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On some bivariate integer-valued distributions on Z^2

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Abstract. We proposed and studied a new bivariate random sign transformation of nonnegative bivariate integer-valued distributions. This transformation develops new bivariate integer-valued distributions on Z^2. We applied the new transformation to the bivariate Poisson and the bivariate geometric distributions. As an illustration, we fitted a real-life data set developed based on the results of the 2019 UEFA Europa League using the new distributions.

Keywords: Bivariate RST; ML estimators; MM estimators; Monte Carlo simulations.

1 Introduction

The development of nonnegative integer-valued bivariate distributions has received considerable attention in the literature. For important results and reviews on this topic, we refer the reader to Kocherlakota and Kocherlakota (1992), Johnson et al. (1997), Lai (2006) and Sarabia Alegría and Gómez Déniz (2008). Some recent important results in this area include Odhah (2013), Genest and Mesfioui (2014), Bulla et al. (2015), Omair et al. (2016) and Karlis and Mamode Khan (2023).

Chesneau et al. (2018) noted that changes in intra-daily stock prices take both positive and negative integer values and that the price change is therefore characterized by discrete positive and negative jumps. This motivated Chesneau et al. (2018) to propose and study some bivariate integer-valued distributions on Z^2. Omair et al. (2022) proposed some bivariate integer-valued distributions on Z^2 and applied their models to fit the two real life data sets; the difference in the number of casualties to the number of employees on duty on railroads and the difference in the number of goals scored in the English Premier League in different years.

In this paper, we proposed and studied a new bivariate random sign transformation (BRST) of nonnegative bivariate integer valued distributions. The BRST is an extension of the random

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sign transformation (RST) of Aly (2018). The BRST is also a generalization of the transformation used in Chesneau et al. (2018). We used the BRST to introduce and study new families of bivariate integer-valued distributions on Z^2. The first family is developed based on the bivariate Poisson distribution (BPD). The second family is developed based on the bivariate geometric distribution (BGD).

In Section 2, we review some important bivariate integer-valued distributions. In Sections 3, we introduce and study three versions of the BRST. In Section 4, we apply the transformations of Sections 3 to the BPD. In Section 5, we apply the transformations of Sections 3 to the BGD. In Section 6, we report the results of Monte Carlo simulation studies conducted to evaluate the estimators of the parameters of the models of Sections 4 and 5. In Section 7, we apply the models of Sections 4 and 5 to a real life data set developed based on the results of the 2019 UEFA Europa League.

A random vector (or variable) will be denoted by RV. The probability mass function of discrete RV will be abbreviated by pmf and the joint probability mass function of a discrete RV will be abbreviated by $jpmf$. The univariate Bernoulli distribution with parameter θ will be denoted by $Ber(\theta)$. The geometric distribution with $pmf, g(x) = (1-\theta)^x\theta, x = 0,1,\ldots,0<\theta<1$, will be denoted by $Geo(\theta)$. The Poisson distribution with parameter $\lambda > 0$ will be denoted by $Poi(\lambda)$.

2 Some bivariate integer-valued distributions

2.1 Some bivariate Bernoulli distributions

Definition 1. *Assume that* $\underline{\beta} = (\beta_{00}, \beta_{01}, \beta_{10}, \beta_{11})$, *where* $0 \le \beta_{ij} \le 1$ *and* $\sum\sum_{i,j=0,1} \beta_{ij} = 1$. *The RV* (U_1,U_2) *with* $jpmf$,

$$P(U_1 = i, U_2 = j) = \beta_{ij}, \qquad 0 \le i, j \le 1 \tag{1}$$

is said to have the Bivariate Bernoulli ($BVBer$*) distribution denoted by* $BVBer(\underline{\beta})$.

Lemma 1. *Assume that* $U_1 \sim Ber(\beta_{11}+\beta_{10}), V_2 \sim Ber(\frac{\beta_{11}}{\beta_{11}+\beta_{10}})$ *and* $V_3 \sim Ber(\frac{\beta_{01}}{1-\beta_{11}-\beta_{10}})$ *are independent. Let*

$$U_2 = U_1V_2 + (1-U_1)V_3, \tag{2}$$

then, (U_1,U_2) *has the* $BVBer(\underline{\beta})$ *distribution of (1).*

Definition 2. *The RV* (U_1,U_2) *with* $jpmf$,

$$g(u_1,u_2) = \pi^{u_1}\overline{\pi}^{1-u_1}\alpha^{(1-u_1)(1-u_2)+u_1u_2}\overline{\alpha}^{u_1(1-u_2)+u_2(1-u_1)}, \qquad u_1,u_2 = 0,1 \tag{3}$$

is said to have the two parameters $BVBer$ *distribution denoted by* $BVBer(\pi,\alpha)$.

Note that the $BVBer(\pi,\alpha)$ of (3) is the special case of (1) when $\beta_{11} = \alpha\pi, \beta_{01} = \overline{\alpha}\overline{\pi}, \beta_{10} = \overline{\alpha}\pi, \beta_{00} = \alpha\overline{\pi}$. To generate (U_1,U_2) from the $BVBer(\pi,\alpha)$ of (3), we independently generate $U_1 \sim Ber(\pi), V_2 \sim Ber(\alpha)$ and $V_3 \sim Ber(\overline{\alpha})$ and use (2).

Note that if $(U_1,U_2) \sim BVBer(\pi,\alpha)$, then $U_1 \sim Ber(\pi), U_2 \sim Ber(\pi\alpha+\overline{\pi}\overline{\alpha}), Cov(U_1,U_2) = \pi\overline{\pi}(2\alpha-1)$ and U_1,U_2 are independent if and only if $\alpha = 0.5$.

Definition 3. *The RV* (U_1,U_2) *with* $jpmf$,

$$g(u_1,u_2)=\frac{1}{2}\beta^{1-u_1-u_2+2u_1u_2}\overline{\beta}^{u_1+u_2-2u_1u_2}, \qquad 0\le\beta\le 1, u_1,u_2=0,1, \tag{4}$$

is said to have the one-parameter $BVBer$ *distribution denoted by* $BVBer(\beta)$.

Note that $BVBer(\beta)$ is the special case of $BVBer(\underline{\beta})$ when $\beta_{00}=\beta_{11}=\frac{1}{2}\beta$ and $\beta_{01}=\beta_{10}=\frac{1}{2}\overline{\beta}$. Note also that if $(U_1,U_2)\sim BVBer(\beta)$, then $U_i\sim Ber(\frac{1}{2}), i=1,2, Cov(U_1,U_2)=\frac{2\beta-1}{4}$ and U_1,U_2 are independent if and only if $\beta=\frac{1}{2}$.

To generate one realization (U_1,U_2) from $BVBer(\beta)$ of (4), we independently generate $U_1\sim Ber(\frac{1}{2}), V_2\sim Ber(\beta)$ and $V_3\sim Ber(\overline{\beta})$ and use (2).

2.2 The BPD

Assume that $W_j\sim Poi(\lambda_j), j=1,2,3$ are independent RV. It is well known that $(X_1=W_1+W_3, X_2=W_2+W_3)$ has the bivariate Poisson distribution $(BPD(\underline{\lambda}))$ with $jpmf$

$$p(s,t;\underline{\lambda})=e^{-(\lambda_1+\lambda_2+\lambda_3)}\frac{\lambda_1^s}{s!}\frac{\lambda_2^t}{t!}\sum_{i=0}^{\min(s,t)}\binom{s}{i}\binom{t}{i}i!\left(\frac{\lambda_3}{\lambda_1\lambda_2}\right)^i, \qquad s,t=0,1,2,\ldots. \tag{5}$$

Note that

$$E(X_i)=Var(X_i)=\lambda_i+\lambda_3 \text{ and } Cov(X_1,X_2)=\lambda_3. \tag{6}$$

For a comprehensive treatment of the BPD, we refer to Kocherlakota and Kocherlakota (1992) and Johnson et al. (1997). The $jpmf$ of (5) can be computed by using the R function "pbivpois" of Karlis and Ntzoufras (2005). Let $(X_{1,i},X_{2,i}), i=1,2,\ldots,n$ be a random sample from (5). The MLE of λ_1,λ_2 and λ_3 can be obtained by using the R function "simple.bp" of Karlis and Ntzoufras (2005). The method of moments estimators (MME) of λ_1,λ_2 and λ_3 are given by

$$\widetilde{\lambda}_j=\overline{X}_j-\widetilde{\lambda}_3, \qquad j=1,2, \tag{7}$$

and

$$\widetilde{\lambda}_3=\frac{1}{n}\sum_{i=1}^{n}\left(X_{1,i}-\overline{X}_1\right)\left(X_{2,i}-\overline{X}_2\right). \tag{8}$$

2.3 The BGD of Phatak and Sreehari

The RV (X_1,X_2) with $jpmf$,

$$q(s,t;\underline{\theta})=\binom{s+t}{s}\delta_1^s\delta_2^t\left(1-\delta_1-\delta_2\right), \qquad s,t=0,1,2,\ldots \tag{9}$$

where $0<\delta_1,\delta_2<1$ and $0<1-\delta_1-\delta_2<1$, is said to follow the BGD of Phatak and Sreehari (1981), denoted by $BGD(\underline{\delta})$.

