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Research Article

Mathematical modeling, analysis, and optimal control of the cochineal insect impact on cacti plants

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Abstract

We propose a mathematical model, SIM, that describes the dynamics of cochineal insect spread among cacti and examines the effects of various control strategies. The model is analyzed for the existence and uniqueness of solutions, and we investigate the equilibrium points and stability of the system using both local and global stability analyses. By performing numerical simulations in MATLAB, we validate our theoretical findings. Furthermore, we propose an optimal control strategy to minimize

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the cochineal population in cacti fields. The optimal control problem is formulated using Pontryagin's maximum principle, and the corresponding optimality system is solved iteratively. Our study compares three control strategies: cutting and burning infected cacti, insecticide spraying, and a combined approach. The results demonstrate that the combined strategy is the most effective in reducing the cochineal population. This research provides valuable insights into managing cochineal infestations and offers practical recommendations for farmers to control the spread of these pests.

AMS subject classifications (2020): Primary 03C45; Secondary 90C31, 35F21.

Keywords: Mathematical modeling; Stability analysis; Optimal control; Cochineal insect; Cacti plants.

1 Introduction

The cochineal insect, **Dactylopius coccus**, is a well-known pest that significantly affects cacti and other succulents. It is notorious for feeding on plant sap, weakening their health, and, in severe cases, leading to the death of the host plant. These infestations are of considerable economic concern, especially in regions where cacti are a major agricultural crop. The damage caused by cochineal infestations includes dehydration, discoloration, and, ultimately, the destruction of entire crops. Cochineal is particularly harmful to cacti, which are otherwise resilient [13, 1].

Historically, cochineal insects were highly valued for their use in producing carmine dye, a pigment that has been utilized for centuries in the textile industry and art. The ability of these insects to produce carmine made them a valuable commodity in pre-Columbian Mesoamerica, where they were a significant economic resource. During the colonial era, the Spanish established a monopoly on cochineal trade, making it one of the most important exports from the Americas to Europe. The cultivation of cochineal and its host plants later spread globally, particularly to Mediterranean regions and parts of Asia, following its introduction by European colonial powers [8].

Mathematical modeling has become an essential tool for understanding the dynamics of cochineal infestations and assessing the effectiveness of different control measures. Early mathematical models of pest populations date back to the works of Verhulst and Lotka, who developed foundational models describing population growth and interactions [17, 12]. More recently, mathematical models have been applied to agricultural pest spread, including cochineal. These models typically take the form of differential equations that describe the dynamics of both the pest and its host plants [14, 10]. For example, the SIR (Susceptible-Infected-Removed) model, widely used in epidemiology, has been adapted to model the spread of various agricultural pests, including cochineal [5, 4].

However, most previous models have focused on the population dynamics of the pest or the biological aspects of infestation, often neglecting the effects of control measures and external interventions. In this paper, we propose a new mathematical model, denoted as the SIM model, that describes the spread of cochineal insects on cacti while incorporating different types of control strategies. Our model builds upon the classical SIR framework by including terms that represent the impact of interventions such as chemical treatments, removal of infected plants, and natural predation. We also conduct a stability analysis to examine the system's behavior under various conditions and determine the effects of control measures on pest spread [16, 9].

The novelty of our work lies in integrating optimal control theory with pest management strategies. Using Pontryagin's maximum principle, we derive optimal control strategies for minimizing the spread of cochineal while considering resource constraints. This approach enables us to evaluate the most efficient allocation of resources for pest control and optimize management efforts [15, 6].

This study contributes to the scientific literature by offering a more comprehensive approach to understanding cochineal infestations. In addition to providing theoretical insights into the dynamics of cochineal spread, our model offers practical guidance for pest management in agricultural settings. The findings have significant implications for policymakers and farmers, suggesting effective strategies for reducing the impact of cochineal infestations and preventing their spread to new regions.

The remainder of the paper is structured as follows: Section 2 introduces the proposed mathematical model, Section 3 analyzes its fundamental properties and equilibrium points, and Section 4 presents a stability analysis. Section 5 provides numerical simulations to validate the theoretical results, while Section 6 discusses optimal control strategies for managing cochineal infestations. In Section 7, optimal control numerical results are considered. Finally, Section 8 concludes with a summary of our findings and potential applications.

2 Formulation of the mathematical model

Before introducing our model, we first examine the biology and reproductive behavior of the cochineal insect, as understanding these aspects is essential for modeling its spread among cacti.

Cochineal insects are soft-bodied, oval-shaped organisms. Males possess wings, while females remain wingless. After mating, females lay eggs that rapidly hatch into tiny larvae. These larvae secrete a white waxy substance on their bodies, which serves as protection against water loss and excessive sunlight. This waxy coating appears on cacti as white, cotton-like masses. The larvae migrate to the edges of cactus pads, where wind currents carry the wax threads, dispersing them onto adjacent plants.

Cochineal infestations can cause significant agricultural losses, as these insects feed on cacti by extracting their fluids. In severe cases, this leads to plant desiccation and eventual death. In most instances, if a cactus is infested, it either perishes or must be removed—farmers typically resort to cutting, burying, or burning infected plants to prevent further spread. The insect's rapid dispersion is facilitated by wind, as well as by transportation via farming equipment, trucks, livestock wool, and the movement of animals and people from infested areas.

Based on these observations, we propose a mathematical model, denoted as SIM, to describe the spread of cochineal insects among cacti. We categorize the total cacti population, N, into two compartments:

- **S**: Represents the healthy (susceptible) cacti plants in the region.

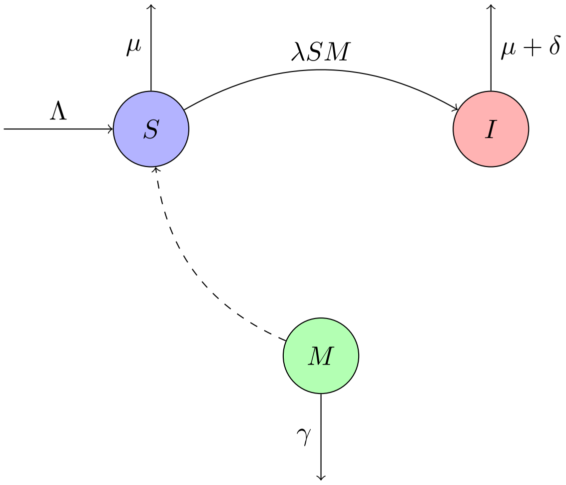

Figure 1: Compartments of cacti population

- **I**: Represents the infected cacti plants affected by the cochineal insect.

