{L(0|R_i)\pi_6(R_i)}{\int_0^1 L(0|R_i)\pi_6(R_i)\mathrm{d}R_i} = \frac{\displaystyle\int_0^1\int_1^c a\frac{R_i^{s_i+a-1}(1-R_i)^{b-1}}{B(a,b)}\mathrm{d}a\mathrm{d}b}{\displaystyle\int_0^1\int_1^c a\frac{B(a+s_i,b)}{B(a,b)}\mathrm{d}a\mathrm{d}b}, \quad i=1,2,...,k.$$

We therefore use the square error loss function, the corresponding hierarchical Bayesian estimator of R_i can be presented as, respectively,

$$\hat{R}_{iHB1} = \int_0^1 R_i h_1(R_i|s_i)\mathrm{d}R_i = \frac{\int_0^1 \int_1^c (c-a)\frac{B(a+s_i+1)}{B(a,b)}\mathrm{d}a\mathrm{d}b}{\int_0^1 \int_1^c (c-a)\frac{B(a+s_i,b)}{B(a,b)}\mathrm{d}a\mathrm{d}b},\ \ i=1,2,...,k,$$

$$\hat{R}_{iHB2} = \int_0^1 R_i h_2(R_i|s_i)\mathrm{d}R_i = \frac{\int_0^1 \int_1^c \frac{B(a+s_i+1)}{B(a,b)}\mathrm{d}a\mathrm{d}b}{\int_0^1 \int_1^c \frac{B(a+s_i,b)}{B(a,b)}\mathrm{d}a\mathrm{d}b},\ \ i=1,2,...,k,$$

$$\hat{R}_{iHB3} = \int_0^1 R_i h_3(R_i|s_i)\mathrm{d}R_i = \frac{\int_0^1 \int_1^c a\frac{B(a+s_i+1)}{B(a,b)}\mathrm{d}a\mathrm{d}b}{\int_0^1 \int_1^c a\frac{B(a+s_i,b)}{B(a,b)}\mathrm{d}a\mathrm{d}b},\ \ i=1,2,...,k.$$

Appendix C

The proof of Theorem 3. For the three different prior densities function $\pi_1(a,b), \pi_2(a,b), \pi_3(a,b)$ of hyper-parameters a and b shown in (3) to (5), respectively, taking use of the Definition 2 and formula (9) of the Corollary 2, the one-sided M-Bayesian lower credible limits of R_i are given by, respectively

$$\hat{R}_{iMBL1} = \iint_D \hat{R}_{iB}(a,b)\pi_1(a,b)\mathrm{d}a\mathrm{d}b = \frac{2}{(c-1)^2}\int_0^1\int_1^c \frac{(c-a)(a+s_i)}{a+s_i+bF_{1-\alpha}(2b,2(a+s_i))}\mathrm{d}a\mathrm{d}b,\ \ i=1,2,..,k,$$

$$\hat{R}_{iMBL2} = \iint_D \hat{R}_{iB}(a,b)\pi_2(a,b)\mathrm{d}a\mathrm{d}b = \frac{1}{c-1}\int_0^1\int_1^c \frac{a+s_i}{a+s_i+bF_{1-\alpha}(2b,2(a+s_i))}\mathrm{d}a\mathrm{d}b,\ \ i=1,2,..,k,$$

$$\hat{R}_{iMBL3} = \iint_D \hat{R}_{iB}(a,b)\pi_3(a,b)\mathrm{d}a\mathrm{d}b = \frac{2}{c^2-1}\int_0^1\int_1^c \frac{a(a+s_i)}{a+s_i+bF_{1-\alpha}(2b,2(a+s_i))}\mathrm{d}a\mathrm{d}b,\ \ i=1,2,..,k.$$

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