Note that for the $BGD(\underline{\delta})$, the following results hold (see, Krishna and Pundir (2009) and Hogg et al. (2005)):

1. $X_1 \sim Geo\left(\frac{1-\delta_1-\delta_2}{1-\delta_2}\right)$ and $X_2 \sim Geo\left(\frac{1-\delta_1-\delta_2}{1-\delta_1}\right)$.

2. Let $(X_{1,i}, X_{2,i}), i = 1,2,\ldots,n$ be a random sample from (9).

   (a) The *MLE* of δ_1 and δ_2 are given by

$$\widehat{\delta}_j = \frac{\overline{X}_j}{1+\overline{X}_1+\overline{X}_2}, \qquad j = 1,2, \tag{10}$$

   where $\overline{X}_j = \frac{1}{n}\sum_{i=1}^{n} X_{j,i}, j = 1,2.$

   (b) Using the result that $q(0,0;\underline{\delta}) = 1-\delta_1-\delta_2$, the *MME* of δ_1 and δ_2 are obtained as follows:

$$\widetilde{\delta}_j = \frac{\overline{X}_j \times \sum_{i=1}^{n} I(X_{1,i}=0, X_{2,i}=0)}{n}, \qquad j = 1,2. \tag{11}$$

3. We may generate a realization (X_1, X_2) from the $BGD(\underline{\delta})$ as follows:

   (a) Generate X_2 from $Geo(1-\frac{\delta_2}{1-\delta_1})$.

   (b) Given that $X_2 = y$, generate $V_1, V_2, \ldots, V_{y+1}$ independently from $Geo(1-\delta_1)$ and set $X_1 = \sum_{i=1}^{y+1} V_i$.

3 The BRST

Definition 4. *Assume that (U_1,U_2) has a BVBer distribution, that (X_1,X_2) is a nonnegative integer-valued RV independent of (U_1,U_2) with jpmf, $f(s,t)$. Let $f_i(\cdot)$ be the marginal pmf of $X_i, i = 1,2$. Then, the BRST of (X_1,X_2) is defined as*

$$Z_i = (2U_i - 1)X_i, \qquad i = 1,2. \tag{12}$$

3.1 BRST based on the $BVBer(\underline{\beta})$

Assume that $(U_1,U_2) \sim BVBer(\underline{\beta})$ of (1). In this case, the *jpmf* of Z_1 and Z_2 is given as follows:

$$h(0,0) = f(0,0), \tag{13}$$

$$h(s,0) = \begin{cases} (\beta_{11}+\beta_{10})f(s,0), & s = 1,2,\ldots \\ (\beta_{00}+\beta_{01})f(-s,0), & s = -1,-2,\ldots \end{cases} \tag{14}$$

$$h(0,t) = \begin{cases} (\beta_{11}+\beta_{01})f(0,t), & t = 1,2,\ldots \\ (\beta_{00}+\beta_{10})f(0,-t), & t = -1,-2,\ldots \end{cases} \tag{15}$$

and

$$h(s,t) = f(|s|,|t|) \times \begin{cases} \beta_{00}, & s,t = -1,-2,\ldots \\ \beta_{10}, & s = 1,2,\ldots,t = -1,-2,\ldots \\ \beta_{01}, & s = -1,-2,\ldots,t = 1,2,\ldots \\ \beta_{11}, & s,t = 1,2,\ldots. \end{cases} \tag{16}$$

For $i = 1,2$, the marginal *pmf* of Z_i is given by

$$h_i(s) = \begin{cases} (\beta_{11}+\beta_{10})\, f_i(s), & s = 1,2,\ldots \\ f_i(0), & s = 0, \\ (\beta_{00}+I(i=1)\beta_{01}+I(i=2)\beta_{10})\, f_i(-s), & s = -1,-2,\ldots. \end{cases} \tag{17}$$

Lemma 2. *It holds that*

$$E(Z_1^n Z_2^m) = E(X_1^n X_2^m) \times \begin{cases} 1, & \text{if } m \text{ and } n \text{ are even}, \\ (2\beta_{11}+2\beta_{10}-1), & \text{if } m \text{ is even and } n \text{ is odd}, \\ (2\beta_{11}+2\beta_{01}-1), & \text{if } m \text{ is odd and } n \text{ is even}, \\ (1-2\beta_{10}-2\beta_{01}), & \text{if } m \text{ and } n \text{ are odd}. \end{cases} \tag{18}$$

Proof. Note that (18) follows from the result that for $m,n = 0,1,2,\ldots$, we have

$$Z_1^n Z_2^m = X_1^n X_2^m \times \begin{cases} 1, & \text{if } m \text{ and } n \text{ are even}, \\ (2U_1-1), & \text{if } m \text{ is even and } n \text{ is odd}, \\ (2U_2-1), & \text{if } m \text{ is odd and } n \text{ is even}, \\ (2U_1-1)(2U_2-1), & \text{if } m \text{ and } n \text{ are odd}. \end{cases}$$

□

Corollary 1. *By (18), we have*

$$E(Z_1^n) = E(X_1^n) \times \begin{cases} 1, & \text{if } n \text{ is even}, \\ (2\beta_{11}+2\beta_{10}-1), & \text{if } n \text{ is odd}, \end{cases} \tag{19}$$

$$E(Z_2^m) = E(X_2^m) \times \begin{cases} 1, & \text{if } m \text{ is even}, \\ (2\beta_{11}+2\beta_{01}-1), & \text{if } m \text{ is odd}, \end{cases} \tag{20}$$

and

$$E(Z_1 Z_2) = (1-2\beta_{10}-2\beta_{01})\, E(X_1 X_2). \tag{21}$$

Hence, by (19)-(21), for $i = 1,2$,

$$E(Z_i^2) = E(X_i^2),$$

$$E(Z_i) = (2\beta_{11}+2I(i=1)\beta_{10}+2I(i=2)\beta_{01}-1)\, E(X_i), \tag{22}$$

$$\begin{aligned} Var(Z_i) =& Var(X_i) + 4\,(\beta_{11}+I(i=1)\beta_{10}+I(i=2)\beta_{01}) \\ & \times (1-\beta_{11}-I(i=1)\beta_{10}-I(i=2)\beta_{01})\,(E(X_i))^2 \end{aligned} \tag{23}$$

and

$$\begin{aligned} Cov(Z_1,Z_2) =& (1-2\beta_{10}-2\beta_{01})\, Cov(X_1,X_2) \\ & + 4E(X_1)E(X_2)\,(\beta_{11}-(\beta_{11}+\beta_{10})(\beta_{11}+\beta_{01})). \end{aligned} \tag{24}$$

Corollary 2. *In the special case when U_1 and U_2 are independent (i.e., when $\beta_{11} = \alpha_1\alpha_2, \beta_{10} = \alpha_1\overline{\alpha}_2, \beta_{01} = \overline{\alpha}_1\alpha_2$ and $\beta_{00} = \overline{\alpha}_1\overline{\alpha}_2$ with $0 < \alpha_1, \alpha_2 < 1$) (18)-(24) reduce to the corresponding results of Chesneau et al. (2018).*

Let $EN(W_1, W_2)$ be Shannon's entropy (see Shannon (1951)) of the RV (W_1, W_2) and let $EN(V)$ be Shannon's entropy of the RV V. Then, we have the following lemma.

Lemma 3. *It holds that*

$$\begin{aligned} EN(Z_1, Z_2) &= EN(X_1, X_2) + EN(U_1, U_2)\{1 - f_1(0) - f_2(0) + f(0,0)\} \\ &\quad + (f_1(0) - f(0,0))\,EN(U_2) + (f_2(0) - f(0,0))\,EN(U_1). \end{aligned} \tag{25}$$

Proof.

$$EN(Z_1, Z_2) = -\sum_{i=-\infty}^{\infty}\sum_{j=-\infty}^{\infty} h(i,j)\ln h(i,j) = \sum_{r=1}^{9} S_r, \tag{26}$$

where

$$S_1 = -f(0,0)\ln f(0,0), \tag{27}$$

$$\begin{aligned} &S_2 = -\sum_{i=1}^{\infty}\sum_{j=1}^{\infty} h(i,j)\ln h(i,j), \quad S_3 = -\sum_{i=-1}^{-\infty}\sum_{j=1}^{\infty} h(-i,j)\ln h(-i,j), \\ &S_4 = -\sum_{i=1}^{\infty}\sum_{j=-1}^{-\infty} h(i,-j)\ln h(i,-j), \quad S_5 = -\sum_{i=-1}^{-\infty}\sum_{j=-1}^{-\infty} h(-i,-j)\ln h(-i,-j), \\ &S_6 = -\sum_{j=1}^{\infty} h(0,j)\ln h(0,j), \; S_7 = -\sum_{j=-1}^{-\infty} h(0,-j)\ln h(0,-j), \\ &S_8 = -\sum_{i=1}^{\infty} h(i,0)\ln h(i,0), \quad \text{and} \quad S_9 = -\sum_{j=-1}^{-\infty} h(-i,0)\ln h(-i,0). \end{aligned}$$