Additionally, we introduce a third compartment to represent the cochineal insect population, denoted as **M**.

In Figure 1, λ is the rate of cochineal insect encounters with cacti (measured in [cochineal/day]), γ is the cochineal insect mortality rate (measured in [cochineal/day]), δ is the death rate of cacti due to infection (measured in [cacti/day]), β is the growth rate of the cochineal insect population (measured in [cochineal/day] and $\beta > 0$), K is the carrying capacity of the cochineal insect population (measured in [cochineal/day] and $K > 0$), Λ is the recruitment rate of cacti (measured in [cacti/day]), and μ is the natural mortality rate of cacti (measured in [cacti/day]). Moreover,

$$\begin{cases} \frac{dS}{dt} = \Lambda - \lambda SM - \mu S, \\ \frac{dI}{dt} = \lambda SM - \mu I - \delta I, \\ \frac{dM}{dt} = \beta M \left(1 - \frac{M}{K}\right) - \gamma M, \end{cases} \tag{1}$$

with the following nonnegative initial conditions: $S(0) \geqslant 0, I(0) \geqslant 0, M(0) \geqslant 0$.

3 Model basic properties

The provided model describes the populations of both cacti and Cochineal insects. Therefore, it becomes necessary to investigate some fundamental properties of the system (1) such as the existence, positivity, and boundedness of the solutions.

3.1 Positivity of solutions

Theorem 1. If $S(0) \geq 0$, $I(0) \geq 0$ and $M(0) \geq 0$, then the solutions $S(t)$, $I(t)$, and $M(t)$ of system (1) are positive for all $t \geq 0$.

Proof. It follows from the first equation of system (1) that

$$\frac{\mathrm{d}S}{\mathrm{d}t} = \Lambda - \lambda SM - \mu S \geq -\left(\lambda M + \mu\right) S. \tag{2}$$

Then, we have

$$\frac{\mathrm{d}S}{S} \geq -\left(\lambda M + \mu\right) \mathrm{d}t. \tag{3}$$

Integrating this inequality from 0 to t gives

$$S(t) \geq S_b(0) e^{-\int_0^t (\lambda M + \mu) ds}.$$

That implies

$$S(t) \geq 0 \qquad \text{for all } t \geq 0.$$

Similarly, we prove that $I(t) \geq 0$ and $M(t) \geq 0$ for all $t \geq 0$.

□

3.2 Boundedness of the solutions

Theorem 2. Let $C = \max\{M(0), K\}$. Then the set

$$\Gamma = \left\{(S, I) \in \mathbb{R}_+^2 : N(t) \leq \frac{\Lambda}{\mu}\right\} \times \left\{M \in \mathbb{R}_+ : M(t) \leq C\right\}$$

is positively invariant under system (1) with nonnegative initial conditions $S(0)$, $I(0)$, and $M(0)$.

Proof. From the three equations of the system (1), we have

$$\frac{dN(t)}{dt} = \Lambda - \mu N(t) - \delta I \leq \Lambda - \mu N(t). \tag{4}$$

Then,

$$N(t) \leq N(0)e^{-\mu t} + \frac{\Lambda}{\mu}\left[1 - e^{-\mu t}\right]. \tag{5}$$

If we take limit $t \to \infty$, we have $0 \leq N(t) \leq \frac{\Lambda}{\mu}$.

From the last equation of (1), we have

$$\frac{\mathrm{d}M}{\mathrm{d}t} \leq \beta M \left(1 - \frac{M}{K}\right). \tag{6}$$

Hence, employing a typical comparison approach yields $\limsup_{t\to\infty} M(t) \leq K$.

Finally, the set Γ is positively invariant for the system (1). $\square$

3.3 Existence of solutions

Theorem 3. The system (1) that satisfies a given initial condition $(S(0), I(0), M(0))$ has a unique solution.

Proof. Model (1) can be reformulated in matrix form as follows:
Let $X(t) = (S, I, M)^T$ and $F(X(t)) = \left(\frac{dS}{dt}, \frac{dI}{dt}, \frac{dM}{dt}\right)^T$.

The system (1) can be rewritten as follows:

$$F(X(t)) = AX + B(X(t)),$$

where

$$A = \begin{pmatrix} -\mu & 0 & 0 \\ 0 & -(\mu+\delta) & 0 \\ 0 & 0 & \beta - \gamma \end{pmatrix}$$

and

$$B(X(t)) = \begin{pmatrix} \Lambda - \lambda SM \\ \lambda SM \\ -\frac{\beta M^2}{K} \end{pmatrix}.$$

Let X_1 and X_2 be solutions of (1). Then

$$\begin{aligned}
|B(X_1) - B(X_2)| \leq & \mid \lambda(S_2M_2 - S_1M_1)| + \mid \lambda(S_1M_1 - S_2M_2)| + \mid \frac{\beta}{K}(M_2^2 - M_1^2)| \\
\leq & \mid \lambda(S_2M_2 - S_2M_1)| + \mid \lambda(S_2M_1 - S_1M_1)| \\
& + \mid \lambda(S_1M_1 - S_1M_2)| + \mid \lambda(S_1M_2 - S_2M_2)| + \mid \frac{\beta}{K}(M_2^2 - M_1^2)| \\
\leq & \lambda S_2 \mid M_2 - M_1| + \lambda M_1 \mid S_2 - S_1| + \lambda S_1 \mid M_1 - M_2| \\
& + \lambda M_2 \mid S_1 - S_2| + \frac{\beta}{K} \mid M_2 - M_1| \mid M_2 + M_1| \\
\leq & \frac{\lambda\Lambda}{\mu} \mid M_2 - M_1| + \lambda C \mid S_2 - S_1| \\
& + \frac{\lambda\Lambda}{\mu} \mid M_1 - M_2| + \lambda C \mid S_1 - S_2| + \frac{2\beta C}{K} \mid M_2 - M_1| \\
\leq & \left(\frac{2\lambda\Lambda}{\mu} + \frac{2\beta C}{K}\right) \mid M_1 - M_2| + 2\lambda C \mid S_1 - S_2| \\
\leq & N \left\|X_1 - X_2\right\|,
\end{aligned}$$

where

$$N = \max\left(\frac{2\lambda\Lambda}{\mu} + \frac{2\beta C}{K}, 2\lambda C, \|A\|\right).$$

Therefore,

$$\left\|F(X_1) - F(X_2)\right\| \leq N \left\|X_1 - X_2\right\|.$$

Thus, it follows that the function F is uniformly Lipschitz continuous, and the restriction on $S(t) \geq 0$, $I(t) \geq 0$, and $M(t) \geq 0$ in $\mathbb{R}_+^3$. Therefore, a solution of the model (1) exists [3]. □

4 Stability analysis of the model

In this section, we explore the stability characteristics of model (1) for both the cochineal insect-free equilibrium point denoted as E_0 and the cochineal insect endemic equilibrium point denoted as E^*.