For S_2, we have

$$S_2 = -\beta_{11}\ln\beta_{11}\sum_{i=1}^{\infty}\sum_{j=1}^{\infty} f(i,j) - \beta_{11}\sum_{i=1}^{\infty}\sum_{j=1}^{\infty} f(i,j)\ln f(i,j) = -\beta_{11}T_1\ln\beta_{11} + \beta_{11}T_2,$$

where

$$T_1 = 1 - f_1(0) - f_2(0) + f(0,0) \quad \text{and} \quad T_2 = -\sum_{i=1}^{\infty}\sum_{j=1}^{\infty} f(i,j)\ln f(i,j).$$

For S_3, we have

$$\begin{aligned} S_3 &= -\sum_{i=-1}^{-\infty}\sum_{j=1}^{\infty} \beta_{01} f(-i,j)\{\ln\beta_{01} + \ln f(-i,j)\} \\ &= -\beta_{01}\ln\beta_{01}\sum_{i=1}^{\infty}\sum_{j=1}^{\infty} f(i,j) - \beta_{01}\sum_{i=1}^{\infty}\sum_{j=1}^{\infty} f(i,j)\ln f(i,j) \\ &= -\beta_{01}\ln\beta_{01}T_1 + \beta_{01}T_2. \end{aligned}$$

Similarly,

$$S_4 = -\beta_{10}\ln\beta_{10}T_1 + \beta_{10}T_2 \quad \text{and} \quad S_5 = -\beta_{00}\ln\beta_{00}T_1 + \beta_{00}T_2.$$

Hence,

$$\sum_{r=2}^{5} S_r = EN(U_1,U_2)T_1 + T_2. \tag{28}$$

We can show that

$$T_2 = EN(X_1,X_2) - f(0,0)\ln f(0,0) + \sum_{i=0}^{\infty}[f(0,i)\ln f(0,i) + f(i,0)\ln f(i,0)]. \tag{29}$$

For S_6, we have

$$\begin{aligned}
S_6 &= -\sum_{j=1}^{\infty}(\beta_{11}+\beta_{01})\, f(0,j)\,\{\ln(\beta_{11}+\beta_{01}) + \ln f(0,j))\} \\
&= -(\beta_{11}+\beta_{01})\ln(\beta_{11}+\beta_{01})\sum_{j=1}^{\infty} f(0,j) - (\beta_{11}+\beta_{01})\sum_{j=1}^{\infty} f(0,j)\ln f(0,j) \\
&= -(\beta_{11}+\beta_{01})\ln(\beta_{11}+\beta_{01})\,(f_1(0) - f(0,0)) + (\beta_{11}+\beta_{01})\, f(0,0)\ln f(0,0) \\
&\quad -(\beta_{11}+\beta_{01})\sum_{i=0}^{\infty} f(0,i)\ln f(0,i).
\end{aligned}$$

For S_7, we have

$$\begin{aligned}
S_7 &= -(\beta_{00}+\beta_{10})\ln(\beta_{00}+\beta_{10})\,(f_1(0) - f(0,0)) + (\beta_{00}+\beta_{10})\, f(0,0)\ln f(0,0) \\
&\quad -(\beta_{00}+\beta_{10})\sum_{i=0}^{\infty} f(0,i)\ln f(0,i).
\end{aligned}$$

Hence

$$S_6 + S_7 = -\sum_{i=0}^{\infty} f(0,i)\ln f(0,i) + (f_1(0) - f(0,0))\, EN(U_2) + f(0,0)\ln f(0,0). \tag{30}$$

Similarly,

$$S_8 + S_9 = -\sum_{i=0}^{\infty} f(i,0)\ln f(i,0) + (f_2(0) - f(0,0))\, EN(U_1) + f(0,0)\ln f(0,0). \tag{31}$$

By (26)-(31), we obtain (25).

□

3.1.1 Maximum likelihood estimators (MLE)

Assume that $\underline{\theta}$ is the unknown parameter vector in the *jpmf* of (X_1,X_2) and hence in the *jpmf* of (Z_1,Z_2). In what follows, the presence of $\underline{\theta}$ will be made explicit in both f and h. Let $(Z_{1,i},Z_{2,i}), i = 1,2,\ldots,n$ be a random sample from $h(\cdot,\cdot;\underline{\theta})$ of (13)-(16). Define

$$n_0 = \sum_{i=1}^{n} I(Z_{1,i}=0, Z_{2,i}=0), \quad n_{+,0} = \sum_{i=1}^{n} I(Z_{1,i}>0, Z_{2,i}=0), \tag{32}$$

$$n_{-,0} = \sum_{i=1}^{n} I\left(Z_{1,i} < 0, Z_{2,i} = 0\right), \quad n_{0,+} = \sum_{i=1}^{n} I\left(Z_{1,i} = 0, Z_{2,i} > 0\right), \tag{33}$$

$$n_{0,-} = \sum_{i=1}^{n} I\left(Z_{1,i} = 0, Z_{2,i} < 0\right), \quad n_{+,+} = \sum_{i=1}^{n} I\left(Z_{1,i} > 0, Z_{2,i} > 0\right), \tag{34}$$

$$n_{-,+} = \sum_{i=1}^{n} I\left(Z_{1,i} < 0, Z_{2,i} > 0\right), \quad n_{+,-} = \sum_{i=1}^{n} I\left(Z_{1,i} > 0, Z_{2,i} < 0\right), \tag{35}$$

and

$$n_{-,-} = \sum_{i=1}^{n} I\left(Z_{1,i} < 0, Z_{2,i} < 0\right). \tag{36}$$

We can show that

$$E(n_0) = nP\left(Z_1 = 0, Z_2 = 0\right) = nf(0,0;\underline{\theta}), \tag{37}$$

$$E\left(n_{+,0}\right) = n\left(\beta_{11} + \beta_{10}\right)\left(f_2(0;\underline{\theta}) - f(0,0;\underline{\theta})\right), \tag{38}$$

$$E\left(n_{-,0}\right) = n\left(1 - \beta_{11} - \beta_{10}\right)\left(f_2(0;\underline{\theta}) - f(0,0;\underline{\theta})\right), \tag{39}$$

$$E\left(n_{0,+}\right) = n\left(\beta_{11} + \beta_{01}\right)\left(f_1(0;\underline{\theta}) - f(0,0;\underline{\theta})\right), \tag{40}$$

$$E\left(n_{0,-}\right) = n\left(1 - \beta_{11} - \beta_{01}\right)\left(f_1(0;\underline{\theta}) - f(0,0;\underline{\theta})\right), \tag{41}$$

$$E\left(n_{+,+}\right) = n\beta_{11}\left(1 - f_1(0;\underline{\theta}) - f_2(0;\underline{\theta}) + f(0,0;\underline{\theta})\right), \tag{42}$$

$$E\left(n_{-,+}\right) = n\beta_{01}\left(1 - f_1(0;\underline{\theta}) - f_2(0;\underline{\theta}) + f(0,0;\underline{\theta})\right), \tag{43}$$

$$E\left(n_{+,-}\right) = n\beta_{10}\left(1 - f_1(0;\underline{\theta}) - f_2(0;\underline{\theta}) + f(0,0;\underline{\theta})\right), \tag{44}$$

and

$$E\left(n_{-,-}\right) = n\left(1 - \beta_{11} - \beta_{10} - \beta_{01}\right)\left(1 - f_1(0;\underline{\theta}) - f_2(0;\underline{\theta}) + f(0,0;\underline{\theta})\right). \tag{45}$$

Lemma 4. *Assume that $\underline{T}(\underline{X}_1, \underline{X}_2)$ is the MLE of $\underline{\theta}$ based on a random sample from $f(\cdot,\cdot;\underline{\theta})$, and let $I_{X_1,X_2}(\underline{\theta})$ be the corresponding Fisher Information Matrix. Let $(Z_{1,i}, Z_{2,i}), i = 1,2,\ldots,n$ be a random sample from $h(\cdot,\cdot;\underline{\theta})$ and let $n_{+,0},\ldots,n_{-,-}$ be as in (32)-(36). Then, for the MLE of $\underline{\theta}, \beta_{11}, \beta_{10}$ and β_{01}, we have*

1.
$$\widehat{\underline{\theta}} = \underline{T}(\underline{|Z_1|}, \underline{|Z_2|}).$$

2. *$\widehat{\beta}_{11}, \widehat{\beta}_{10}$ and $\widehat{\beta}_{01}$ are obtained by maximizing*

$$\begin{aligned} l_1 = & n_{+,0} \ln\left(\beta_{11} + \beta_{10}\right) + n_{-,0} \ln\left(1 - \beta_{11} - \beta_{10}\right) + n_{0,+} \ln\left(\beta_{11} + \beta_{01}\right) \\ & + n_{0,-} \ln\left(1 - \beta_{11} - \beta_{01}\right) + n_{+,+} \ln \beta_{11} + n_{-,+} \ln \beta_{01} \\ & + n_{+,-} \ln \beta_{10} + n_{-,-} \ln\left(1 - \beta_{11} - \beta_{10} - \beta_{01}\right) \end{aligned} \tag{46}$$

subject to the constraints,

$$0 \le \beta_{11} \le 1, \quad 0 \le \beta_{10} \le 1, \quad 0 \le \beta_{01} \le 1, \text{ and } 0 \le \beta_{11} + \beta_{10} + \beta_{01} \le 1.$$

3.

$$\sqrt{n}\begin{pmatrix}\widehat{\beta}_{11}-\beta_{11}\\ \widehat{\beta}_{10}-\beta_{10}\\ \widehat{\beta}_{01}-\beta_{01}\\ \underline{\widehat{\theta}}-\underline{\theta}\end{pmatrix}\xrightarrow{D} MVN(\underline{0},\begin{bmatrix}\Sigma_1^{-1} & \underline{0}\\ \underline{0} & I_{X_1,X_2}^{-1}(\underline{\theta})\end{bmatrix}], \tag{47}$$

where

$$\sum\nolimits_1=\begin{bmatrix}\sigma_1 & \sigma_{12} & \sigma_{13}\\ \sigma_{12} & \sigma_2 & \sigma_{23}\\ \sigma_{13} & \sigma_{23} & \sigma_3\end{bmatrix},$$

$$\sigma_{23}=\frac{1-f_1(0;\underline{\theta})-f_2(0;\underline{\theta})+f(0,0;\underline{\theta})}{1-\beta_{11}-\beta_{10}-\beta_{01}},$$

$$\begin{aligned}\sigma_1=&\frac{f_2(0;\underline{\theta})-f(0,0;\underline{\theta})}{(\beta_{11}+\beta_{10})(1-\beta_{11}-\beta_{10})}+\frac{f_1(0;\underline{\theta})-f(0,0;\underline{\theta})}{(\beta_{11}+\beta_{01})(1-\beta_{11}-\beta_{01})}\\ &+\frac{(1-\beta_{10}-\beta_{01})\sigma_{23}}{\beta_{11}},\end{aligned}$$

$$\sigma_2=\frac{f_2(0;\underline{\theta})-f(0,0;\underline{\theta})}{(\beta_{11}+\beta_{10})(1-\beta_{11}-\beta_{10})}+\frac{(1-\beta_{11}-\beta_{01})\sigma_{23}}{\beta_{10}},$$

$$\sigma_3=\frac{f_1(0;\underline{\theta})-f(0,0;\underline{\theta})}{(\beta_{11}+\beta_{01})(1-\beta_{11}-\beta_{01})}+\frac{(1-\beta_{11}-\beta_{10})\sigma_{23}}{\beta_{01}},$$

$$\sigma_{12}=\frac{f_2(0;\underline{\theta})-f(0,0;\underline{\theta})}{(\beta_{11}+\beta_{10})(1-\beta_{11}-\beta_{10})}+\sigma_{23},$$

and

$$\sigma_{13}=\frac{f_1(0;\underline{\theta})-f(0,0;\underline{\theta})}{(\beta_{11}+\beta_{01})(1-\beta_{11}-\beta_{01})}+\sigma_{23}.$$

Proof. The log-likelihood function (Log-LF) of the sample is given by

$$l=l_1+l_2, \tag{48}$$

where l_1 is as in (46) and

$$l_2=\sum_{i=1}^{n}\ln f(|z_{1,i}|,|z_{2,i}|;\underline{\theta}). \tag{49}$$

It is clear from (48),(46), and (49) that the MLE of $\underline{\theta}$ is obtained by maximizing l_2 and the MLE of $\beta_{11}\beta_{10}$ and β_{01} are obtained by maximizing l_1 subject to the constraints,

$$0\le\beta_{11}\le 1,\ 0\le\beta_{10}\le 1,\ 0\le\beta_{01}\le 1,\ \text{and } 0\le\beta_{11}+\beta_{10}+\beta_{01}\le 1.$$

We will use the R function “constrOptim” to obtain the MLE of $\beta_{11}\beta_{10}$ and β_{01}. To obtain $\sum_1$ of (47) we use (37)-(45) and the following results:

$$\begin{aligned}\frac{\partial^2 l}{\partial\beta_{11}^2}=\frac{\partial^2 l_1}{\partial\beta_{11}^2}=&-\frac{n_{+,0}}{(\beta_{11}+\beta_{10})^2}-\frac{n_{-,0}}{(1-\beta_{11}-\beta_{10})^2}-\frac{n_{0,+}}{(\beta_{11}+\beta_{01})^2}\\ &-\frac{n_{0,-}}{(1-\beta_{11}-\beta_{01})^2}-\frac{n_{+,+}}{\beta_{11}^2}-\frac{n_{-,-}}{(1-\beta_{11}-\beta_{10}-\beta_{01})^2},\end{aligned}$$

$$\frac{\partial^2 l}{\partial \beta_{10}^2} = \frac{\partial^2 l_1}{\partial \beta_{10}^2} = -\frac{n_{+,0}}{(\beta_{11}+\beta_{10})^2} - \frac{n_{-,0}}{(1-\beta_{11}-\beta_{10})^2} - \frac{n_{+,-}}{\beta_{10}^2} - \frac{n_{-,-}}{(1-\beta_{11}-\beta_{10}-\beta_{01})^2},$$

$$\frac{\partial^2 l}{\partial \beta_{01}^2} = \frac{\partial^2 l_1}{\partial \beta_{01}^2} = -\frac{n_{0,+}}{(\beta_{11}+\beta_{01})^2} - \frac{n_{0,-}}{(1-\beta_{11}-\beta_{01})^2} - \frac{n_{-,+}}{\beta_{01}^2} - \frac{n_{-,-}}{(1-\beta_{11}-\beta_{10}-\beta_{01})^2},$$

$$\frac{\partial^2 l}{\partial \beta_{10}\partial \beta_{11}} = \frac{\partial^2 l_1}{\partial \beta_{10}\partial \beta_{11}} = -\frac{n_{+,0}}{(\beta_{11}+\beta_{10})^2} - \frac{n_{-,0}}{(1-\beta_{11}-\beta_{10})^2} - \frac{n_{-,-}}{(1-\beta_{11}-\beta_{10}-\beta_{01})^2},$$

$$\frac{\partial^2 l}{\partial \beta_{01}\partial \beta_{11}} = \frac{\partial^2 l_1}{\partial \beta_{01}\partial \beta_{11}} = -\frac{n_{0,+}}{(\beta_{11}+\beta_{01})^2} - \frac{n_{0,-}}{(1-\beta_{11}-\beta_{01})^2} - \frac{n_{-,-}}{(1-\beta_{11}-\beta_{10}-\beta_{01})^2},$$

and

$$\frac{\partial l}{\partial \beta_{10}\partial \beta_{01}} = \frac{\partial l_1}{\partial \beta_{10}\partial \beta_{01}} = -\frac{n_{-,-}}{(1-\beta_{11}-\beta_{10}-\beta_{01})^2}.$$

□

3.1.2 Method of moments estimators (MME)

Lemma 5. *Assume that $\underline{\widetilde{T}}(\cdot,\cdot)$ is the MME of $\underline{\theta}$ based on a random sample from $f(\cdot,\cdot;\underline{\theta})$. Then, for the MME of $\underline{\theta}, \beta_{11}, \beta_{10}$ and β_{01} we have*

$$\underline{\widetilde{\theta}} = \underline{\widetilde{T}}(\underline{|Z_1|}, \underline{|Z_2|}), \tag{50}$$

$$\widetilde{\beta}_{10} = \frac{1}{4}\left\{1 + A_1 - A_2 - C\right\}, \tag{51}$$

$$\widetilde{\beta}_{01} = \frac{1}{4}\left\{1 - A_1 + A_2 - C\right\}, \tag{52}$$

and

$$\widetilde{\beta}_{11} = \frac{1}{4}\left\{1 + A_1 + A_2 + C\right\}, \tag{53}$$

where

$$A_j = \frac{\sum_{i=1}^n Z_{j,i}}{\sum_{i=1}^n |Z_{j,i}|}, \qquad j = 1,2, \tag{54}$$

and

$$C = \frac{\sum_{i=1}^n Z_{1,i} Z_{2,i}}{\sum_{i=1}^n |Z_{1,i} Z_{2,i}|}. \tag{55}$$

Proof. Note that $\widetilde{\beta}_{11}, \widetilde{\beta}_{10}$ and $\widetilde{\beta}_{01}$ are obtained as follows. We start by solving

$$\frac{1}{n}\sum_{i=1}^n Z_{1,i} = (2\beta_{11} + 2\beta_{10} - 1)\left\{E(X_1)\Big|_{\underline{\theta}=\underline{\widetilde{\theta}}}\right\}, \tag{56}$$

$$\frac{1}{n}\sum_{i=1}^n Z_{2,i} = (2\beta_{11} + 2\beta_{01} - 1)\left\{E(X_2)\Big|_{\underline{\theta}=\underline{\widetilde{\theta}}}\right\}, \tag{57}$$

and

$$\frac{1}{n}\sum_{i=1}^{n} Z_{1,i}Z_{2,i} = (1-2\beta_{01}-2\beta_{10})\left\{E(X_1X_2)\Big|_{\underline{\theta}=\underline{\widetilde{\theta}}}\right\}. \tag{58}$$