The equilibrium point of model (1) satisfies the following equations:

$$\Lambda - \lambda SM - \mu S = 0, \tag{7}$$

$$\lambda SM - \mu I - \delta I = 0, \tag{8}$$

$$\beta M\left(1 - \frac{M}{K}\right) - \gamma M = 0. \tag{9}$$

From (9), we have

$$M\left(\beta\left(1 - \frac{M}{K}\right) - \gamma\right) = 0 \Rightarrow M = 0 \quad \text{or} \quad M = K - \frac{\gamma K}{\beta}.$$

For $M = 0$: (7) $\Rightarrow \Lambda - \mu S = 0 \Rightarrow S = \frac{\Lambda}{\mu}$, (8) $\Rightarrow I = 0$, then $E_0 = \left(\frac{\Lambda}{\mu}, 0, 0\right)$.

For $M = M^* = K - \frac{\gamma K}{\beta} = K(1 - \frac{1}{\Re_0})$: From (7), we have

$$S^* = \frac{\Lambda\beta}{\lambda K(\beta - \gamma) + \mu\beta} = \frac{\Lambda}{\lambda K(1 - \frac{1}{\Re_0}) + \mu}.$$

From (8), we have

$$I^* = \frac{\lambda\Lambda K(\beta - \gamma)}{(\mu + \delta)\left(\lambda K(\beta - \gamma) + \mu\beta\right)} = \frac{\lambda\Lambda K(1 - \frac{1}{\Re_0})}{(\mu + \delta)\left(\lambda K(1 - \frac{1}{\Re_0}) + \mu\right)}.$$

Therefore,

$$E^* = \left(S^*, I^*, M^*\right),$$

where $\Re_0$ represents the basic reproduction number, defined as

$$\Re_0 = \frac{\beta}{\gamma}.$$

In epidemiology, the basic reproduction number $\Re_0$ signifies the average number of secondary infections caused by one infected individual in a population entirely susceptible. We calculated this value utilizing the next-generation matrix method as described in [2, 18]

Indeed, if we let $x = (S, I, M)$, then the model (1) can be expressed as

$$\frac{\mathrm{d}x}{\mathrm{d}t} = \mathscr{F}(x) - \vartheta(x),$$

where

$$\mathscr{F}(x) = \begin{pmatrix} \Lambda \\ \lambda SM \\ \beta M\left(1-\frac{M}{K}\right) \end{pmatrix},$$

$$\vartheta(x) = \begin{pmatrix} \lambda SM + \mu S \\ \mu I + \delta I \\ \gamma M \end{pmatrix}.$$

The Jacobian matrices of $\mathscr{F}(x)$ and $\vartheta(x)$ at the free equilibrium E_0 are, respectively,

$$D\mathscr{F}(E_0) = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & \frac{\lambda\Lambda}{\mu} \\ 0 & 0 & \beta \end{pmatrix},$$

$$D\vartheta(E_0) = \begin{pmatrix} \mu & 0 & \frac{\lambda\Lambda}{\mu} \\ 0 & \mu+\delta & 0 \\ 0 & 0 & \gamma \end{pmatrix}.$$

where

$$F = \begin{pmatrix} 0 & \frac{\lambda\Lambda}{\mu} \\ 0 & \beta \end{pmatrix},$$

$$V = \begin{pmatrix} \mu+\delta & 0 \\ 0 & \gamma \end{pmatrix}.$$

Finally, we have

$$\Re_0 = \rho\left(FV^{-1}\right) = \frac{\beta}{\gamma}.$$

4.1 Local stability

In this section, we analyze the local stability of both the free equilibrium point E_0 and the endemic equilibrium point E^*.

Theorem 4. If $\Re_0 < 1$, then E_0 is locally asymptotically stable.

Proof. The Jacobian matrix calculated for system (1) at E_0 is given by

$$J_{E_0} = \begin{pmatrix} -\mu & 0 & \frac{-\lambda\Lambda}{\mu} \\ 0 & -\mu-\delta & \frac{\lambda\Lambda}{\mu} \\ 0 & 0 & \beta-\gamma \end{pmatrix}.$$

We put $d_1 = -\mu < 0$, $d_2 = -\mu - \delta$ and $d_3 = \beta - \gamma = \gamma(\Re_0 - 1)$. Then, let us consider the matrix:

$$T_1 = \begin{pmatrix} d_1 - r & 0 & \frac{-\lambda\Lambda}{\mu} \\ 0 & d_2 - r & \frac{\lambda\Lambda}{\mu} \\ 0 & 0 & d_3 - r \end{pmatrix}.$$

Hence, the eigenvalues of J_{E_0} are $r_1 = d_1$, $r_2 = d_2$ and $r_3 = d_3$. Clearly, all the eigenvalues of the characteristic equation are negative if $\Re_0 < 1$. Then, E_0 is locally asymptotically stable.

□

Theorem 5. If $\Re_0 > 1$, then E^* is locally asymptotically stable.

Proof. Let J_E^* be the Jacobian matrix of the system (1) evaluated at E_1,

$$J_{E^*} = \begin{pmatrix} -\lambda M^* - \mu & 0 & -\lambda S^* \\ \lambda M^* & -\mu - \delta & \lambda S^* \\ 0 & 0 & \beta - \frac{2\beta}{K} M^* - \gamma \end{pmatrix}.$$

We put $d_1 = -\lambda M^* - \mu < 0$, $d_2 = -\lambda M^* - \mu$ and $d_3 = \beta - \gamma - \frac{2\beta}{K} M^*$. Then, we consider the following matrix:

$$T_2 = \begin{pmatrix} d_1 - r & 0 & -\lambda S^* \\ \lambda M^* & d_2 - r & \lambda S^* \\ 0 & 0 & d_3 - r \end{pmatrix}.$$

The characteristic equation of T_2 can be written as

$$(d_1 - r)(d_2 - r)(d_3 - r) = 0.$$

Hence, the eigenvalues of J_{E^*} are $r_1 = -d_1$, $r_2 = -d_2$, and $r_3 = d_3$, where

$$d_3 = \beta - \gamma - \frac{2\beta}{K} M^* = \gamma \left(1 - \Re_0\right).$$

Clearly, all the eigenvalues of the characteristic equation are negative if $\Re_0 > 1$. Then, E^* is locally asymptotically stable if $\Re_0 > 1$, proving the desired result.