Note that

$$X_1X_2 = |Z_1Z_2| \text{ and } E(X_1X_2) = E(|Z_1Z_2|).$$

Hence, (58) can be replaced by

$$\frac{1}{n}\sum_{i=1}^{n} Z_{1,i}Z_{2,i} = (1-2\beta_{01}-2\beta_{10})\,\frac{1}{n}\sum_{i=1}^{n} |Z_{1,i}Z_{2,i}|. \tag{59}$$

We can show that (56), (57), and (59) are, respectively, equivalent to

$$A_1 = 2\widetilde{\beta}_{11} + 2\widetilde{\beta}_{10} - 1, \tag{60}$$

$$A_2 = 2\widetilde{\beta}_{11} + 2\widetilde{\beta}_{01} - 1, \tag{61}$$

and

$$C = 1 - 2\widetilde{\beta}_{01} - 2\widetilde{\beta}_{10}. \tag{62}$$

By solving (60)-(62), we obtain (51)-(53). □

Consider the special case when X_1 and X_2 are independent. In this case, the MME estimators of β_{10}, β_{01} and β_{11} are as given in (51)-(53) after replacing C of (55) with

$$C_1 = \frac{n\sum_{i=1}^{n} Z_{1,i}Z_{2,i}}{\left(\sum_{i=1}^{n}|Z_{1,i}|\right)\left(\sum_{i=1}^{n}|Z_{2,i}|\right)}.$$

3.2 BRST based on the $BVBer(\pi,\alpha)$ distribution

Assume that (U_1,U_2) has the $BVBer(\pi,\alpha)$ distribution of (3). Define (Z_1,Z_2) as in (12). Then, the $jpmf$ of Z_1 and Z_2 is given as follows:

$$h(0,0) = f(0,0), \tag{63}$$

$$h(s,0) = \begin{cases} \pi f(s,0), & s = 1,2,\ldots, \\ \overline{\pi} f(-s,0), & s = -1,-2,\ldots \end{cases} \tag{64}$$

$$h(0,t) = \begin{cases} (\alpha\pi + \overline{\alpha\pi})\, f(0,t), & t = 1,2,\ldots, \\ (\overline{\alpha}\pi + \alpha\overline{\pi})\, f(0,-t), & t = -1,-2,\ldots \end{cases} \tag{65}$$

and

$$h(s,t) = f(|s|,|t|) \times \begin{cases} \overline{\pi}\alpha, & s,t = -1,-2,\ldots \\ \overline{\alpha}\pi, & s = 1,2,\ldots,t = -1,-2,\ldots \\ \overline{\alpha\pi}, & s = -1,-2,\ldots,t = 1,2,\ldots \\ \alpha\pi, & s,t = 1,2,\ldots. \end{cases} \tag{66}$$

The marginal $pmf's$ of Z_1 and Z_2 are given by

$$h_1(s) = \begin{cases} \pi f_1(s), & s = 1,2,\ldots \\ f_1(0), & s = 0, \\ \overline{\pi} f_1(-s), & s = -1,-2,\ldots \end{cases} \tag{67}$$

and

$$h_2(t) = \begin{cases} (\alpha\pi + \overline{\alpha\pi})\, f_2(t), & t = 1,2,\ldots, \\ f_2(0), & t = 0, \\ (\overline{\alpha}\pi + \alpha\overline{\pi})\, f_2(-t), & t = -1,-2,\ldots. \end{cases} \tag{68}$$

Lemma 6. *It holds that*

$$E(Z_1^n Z_2^m) = E(X_1^n X_2^m) \times \begin{cases} 1, & \text{if } m \text{ and } n \text{ are even,} \\ (2\pi - 1), & \text{if } m \text{ is even and } n \text{ is odd,} \\ (2\alpha - 1)(2\pi - 1), & \text{if } m \text{ is odd and } n \text{ is even,} \\ (2\alpha - 1), & \text{if } m \text{ and } n \text{ are odd,} \end{cases}$$

$$E(Z_1^n) = E(X_1^n) \times \begin{cases} 1, & \text{if } n \text{ is even,} \\ (2\pi - 1), & \text{if } n \text{ is odd,} \end{cases}$$

$$E(Z_2^m) = E(X_2^m) \times \begin{cases} 1, & \text{if } m \text{ is even,} \\ (2\alpha - 1)(2\pi - 1), & \text{if } m \text{ is odd,} \end{cases}$$

$$E(Z_1) = (2\pi - 1)\, E(X_1),$$

$$E(Z_2) = (2\alpha - 1)(2\pi - 1)\, E(X_2),$$

$$Var(Z_1) = Var(X_1) + 4\pi\overline{\pi}\, (E(X_1))^2,$$

$$Var(Z_2) = Var(X_2) + 2\,(\alpha\overline{\pi} + \overline{\alpha}\pi)\,(E(X_2))^2,$$

$$E(Z_1 Z_2) = (2\alpha - 1)\, E(X_1 X_2),$$

and

$$Cov(Z_1, Z_2) = (2\alpha - 1)\left\{Cov(X_1, X_2) + 4\pi\overline{\pi} E(X_1) E(X_2)\right\}.$$

3.2.1 MLE estimators

Assume that $(Z_{1,i}, Z_{2,i}), i = 1,2,\ldots,n$ is a random sample from $h(\cdot,\cdot;\underline{\theta})$ of (63)-(66). Let l_2 be as in (49) and let $n_{\pm,0}, n_{\pm,-}, n_{\pm,+}$ be as in (32)-(36) and

$$n_{\pm,\cdot} = n_{\pm,0} + n_{\pm,-} + n_{\pm,+}.$$

The Log-LF of the sample is given by

$$l_3 = l_2 + l_4,$$

where

$$\begin{aligned} l_4 =& n_{+,\cdot} \ln\pi + n_{-,\cdot} \ln\overline{\pi} + (n_{+,+} + n_{-,-}) \ln(\alpha) + (n_{-,+} + n_{+,-}) \ln\overline{\alpha} \\ & + n_{0,+} \ln(\alpha\pi + \overline{\alpha\pi}) + n_{0,-} \ln(\alpha\overline{\pi} + \overline{\alpha}\pi). \end{aligned}$$

Lemma 7. *Assume that $\underline{T}(\cdot,\cdot)$ is the MLE of $\underline{\theta}$ based on a random sample from $f(\cdot,\cdot;\underline{\theta})$, and let $I_{X_1,X_2}(\underline{\theta})$ be the corresponding Fisher information Matrix. Let $\widehat{\underline{\theta}},\widehat{\pi}$ and $\widehat{\alpha}$ be the MLE of $\underline{\theta},\pi$ and α. Then,*

1.

$$\widehat{\underline{\theta}} = \underline{T}(\underline{|Z_1|},\underline{|Z_2|}).$$

2. $\widehat{\pi}$ and $\widehat{\alpha}$ are obtained by maximizing l_4 subject to the constraints,

$$0 \le \alpha \le 1 \text{ and } 0 \le \pi \le 1.$$

3. As $n \longrightarrow \infty$,

$$\sqrt{n}\begin{pmatrix} \widehat{\pi}-\pi \\ \widehat{\alpha}-\alpha \\ \widehat{\underline{\theta}}-\underline{\theta} \end{pmatrix} \xrightarrow{D} MVN(\underline{0}, \begin{bmatrix} \sum_2^{-1} & \underline{0} \\ \underline{0} & I^{-1}_{X_1,X_2}(\underline{\theta}) \end{bmatrix},$$

where

$$\sum\nolimits_2 = \begin{bmatrix} \sigma_1^* & \sigma_{12}^* \\ \sigma_{12}^* & \sigma_2^* \end{bmatrix},$$

$$\sigma_1^* = \frac{(f_1(0;\underline{\theta}_1)-f(0,0;\underline{\theta}_1,\underline{\theta}_2))\,(2\alpha-1)^2}{(\alpha\pi+\overline{\alpha}\overline{\pi})\,(\alpha\overline{\pi}+\overline{\alpha}\pi)} + \frac{1-f_1(0;\underline{\theta}_1)}{\pi\overline{\pi}},$$

$$\sigma_2^* = \frac{(f_1(0;\underline{\theta}_1)-f(0,0;\underline{\theta}_1,\underline{\theta}_2))\,(2\pi-1)^2}{(\alpha\pi+\overline{\alpha}\overline{\pi})\,(\alpha\overline{\pi}+\overline{\alpha}\pi)} + \frac{1-f_1(0;\underline{\theta}_1)-f_2(0;\underline{\theta}_2)+f(0,0;\underline{\theta}_1,\underline{\theta}_2))}{\alpha\overline{\alpha}},$$

and

$$\sigma_{12}^* = \frac{(f_1(0;\underline{\theta}_1)-f(0,0;\underline{\theta}_1,\underline{\theta}_2))\,(2\alpha-1)\,(2\pi-1)}{(\alpha\pi+\overline{\alpha}\overline{\pi})\,(\alpha\overline{\pi}+\overline{\alpha}\pi)}.$$

Proof. To obtain $\sum_2$, we use (38)-(45), (69)-(71),

$$\frac{\partial^2 l_4}{\partial\alpha\partial\pi} = 2\left\{\frac{n_{0,+}}{\alpha\pi+\overline{\alpha}\overline{\pi}} - \frac{n_{0,-}}{\alpha\overline{\pi}+\overline{\alpha}\pi}\right\} - (2\alpha-1)\,(2\pi-1)\left\{\frac{n_{0,+}}{(\alpha\pi+\overline{\alpha}\overline{\pi})^2} + \frac{n_{0,-}}{(\alpha\overline{\pi}+\overline{\alpha}\pi)^2}\right\}, \tag{69}$$

$$\frac{\partial^2 l_4}{\partial\pi^2} = -\frac{n_{+,\cdot}}{\pi^2} - \frac{n_{-,\cdot}}{\overline{\pi}^2} - (2\alpha-1)^2\left\{\frac{n_{0,+}}{(\alpha\pi+\overline{\alpha}\overline{\pi})^2} + \frac{n_{0,-}}{(\alpha\overline{\pi}+\overline{\alpha}\pi)^2}\right\}, \tag{70}$$

and

$$\frac{\partial^2 l_4}{\partial\alpha^2} = -(2\pi-1)^2\left\{\frac{n_{0,+}}{(\alpha\pi+\overline{\alpha}\overline{\pi})^2} + \frac{n_{0,-}}{(\alpha\overline{\pi}+\overline{\alpha}\pi)^2}\right\} - \frac{n_{+,+}+n_{-,-}}{\alpha^2} - \frac{n_{-,+}+n_{+,-}}{\overline{\alpha}^2}. \tag{71}$$

□

3.2.2 MME estimators

Lemma 8. *Assume that* $\underline{\widetilde{T}}(\cdot,\cdot)$ *is the MME of* $\underline{\theta}$ *based on a random sample from* $f(\cdot,\cdot;\underline{\theta})$*. For the MME of* $\underline{\theta}, \pi$*, and* α*, we have*

$$\underline{\widetilde{\theta}} = \underline{\widetilde{T}}(\underline{|Z_1|}, \underline{|Z_2|}),$$

$$\widetilde{\pi} = \frac{\sum_{i=1}^{n} Z_{1,i} + \sum_{i=1}^{n} |Z_{1,i}|}{2\sum_{i=1}^{n} |Z_{1,i}|}, \tag{72}$$

and

$$\widetilde{\alpha} = \frac{1}{2}\left\{ \frac{(\sum_{i=1}^{n} |Z_{1,i}|) \times (\sum_{i=1}^{n} Z_{2,i})}{(\sum_{i=1}^{n} Z_{1,i}) \times (\sum_{i=1}^{n} |Z_{2,i}|)} + 1 \right\}. \tag{73}$$

Note that $\widetilde{\pi}$ *and* $\widetilde{\alpha}$ *are obtained by solving*

$$\sum_{i=1}^{n} Z_{1,i} = (2\pi - 1) \sum_{i=1}^{n} |Z_{1,i}|$$

and

$$\sum_{i=1}^{n} Z_{2,i} = (2\alpha - 1)(2\pi - 1) \sum_{i=1}^{n} |Z_{2,i}|.$$

3.3 BRST based on $BVBer(\beta)$

Assume that (U_1, U_2) has the $BVBer(\beta)$ distribution of (4). Define (Z_1, Z_2) as in (12). Then, the $jpmf$ of Z_1 and Z_2 is given as follows:

$$h(0,0) = f(0,0), \tag{74}$$

$$h(s,0) = \frac{1}{2} \times \begin{cases} f(s,0), & s = 1,2,\ldots \\ f(-s,0), & s = -1,-2,\ldots \end{cases} \tag{75}$$

$$h(0,t) = \frac{1}{2} \times \begin{cases} f(0,t), & t = 1,2,\ldots \\ f(0,-t), & t = -1,-2,\ldots \end{cases} \tag{76}$$

and

$$h(s,t) = \frac{1}{2} f(|s|,|t|) \times \begin{cases} \beta, & s,t = -1,-2,\ldots \\ \overline{\beta}, & s = 1,2,\ldots,t = -1,-2,\ldots \\ \overline{\beta}, & s = -1,-2,\ldots,t = 1,2,\ldots \\ \beta, & s,t = 1,2,\ldots. \end{cases} \tag{77}$$

The marginal $pmf's$ of Z_1 and Z_2 are given, for $i = 1,2$, by

$$h_i(s) = \begin{cases} \frac{1}{2} f_i(s), & s = 1,2,\ldots \\ f_i(0), & s = 0, \\ \frac{1}{2} f_i(-s), & s = -1,-2,\ldots. \end{cases} \tag{78}$$

Lemma 9. *It holds that*

$$E(Z_1^n Z_2^m) = E(X_1^n X_2^m) \times \begin{cases} 1, & \text{if } m \text{ and } n \text{ are even}, \\ 0, & \text{if } m \text{ is even and } n \text{ is odd}, \\ 0, & \text{if } m \text{ is odd and } n \text{ is even}, \\ 2\beta - 1, & \text{if } m \text{ and } n \text{ are odd}, \end{cases} \tag{79}$$

$$E(Z_i^n) = E(X_i^n) \times \begin{cases} 1, & \text{if } n \text{ is even}, \\ 0, & \text{if } n \text{ is odd}, \end{cases} \qquad i = 1, 2, \tag{80}$$

$$Var(Z_i) = Var(X_i) + (E(X_i))^2, \qquad i = 1, 2, \tag{81}$$

and

$$Cov(Z_1, Z_2) = (2\beta - 1)\{Cov(X_1, X_2) + E(X_1)E(X_2)\}. \tag{82}$$

Proof. Note that for $i = 1, 2$ and $r = 0, 1, 2, \ldots,$

$$Z_i^r = \begin{cases} X_i^r, & \text{if } r \text{ is even}, \\ (2U_i - 1)X_i^r, & \text{if } r \text{ is odd}. \end{cases}$$

Consequently, for $m, n = 0, 1, 2, \ldots,$

$$Z_1^n Z_2^m = X_1^n X_2^m \times \begin{cases} 1, & \text{if } m \text{ and } n \text{ are even}, \\ (2U_1 - 1), & \text{if } m \text{ is even and } n \text{ is odd}, \\ (2U_2 - 1), & \text{if } m \text{ is odd and } n \text{ is even}, \\ (2U_1 - 1)(2U_2 - 1), & \text{if } m \text{ and } n \text{ are odd}. \end{cases}$$

Hence, we obtain (79) and (80). Using (79) and (80) ,we obtain

$$E(Z_1) = E(Z_2) = 0,$$

$$E(Z_i^2) = E(X_i^2), \qquad i = 1, 2,$$

and

$$E(Z_1 Z_2) = (2\beta - 1)E(X_1 X_2).$$

Hence, we obtain (81) and (82). □

Remark 1. 1. *For the MLE, we use the notation of Lemma 4. We can show that the log-LF is given by*

$$l = C + (n_{+,+} + n_{-,-}) \ln\beta + (n_{+,-} + n_{-,+}) \ln\overline{\beta} + l_2,$$

where l_2 is as in (49). Hence, the MLE of $\underline{\theta}$ is as in Lemma 4 and the MLE of β is given by

$$\widehat{\beta} = \frac{n_{+,+} + n_{-,-}}{n_{+,+} + n_{-,-} + n_{+,-} + n_{-,+}}. \tag{83}$$

In addition, as $n \longrightarrow \infty$,

$$\sqrt{n} \begin{pmatrix} \widehat{\beta} - \beta \\ \widehat{\underline{\theta}} - \underline{\theta} \end{pmatrix} \xrightarrow{D} MVN\left(\underline{0}, daig\left\{\beta\overline{\beta}, I^{-1}_{X_1,X_2}(\underline{\theta})\right\}\right).$$

2. *The MME of $\underline{\theta}$ is as in (50) and the MME of β is given by*

$$\widehat{\beta}_m = \frac{1}{2}\left\{\frac{\sum Z_{1,i} Z_{2,i}}{\sum |Z_{1,i} Z_{2,i}|} + 1\right\}. \tag{84}$$

4 BRST of the BPD

Assume that (U_1,U_2) has a $BVBer$ distribution, that (X_1,X_2) has the BPD of (5) and that (X_1,X_2) is independent of (U_1,U_2). In the three models of this section, we may estimate λ_1,λ_2 and λ_3 using the following alternatives:

1. The MLE estimators obtained as in Lemma 3 using the R function "simple.bp" of Karlis and Ntzoufras (2005) on $(|Z_{1,i}|,|Z_{2,i}|), i=1,2,\ldots,n$.