□

4.2 Global stability analysis

Now, our aim is to evaluate the global asymptotic stability of both the free equilibrium E_0 and the endemic equilibrium E^* of model (1).

Theorem 6. If $\Re_0 \leq 1$, then E_0 is globally asymptotically stable on Γ.

Proof. We consider the Lyapunov function $V(S, I, M) = M$. We have

$$\begin{aligned} \dot{V} = & \quad \dot{M} \\ = & \quad \beta M \left(1 - \frac{M}{K}\right) - \gamma M \\ = & \quad \gamma\left(\Re_0 - 1\right) M - \frac{\beta}{K} M^2. \end{aligned}$$

Therefore, it is clearly that if $\Re_0 \leq 1$, then $\dot{V} \leq 0$.

Indeed, if $\Re_0 \leq 1$, then $\dot{V}(S, I, M) = 0 \Leftrightarrow M = 0$.

Hence, according to LaSalle's invariance principle [11], the free equilibrium point E_0 is globally asymptotically stable on Γ. ☐

Theorem 7. If $\Re_0 > 1$, then E^* is globally asymptotically stable on Γ.

Proof. To investigate the global stability of the endemic equilibrium E^*, we examine the Lyapunov function represented as

$$V(S, I, M) = M - M^* \left(1 + \ln(\frac{M}{M^*})\right). \tag{10}$$

Then, the time derivative of the Lyapunov function is given by

$$\begin{aligned} \dot{V} = & \quad \frac{1}{M}\left(M - M^*\right)\dot{M} \\ = & \quad \frac{1}{M}\left(M - M^*\right)\left(\beta M \left(1 - \frac{M}{K}\right) - \gamma M\right) \\ = & \quad \left(M - M^*\right)\left(\beta - \frac{\beta}{K} M - \gamma\right) \\ = & \quad \left(M - M^*\right)\left(\beta - \gamma(\Re_0 - 1)\frac{M}{M^*} - \gamma\right) \\ = & \quad \gamma\left(M - M^*\right)\left(\Re_0 - 1\right)\left(1 - \frac{M}{M^*}\right) \\ = & \quad \gamma\left(1 - \Re_0\right)\frac{\left(M - M^*\right)^2}{M^*}. \end{aligned}$$

Therefore, it is obvious that if $\Re_0 > 1$, then $\dot{V} < 0$.

In addition, if $\Re_0 > 1$, then $\dot{V}(S, I, M) = 0 \Leftrightarrow M = M^*$.

Hence, by LaSalle's invariance principle [11], the endemic equilibrium E^* is globally asymptotically stable on Γ. □

5 Numerical simulation

In this section, we present several numerical solutions of system (1) using approximate parameter values, given the lack of actual data resulting from the absence of statistical studies on this phenomenon. The system (1) was solved using Euler's method. We employed different initial values for each state variable and utilized the parameter set: $\Lambda = 700$ [cacti/day], $\mu = 0.05$ [cacti/day], $\lambda = 0.02$ [cochineal/day], $\delta = 0.03$ [cacti/day], $\beta = 0.02$ [cochineal/day], $\gamma = 0.05$ [cochineal/day], and $K = 5000$ [cochineal/day]. It is noteworthy that we identified the disease-free equilibrium point $E_0 = (14000, 0, 0)$ with $\Re_0 = 0.4 < 1$.

The provided figures illustrate the evolution of the state variables S, I, and M over time (in days), representing the susceptible cacti, infected cacti, and cochineal insects, respectively. Three scenarios with different initial values are depicted in each figure:

- **Figure 2 (S): Evolution of susceptible cacti**
  - $S(0)$ = 150,000 (blue line): The number of susceptible cacti decreases rapidly and stabilizes around 14,000 over time.
  - $S(0)$ = 100,000 (orange line): Similarly, the population decreases and reaches the same steady-state value of 14,000.
  - $S(0)$ = 50,000 (yellow line): Despite a lower initial population, the system stabilizes at the same steady-state level.

  This figure demonstrates that the system consistently converges to $S =$ 14,000 regardless of initial conditions.

- **Figure 2 (I): Evolution of infected cacti**

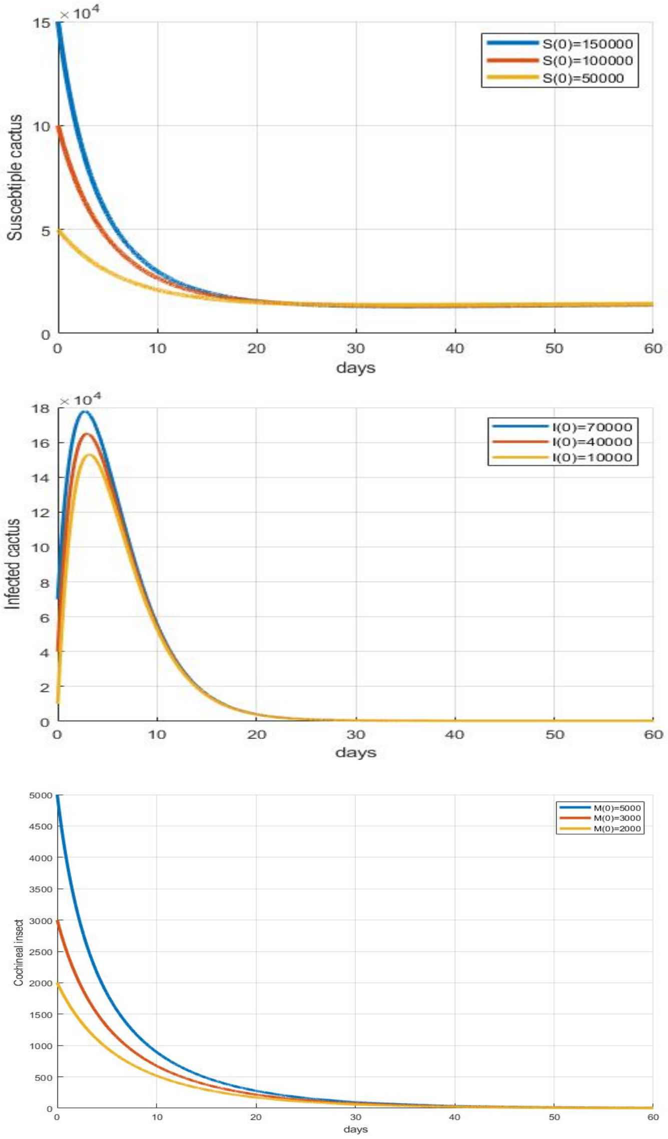

Figure 2: SIM model with a disease-free equilibrium

- $I(0) = 70{,}000$ (blue line): The number of infected cacti increases initially, reaching a peak near 18×10^4 before gradually declining to zero.