2. The MME of (8) and (7) expressed in terms of $(|Z_{1,i}|,|Z_{2,i}|), i=1,2,\ldots,n$.

4.1 Models based on the $BVBer(\underline{\beta})$ distribution

Assume here that (U_1,U_2) has the $BVBer(\underline{\beta})$ distribution of (1). In this case, the $jpmf$ of Z_1 and Z_2 is given as in (13)-(16) after replacing $f(\cdot,\cdot)$ by $p(\cdot,\cdot;\underline{\lambda})$ of (5). The marginal $pmf's$ of Z_1 and Z_2 are given as in (13) and (16) after replacing $f_i(\cdot)$ by the pdf of $Poi(\lambda_i+\lambda_3), i=1,2$. Using (22), (23), (24), and (6) we obtain

$$\begin{aligned}Cov(Z_1,Z_2) &= (1-2\beta_{10}-2\beta_{01})\lambda_3+4(\lambda_1+\lambda_3)(\lambda_2+\lambda_3)\\&\quad\times(\beta_{11}-(\beta_{11}+\beta_{10})(\beta_{11}+\beta_{01})),\end{aligned}$$

$$E(Z_i)=(2\beta_{11}+2I(i=1)\beta_{10}+2I(i=2)\beta_{01}-1)(\lambda_1 I(i=1)+\lambda_2 I(i=2)+\lambda_3),$$

and

$$\begin{aligned}Var(Z_i) &= (\lambda_1 I(i=1)+\lambda_2 I(i=2)+\lambda_3)+4(\beta_{11}+I(i=1)\beta_{10}+I(i=2)\beta_{01})\\&\quad\times(1-\beta_{11}-I(i=1)\beta_{10}-I(i=2)\beta_{01})(\lambda_1 I(i=1)+\lambda_2 I(i=2)+\lambda_3)^2.\end{aligned}$$

For the estimation of β_{11},β_{10}, and β_{01}, we may use the following alternatives:

1. The MLE estimators obtained as in Lemma 3 using the R function "constrOptim".

2. The MME of (51)-(53).

In each of Figures 1 and 2, we give the scatter plot of a random sample of size $n=100$ from the $BRST$ of the BPD for selected values of $\underline{\beta}$ and $\underline{\lambda}$.

4.2 Models based on the $BVBer(\pi,\alpha)$ distribution

Assume that (U_1,U_2) has the $BVBer(\pi,\alpha)$ distribution of (1). In this case, the $jpmf$ of Z_1 and Z_2 is given as in (63)-(66) after replacing $f(\cdot,\cdot)$ by $p(\cdot,\cdot;\underline{\lambda})$ of (5). The marginal $pmf's$ of Z_1 and Z_2 are given as in (67) and (68) after replacing $f_i(\cdot)$ by the pdf of $Poi(\lambda_i+\lambda_3), i=1,2$. In addition,

$$E(Z_1)=(2\pi-1)(\lambda_1+\lambda_3),$$

$$E(Z_2)=(2\alpha-1)(2\pi-1)(\lambda_2+\lambda_3),$$

$$Var(Z_1)=(\lambda_1+\lambda_3)+4\pi\overline{\pi}(\lambda_1+\lambda_3)^2,$$

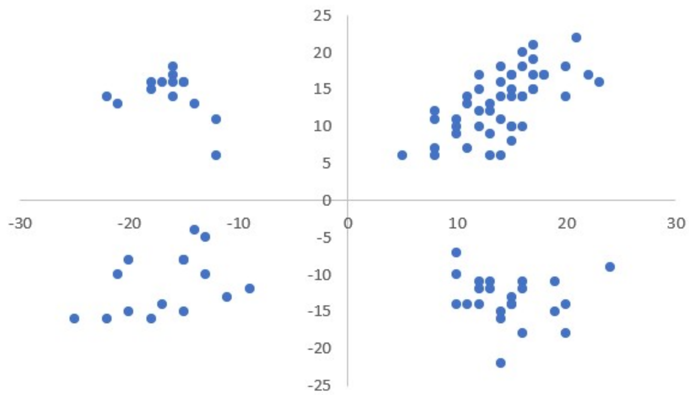

Figure 1: $BRST$ of BPD with $\underline{\beta} = (0.4, 0.22, 0.17), \underline{\lambda} = (9, 7, 2), \rho = 0.137$, and $n = 100$.

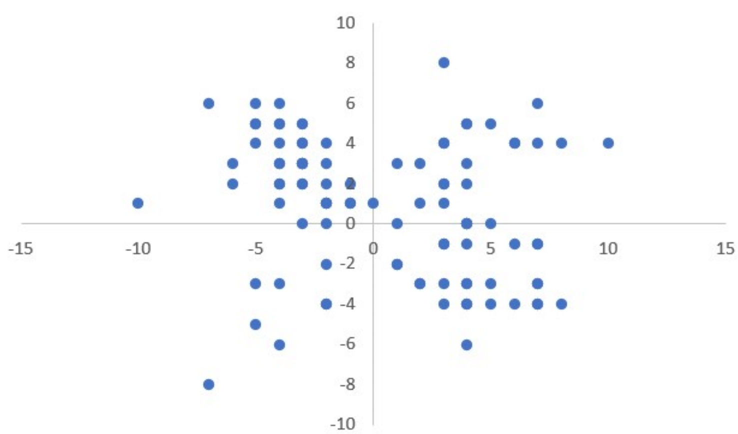

Figure 2: $BRST$ of BPD with $\underline{\beta} = (0.2, 0.3, 0.35)$ $, \underline{\lambda} = (3, 2, 1), \rho = -0.22$, and $n = 100$.

$$Var(Z_2) = (\lambda_2 + \lambda_3) + 2\left(\overline{\alpha}\pi + \alpha\overline{\pi}\right)(\lambda_2 + \lambda_3)^2,$$

and

$$Cov(Z_1, Z_2) = \lambda_3 + 4\pi\overline{\pi}(2\alpha - 1)(\lambda_1 + \lambda_3)(\lambda_2 + \lambda_3).$$

For the estimation of π and α, we may use the following alternatives:

1. The *MLE* estimators obtained as in Lemma 6 using the R function "constrOptim".

2. The *MME* of (72)-(73).

In each of Figures 3 and 4, we give the scatter plot of a random sample of size $n = 100$ from the *BRST* of the *BPD* for selected values of π, α and $\underline{\lambda}$.

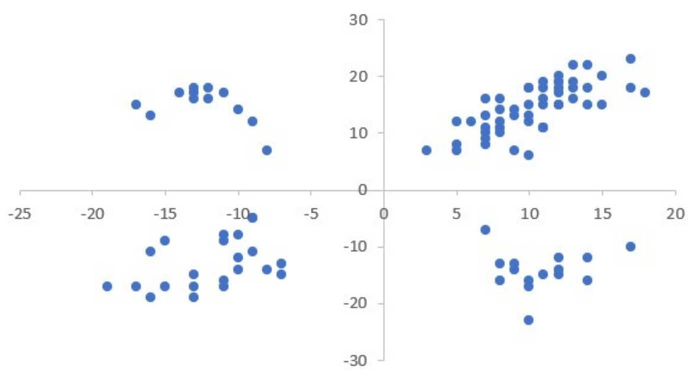

Figure 3: *BRST* of *BPD* with $\pi = 0.6, \alpha = 0.7, \underline{\lambda} = (5, 8, 6), \rho = 0.379$, and $n = 100$.

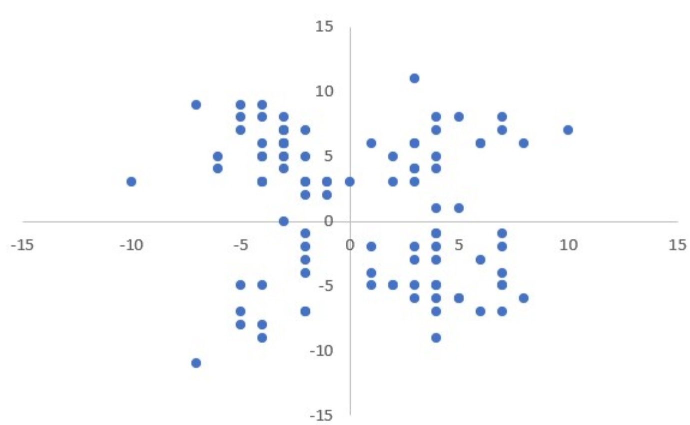

Figure 4: *BRST* of *BPD* with $\pi = 0.5, \alpha = 0.4, \underline{\lambda} = (3, 4, 1), \rho = -0.178$, and $n = 100$.

4.3 Models based on the $BVBer(\beta)$ distribution

Assume that (U_1,U_2) has the $BVBer(\beta)$ distribution of (4). In this case, the $jpmf$ of Z_1 and Z_2 is as given in (74)-(77) after replacing $f(\cdot,\cdot)$ by $p(\cdot,\cdot;\underline{\lambda})$ of (5). The marginal $pmf's$ of Z_1 and Z_2 are given as in (78) after replacing $f_i(\cdot)$ by the pdf of $Poi(\lambda_i+\lambda_3), i=1,2$. In addition,

$$E(Z_1)=E(Z_2)=0,$$

$$Var(Z_i)=(\lambda_i+\lambda_3)+(\lambda_i+\lambda_3)^2, \qquad i=1,2,$$

and

$$Cov(Z_1,Z_2)=(2\beta-1)\left(\lambda_3+(\lambda_1+\lambda_3)(\lambda_2+\lambda_3)\right).$$

For the estimation of β, we may use the following alternatives:

1. The *MLE* estimator of (83).
2. The *MME* of (84).

In each of Figures 5 and 6, we give the scatter plot of a random sample of size $n=100$ from the *BRST* of the *BPD* for selected values of β and $\underline{\lambda}$.