  - $I(0) = 40{,}000$ (orange line): Following a similar pattern, the population rises to a peak before declining to zero.
  - $I(0) = 10{,}000$ (yellow line): Even with smaller initial infections, the population exhibits the same behavior, with infection eradication over time.

  This figure highlights that infection is eliminated in all scenarios as the system stabilizes.

- **Figure 2 (M): Evolution of cochineal insects**
  - $M(0) = 5{,}000$ (blue line): The cochineal insect population decreases steadily, approaching zero within approximately 60 days.
  - $M(0) = 3{,}000$ (orange line): Similarly, the population declines to zero at a slightly faster rate.
  - $M(0) = 2{,}000$ (yellow line): A smaller initial population also diminishes to zero over time.

  This figure demonstrates that the cochineal insect population is eradicated in all scenarios.

Across all figures, the application of the model confirms that the system is globally stable at $E_0 = (14{,}000, 0, 0)$. This stability is achieved regardless of initial conditions, as long as $\Re_0 < 1$. The results highlight the robustness of the system's dynamics in eliminating infections and cochineal insects while stabilizing the susceptible cactus population. According to Theorem 6, E_0 is stable on Γ.

Additionally, considering diverse initial values for each state variable and the set of parameters: $\Lambda = 700$ [cacti/day], $\mu = 0.05$ [cacti/day], $\lambda = 0.02$ [cochineal/day], $\delta = 0.03$ [cacti/day], $\beta = 0.07$ [cochineal/day], $\gamma = 0.01$ [cochineal/day], and $K = 5000$ [cochineal/day], we observe that the endemic equilibrium point $E^* = (20000, 72000, 3000)$ and $\Re_0 = 7 > 1$.

The provided figures illustrate the evolution of the state variables S, I, and M over time (in days), representing the susceptible cacti, infected cacti, and cochineal insects, respectively. Three scenarios with different initial values are depicted in each figure:

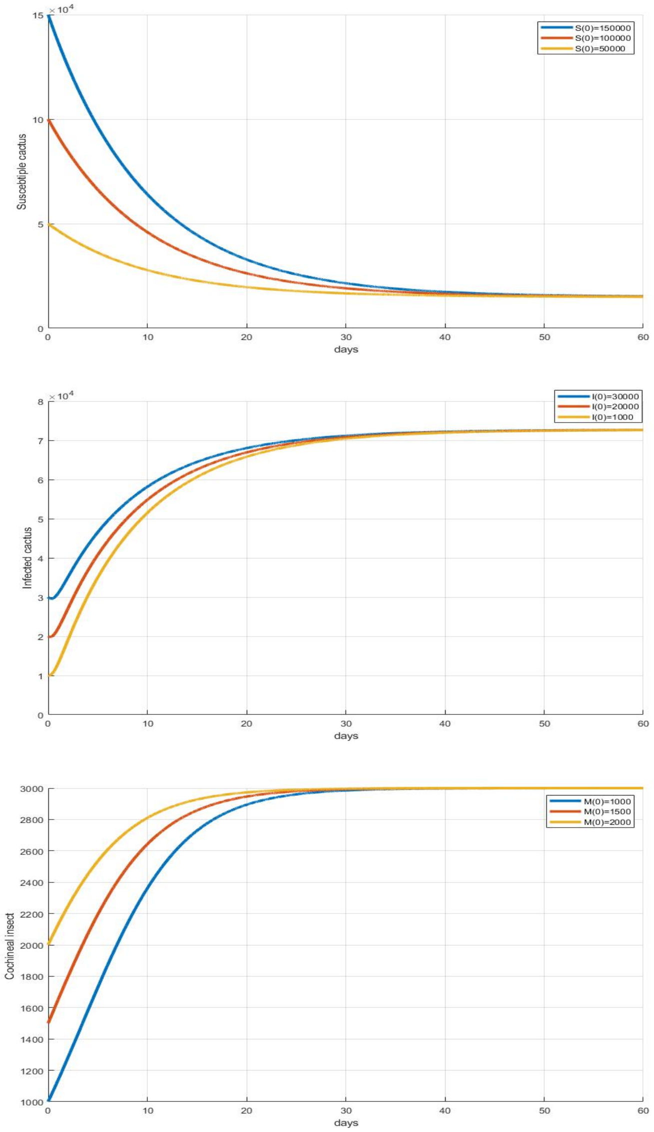

Figure 3: SIM model with an epidemic equilibrium

- **Figure 3 (S): Evolution of Susceptible Cacti**
  - $S(0) = 150{,}000$ (blue line): The susceptible cactus population decreases over time, stabilizing at a value near 20,000.

  - $S(0) = 100{,}000$ (orange line): The population follows a similar trend, reaching the same steady-state value of $S \approx 20{,}000$.
  - $S(0) = 50{,}000$ (yellow line): Despite a lower initial population, the system stabilizes at the same steady-state value.

- **Figure 3 (I): Evolution of Infected Cacti**
  - $I(0) = 30{,}000$ (blue line): The number of infected cacti rises and stabilizes at a value near 72,000.
  - $I(0) = 20{,}000$ (orange line): Similarly, the population stabilizes at $I \approx 72{,}000$ after an initial increase.
  - $I(0) = 10{,}000$ (yellow line): Even with a smaller starting population, the system converges to the same equilibrium.

- **Figure 3 (M): Evolution of Cochineal Insects**
  - $M(0) = 1{,}000$ (blue line): The population of cochineal insects grows steadily, approaching a value near 3,000 over time.
  - $M(0) = 1{,}500$ (orange line): Similarly, the population stabilizes near 3,000 after initial growth.
  - $M(0) = 2{,}000$ (yellow line): Regardless of the higher initial population, the system stabilizes at $M \approx 3{,}000$.

Over time, the system stabilizes with the susceptible cactus population S close to 20,000, the infected cactus population I near 72,000, and the cochineal insect population M at approximately 3,000. This behavior demonstrates that, for $\Re_0 > 1$, the solution curves converge towards the equilibrium point E^*. According to Theorem 7, the equilibrium E^* is stable on Γ.

6 The optimal control problem

Given the ongoing threat of cochineal infestations and their severe economic impact on cactus production, farmers need a cost-effective strategy to control the pest's spread within a specific timeframe. To address this, we develop an optimal control problem that focuses on minimizing the number of infected plants, $I(t)$, while maximizing the number of recovered plants, $R(t)$, over the period $[t_0, t_f]$. Additionally, the model aims to minimize the costs associated with control measures, ensuring a balance between effective pest management and economic feasibility.

A key aspect of our approach is the natural control provided by Hyperaspis trifurcata, a predatory beetle that feeds on cochineal insects. By incorporating this biological control agent into the model, we emphasize the beetle's role in naturally reducing cochineal infestations. Hyperaspis trifurcata offers a sustainable and environmentally friendly alternative to chemical pesticides, as it directly targets the cochineal population, helping to curb its spread.

The system of equations (1) is adjusted to include two control variables, $u_1(t)$ and $u_2(t)$ for $t \in [t_0, t_f]$, as follows:

$$\begin{cases} \frac{dS}{dt} &= \Lambda - \lambda S(t)M(t) - \mu S(t), \\ \frac{dI}{dt} &= \lambda S(t)M(t) - \delta I(t) - \mu I(t) - u_1(t)I(t), \\ \frac{dM}{dt} &= \beta M(t)\left(1 - \frac{M(t)}{K}\right) - \gamma M(t) - u_2(t)M(t), \end{cases} \tag{11}$$

with the initial conditions $S(0) \geq 0$, $I(0) \geq 0$, and $C(0) \geq 0$.

The control $u_1(t)$ represents cutting and burning cactus infected with cochineal, while the control $u_2(t)$ denotes the application of insecticide to combat cochineal.

The problem is to minimize the objective functional:

$$J(u_1, u_2) = I(t_f) + M(t_f) + \int_{t_0}^{t_f} \left[I(t) + M(t) + \frac{C_1}{2}\left(u_1(t)\right)^2 + \frac{C_2}{2}\left(u_2(t)\right)^2 \right] dt, \tag{12}$$

where $C_1 > 0$ and $C_2 > 0$ are chosen to assign the relative importance of $u_1(t)$ and $u_2(t)$ at any given time t, with t_f representing the final time.

In other words, our goal is to find the optimal controls u_1^* and u_2^* such that

$$J(u_1^*, u_2^*) = \min_{(u_1, u_2) \in U} J(u_1, u_2),$$

where U is the set of admissible controls defined by

$$U = \left\{ (u_1(t), u_2(t)) : 0 \leq u_1(t) \leq 1 \ , 0 \leq u_2(t) \leq 1, \ / \ t \in [t_0, t_f] \right\}.$$

6.1 Existence of optimal controls

In this part, we present the theorem that proves the existence of an optimal control $(u_1^*,\, u_2^*)$ minimizing the cost function J.

Theorem 8. There exists an optimal control $(u_1^*, u_2^*) \in U$ such that

$$J(u_1^*, u_2^*) = \min_{(u_1,u_2)\in U} J(u_1, u_2).$$

Proof. To use the existence result in [7], we must check the following properties:

(A_1): The set of controls and the corresponding state variables is nonempty.

(A_2): The control set U is convex and closed.

(A_3): The right-hand side of the state system is bounded by a linear function in the state and control variables.

(A_4): The integral $L(I, M, u_1, u_2)$ of the objective functional is convex on U, and there exist constants $\varkappa_1 > 0$, $\varkappa_2 > 0$, and $\varepsilon > 1$ such that

$$L(I, M, u_1, u_2) \geq -\varkappa_1 + \varkappa_2 \left(|u_1|^2 + |u_2|^2\right)^{\frac{\varepsilon}{2}}.$$

The first condition (A_1) is verified using the result in Lukes. The set U is convex and closed by definition, thus the condition (A_2) is satisfied. Our state system is linear in u_1 and u_2; moreover the solutions of the system are bounded as proved in model (1), hence the condition (A_3) is satisfied. Also, we have the last needed condition (A_4),

$$L(I, M, u_1, u_2) \geq -\varkappa_1 + \varkappa_2 \left(|u_1|^2 + |u_2|^2\right)^{\frac{\varepsilon}{2}}.$$

where

$$\varkappa_1 = 2 \sup_{t\in[t_0,t_f]} (I(t), M(t)), \qquad \varkappa_2 = \inf(\frac{C_1}{2}, \frac{C_2}{2}) \text{ and } \varepsilon = 2,$$

since $C_1 > 0$ and $C_2 > 0$. We conclude that there exists an optimal control $(u_1^*, u_2^*) \in U$ such that

$$J(u_1^*, u_2^*) = \min_{(u_1,u_2)\in U} J(u_1, u_2).$$

□

6.2 Characterization of the optimal controls

In this section, we utilize Pontryagin's principle [15]. The central concept is to introduce the adjoint function, which connects the system of differential equations to the objective functional. This connection leads to the formulation of the Hamiltonian. By applying this principle, the task of determining a control that optimizes the objective functional with a specified initial condition is transformed into the problem of finding a control that optimizes the Hamiltonian pointwise.

To derive the optimal control conditions, we apply Pontryagin's maximum principle such that the Hamiltonian H at time t is defined by

$$H(t) = I(t) + M(t) + \frac{C_1}{2}\left(u_1(t)\right)^2 + \frac{C_2}{2}\left(u_2(t)\right)^2 + \sum_{i=1}^{3} \lambda_i h_i, \tag{13}$$

where h_i is the right side of the system of differential equations (11) of ith state variable.

Theorem 9. Given the optimal controls (u_1^*, u_2^*) and solutions S^*, I^*, and M^* of the corresponding state system (11), there exist adjoint functions $\lambda_1,\ \lambda_2$, and λ_3 satisfying

$$\begin{cases} h_1 = \Lambda - \lambda S(t)M(t) - \mu S(t), \\ h_2 = \lambda S(t)M(t) - \delta I(t) - \mu I(t) - u_1(t)I(t), \\ h_3 = \beta M(t)\left(1 - \frac{M(t)}{K}\right) - \gamma M(t) - u_2(t)M(t), \end{cases} \tag{14}$$

such that the transversality conditions at time t_f are

$$\begin{cases} \lambda_1(t_f) = 0, \\ \lambda_2(t_f) = 1, \\ \lambda_3(t_f) = 1. \end{cases} \tag{15}$$

In addition to that we have, for $t \in [t_0, t_f]$, optimal controls $u_1^*(t)$ and $u_2^*(t)$ are given by

$$\begin{cases} u_1^*(t) = \min\left(1, \max\left(0, \dfrac{\lambda_2}{C_1} I(t)\right)\right), \\ u_2^*(t) = \min\left(1, \max\left(0, \dfrac{\lambda_3}{C_2} M(t)\right)\right). \end{cases} \tag{16}$$

Proof. The Hamiltonian H is defined as follows:

$$H(t) = I(t) + M(t) + \frac{C_1}{2}\left(u_1(t)\right)^2 + \frac{C_2}{2}\left(u_2(t)\right)^2 + \sum_{i=1}^{3} \lambda_i h_i,$$

where

$$\begin{cases} h_1 = \Lambda - \lambda S(t)M(t) - \mu S(t), \\ h_2 = \lambda S(t)M(t) - \delta I(t) - \mu I(t) - u_1(t)I(t), \\ h_3 = \beta M(t)\left(1 - \frac{M(t)}{K}\right) - \gamma M(t) - u_2(t)M(t). \end{cases}$$

For $t \in [t_0, t_f]$, the adjoint equations and transversality conditions can be obtained by using Pontryagin's maximum principle [15] such that

$$\begin{cases} \lambda_1' = -\frac{dH}{dS} = \lambda_1(\lambda M(t) + \mu) - \lambda_2 \lambda M(t), \\ \lambda_2' = -\frac{dH}{dI} = -1 + \lambda_2\left(\delta + \mu + u_1(t)\right), \\ \lambda_3' = -\frac{dH}{dM} = -1 + \lambda_1 \lambda S(t) - \lambda_2 \lambda S(t) - \lambda_3\left(\beta - \frac{2\beta M(t)}{K} - \gamma - u_2(t)\right), \end{cases}$$

for $t \in [t_0, t_f]$, the optimal controls u_1^* and u_2^* can be solved from the optimality condition. We have

$$\frac{dH}{du_1} = C_1 u_1(t) - \lambda_2 I(t) = 0.$$

So

$$u_1(t) = \frac{\lambda_2}{C_1} I(t).$$

We have

$$\frac{dH}{du_2} = C_2 u_2(t) - \lambda_3 M(t) = 0.$$

So

$$u_2(t) = \frac{\lambda_3}{C_2} M(t).$$

By the bounds in U of the controls, it is convenient to obtain u_1^* and u_2^* in the form of (16).

□

7 Optimal control numerical results

In this section, we present the results obtained by solving the optimality system. For our control problem, we define conditions for the state variables and terminal conditions for the adjoint variables. The optimality system is essentially a two-point boundary value problem, with conditions at the initial time step $i = t_0$ and the final time step $i = t_f$. To solve this system, we initially solve the state model, followed by solving the adjoint system in reverse order. In the first iteration, we start with an initial guess for the control variables and update them based on a characterization of the optimal controls before moving on to the next iteration. This process is repeated until the iterates converge. To achieve this, we created a MATLAB code utilizing the following parameters. Given the lack of real-world data, the parameter values were chosen hypothetically. The plots illustrating susceptible, infected, and recovered cochineal individuals both with and without control measures are generated based on these parameter values: $\Lambda = 1000$ [cacti/day], $\mu = 0.05$ [cacti/day], $\lambda = 0.02$ [cochineal/day], $\delta = 0.03$ [cacti/day], $\beta = 0.07$ [cochineal/day], $\gamma = 0.01$ [cochineal/day], and $K = 70000$ [cochineal/day]. When analyzing the graphs, please be aware that solid lines represent cochineal individuals without control measures, whereas dashed lines indicate those with control measures.

7.1 Control Strategy 1: Cutting and burning cactus infected with cochineal

The goal of this approach is to minimize the function (12), with a primary focus on reducing the cochineal population through cutting and burning cactus infected with cochineal. Figure 4 illustrates the effects of this spraying on the cacti plants.

This figure illustrates the evolution of the state variable I, representing the number of infected cacti, over time (in days). Two scenarios are depicted:

- **Without control** u_1 (solid red line): In this scenario, the number of infected cacti increases rapidly and levels off, approaching a steady-state at approximately 7×10^5 after around 50 days.
- **With control** u_1 (dashed red line): When control u_1 is applied, the number of infected cacti grows initially but peaks at a lower value.

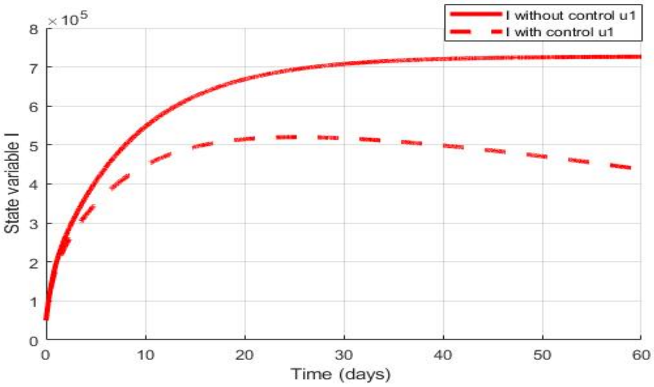

Figure 4: The evolution of the number of infected cacti with and without the control u_1

After reaching the peak, the number steadily declines, indicating that the control measure is effective in reducing the infection.

This comparison demonstrates the significant impact of control u_1 on managing the infection levels among the cacti.

7.2 Control Strategy 2: Effects of insecticide spraying

The main objective of treating cacti infested with cochineal, within the context of a strategy, is to minimize the function (12) while maintaining other control measures at zero. Figure 5 illustrates the cochineal dynamics, taking into account the presence or absence of this control measure.

This figure illustrates the evolution of the state variable M, representing the number of cochineals, over time (in days). Two scenarios are depicted:

- **Without control** u_2 (solid green line): In this scenario, the number of cochineals increases rapidly and levels off, approaching a steady-state at approximately 6×10^5 after around 30 days.
- **With control** u_2 (dashed green line): When control u_2 is applied, the number of cochineals grows initially but stabilizes at a much lower

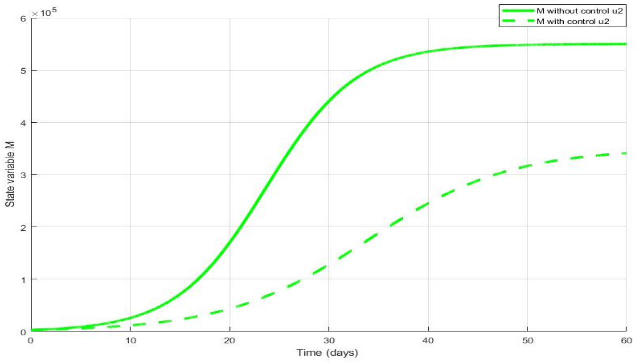

Figure 5: The evolution of the number of cochineals with and without the control u_2

value. This indicates that the control measure effectively limits the growth of the population.

This comparison demonstrates the significant impact of control u_2 in managing the cochineal population.

7.3 Control Strategy 3: Cutting and burning cactus infected with cochineal and insecticide spraying

This strategy aims to minimize the objective function (12) by implementing both control measures. Figure 6 illustrates the disease progression with both controls in effect, compared to the scenario where no control measures are utilized to manage the disease.

This first figure illustrates the evolution of the state variable I, representing the number of infected cacti, over time (in days). Two scenarios are depicted as follows:

- **Without control u_1 and u_2** (solid red line): In this scenario, the number of infected cacti increases rapidly at first and then levels off, approaching a steady-state at approximately 7×10^5 after around 50 days.

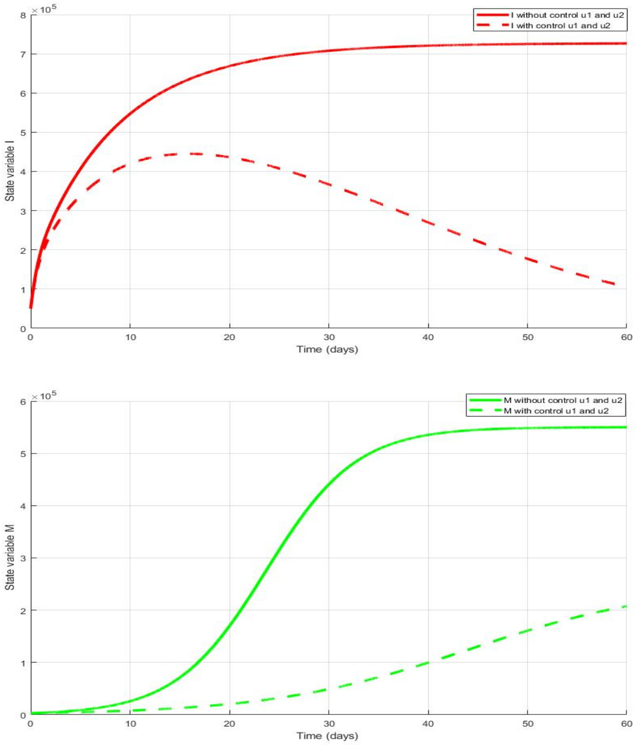

Figure 6: The evolution of the number of infected cacti and cochineals, both with and without u_1 and u_2

- **With control** u_1 **and** u_2 (dashed red line): When control measures are applied, the number of infected cacti initially grows but peaks at a lower value compared to the uncontrolled scenario. After reaching the peak, the number of infected cacti steadily declines, indicating that the controls effectively reduce the infection levels over time.

This comparison highlights the effectiveness of control measures (u_1 and u_2) in managing the spread of the infection among cacti.

The second figure illustrates the evolution of the state variable M, representing the number of cochineals, over time (in days). Two scenarios are depicted:

- **Without control u_1 and u_2** (solid green line): In this scenario, the number of cochineals increases rapidly at first and then levels off, approaching a steady-state at approximately 6×10^5 after around 30 days.
- **With control u_1 and u_2** (dashed green line): When control measures are applied, the number of cochineals grows at a much slower rate and stabilizes at a significantly lower value, suggesting the effectiveness of the controls in limiting the population of cochineals.

This comparison highlights the impact of control measures (u_1 and u_2) in managing the population growth of cochineals.

8 Conclusion

In this paper, we presented a mathematical model, SIM, that describes the spread of the cochineal insect among cacti. We showed that the overall behavior of the model is entirely determined by a critical parameter, the basic reproduction number $\Re_0$. Specifically, we demonstrated that the infection-free equilibrium, E_0, is both locally and globally asymptotically stable when $\Re_0 < 1$, leading to a gradual decline in the cochineal population over time and its eventual eradication. Conversely, when $\Re_0 > 1$, the cochineal population increases. Furthermore, we established that the endemic equilibrium, E^*, is locally and globally asymptotically stable when $\Re_0 > 1$, ensuring the persistence of the infestation.

Our numerical simulations validated the analytical findings regarding the stability of both the infection-free equilibrium E_0 and the endemic equilibrium E^*. These results provide a solid foundation for the development of an optimal control strategy to help farmers minimize losses caused by cochineal infestations in cacti fields.

The model incorporates the dynamics of cochineal infestations and the effects of control measures. We analyzed the system's behavior under three control strategies aimed at managing the cochineal population:

- **Control Strategy 1 (cutting and burning infected cacti, u_1)**: This strategy effectively reduces the number of infected cacti, but the decline is slow, and the population stabilizes at a higher level compared to other strategies. While effective, it requires continuous intervention over an extended period.

- **Control Strategy 2 (insecticide spraying, u_2)**: Insecticide application significantly accelerates the reduction of the cochineal population. The population stabilizes at a lower level more quickly, reflecting the effectiveness of chemical control in mitigating infestation rates.
- **Control Strategy 3 (combined approach, u_1 and u_2)**: The combined application of cutting and burning infected cacti along with insecticide spraying yields the most substantial reduction in the cochineal population. This strategy results in a faster decline and a lower steady-state population, showcasing the synergistic effect of integrating both physical and chemical control measures.

In conclusion, the combined strategy (Control Strategy 3) proves to be the most effective in reducing the cochineal population, offering a more comprehensive and sustainable solution. However, the choice of the most appropriate strategy depends on factors such as the severity of the infestation, available resources, and environmental considerations. The SIM model provides a valuable tool for evaluating these control measures and supporting decision-making in the management of cochineal infestations on cacti.

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