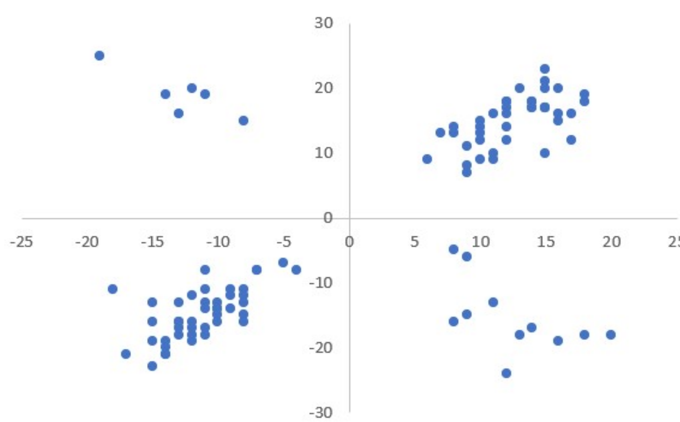

Figure 5: *BRST* of *BPD* with $\beta=0.75, \underline{\lambda}=(4,7,8), \rho=0.594$, and $n=100$.

5 BRST of the BGD

Assume that (U_1,U_2) has a *BVB* distribution, that (X_1,X_2) has the *BGD* with the $jpmf$ of (9) and that (X_1,X_2) is independent of (U_1,U_2). In the three models of this section, we may estimate δ_1 and δ_2 using the following alternatives:

1. The *MLE* estimators obtained as in Lemma 3 using (10).
2. The *MME* of (11) expressed in terms of $(|Z_{1i}|,|Z_{2i}|), i=1,2,\ldots,n$.

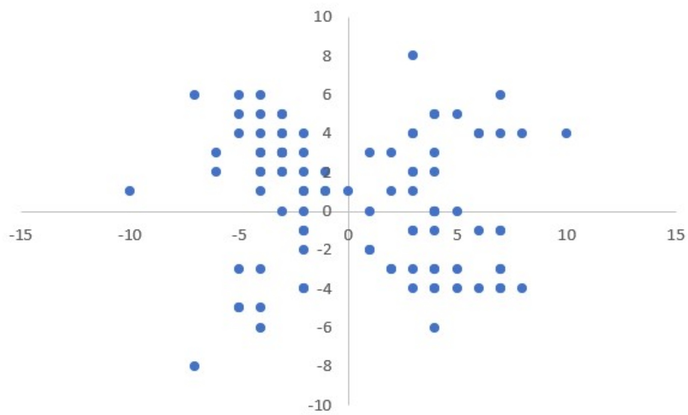

Figure 6: *BRST* of *BPD* with $\beta = 0.4, \underline{\lambda} = (3,2,1), \rho = -0.144$, and $n = 100$.

5.1 Models based on the $BVBer(\underline{\beta})$ distribution

Assume that (U_1,U_2) has the $BVBer(\underline{\beta})$ distribution of (1). In this case, the *jpmf* of Z_1 and Z_2 is given as in (13)-(16) after replacing $f(\cdot,\cdot)$ by the *jpmf* of (9). The marginal *pmf's* of Z_1 and Z_2 are given as in (13) and (16) after replacing $f_1(\cdot)$ by the *pdf* of $Geo\left(\frac{1-\delta_1-\delta_2}{1-\delta_2}\right)$ and $f_2(\cdot)$ by the *pdf* of $Geo\left(\frac{1-\delta_1-\delta_2}{1-\delta_1}\right)$. We can show that

$$E(Z_i) = (2\beta_{11} + 2I(i=1)\beta_{10} + 2I(i=2)\beta_{01} - 1)\,\frac{\delta_i}{\delta_3},$$

$$\begin{aligned} Var(Z_i) =& \left(\frac{\delta_i}{1-\delta_1-\delta_2}\right)\left(1+\frac{\delta_i}{1-\delta_1-\delta_2}\right) + 4\left(\beta_{11} + I(i=1)\beta_{10} + I(i=2)\beta_{01}\right) \\ & \times \left(1 - (\beta_{11} + I(i=1)\beta_{10} + I(i=2)\beta_{01})\right)\left(\frac{\delta_i}{1-\delta_1-\delta_2}\right)^2, \end{aligned}$$

and

$$Cov(Z_1,Z_2) = \frac{\delta_1\delta_2}{(1-\delta_1-\delta_2)^2}\left\{(1-2\beta_{10}-2\beta_{01}) + 4Cov(U_1,U_2)\right\},$$

where

$$Cov(U_1,U_2) = \beta_{11} - (\beta_{11}+\beta_{10})(\beta_{11}+\beta_{01}).$$

For the estimation of β_{11}, β_{10}, and β_{01}, we may use the following alternatives:

1. The *MLE* estimators obtained as in Lemma 3 using the R function "constrOptim".
2. The *MME* of (51)-(53).

In each of Figures 7 and 8, we give the scatter plot of a random sample of size $n = 100$ from the *BRST* of the *BGD* for selected values of $\underline{\beta}$ and $\underline{\theta}$.

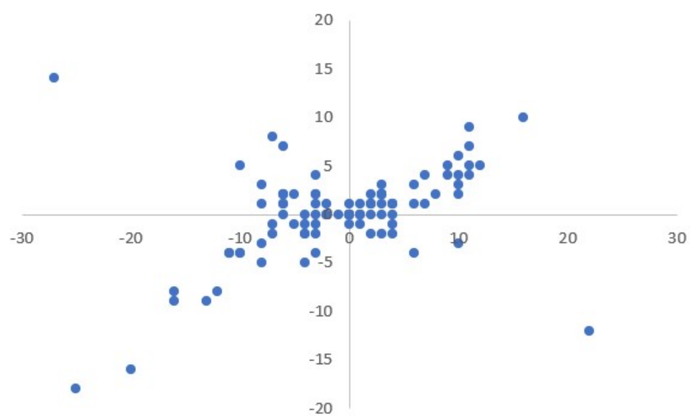

Figure 7: *BRST* of *BGD* with $\underline{\theta} = (0.6, 0.3, 0.1), \underline{\beta} = (0.4, 0.22, 0.17), \rho = 0.3993$, and $n = 100$.

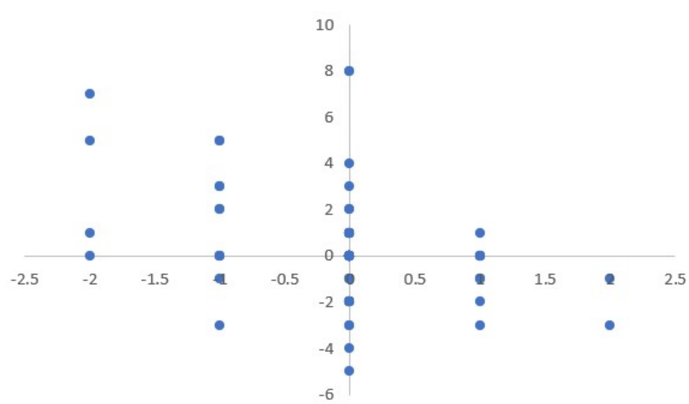

Figure 8: *BRST* of *BGD* with $\underline{\theta} = (0.2, 0.48, 0.32), \underline{\beta} = (0.2, 0.3, 0.35), \rho = -0.387$, and $n = 100$.

5.2 Models based on the $BVBer(\pi,\alpha)$ distribution

Assume that (U_1,U_2) has the $BVBer(\pi,\alpha)$ distribution of (1). In this case, the $jpmf$ of Z_1 and Z_2 is given as in (63)-(66) after replacing $f(\cdot,\cdot)$ by the $jpmf$ of (9). The marginal $pmf's$ of Z_1 and Z_2 are given as in (67) and (68) after replacing $f_1(\cdot)$ by the pdf of $Geo\left(\frac{1-\delta_1-\delta_2}{1-\delta_2}\right)$ and $f_2(\cdot)$ by the pdf of $Geo\left(\frac{1-\delta_1-\delta_2}{1-\delta_1}\right)$. We can show that

$$E(Z_1) = (2\pi-1)\frac{\delta_1}{1-\delta_1-\delta_2},$$

$$E(Z_2) = (2\alpha-1)(2\pi-1)\frac{\delta_2}{1-\delta_1-\delta_2},$$

$$Var(Z_1) = (\frac{\delta_1}{1-\delta_1-\delta_2})(1+\frac{\delta_1}{1-\delta_1-\delta_2})+4\pi\overline{\pi}\left(\frac{\delta_1}{1-\delta_1-\delta_2}\right)^2,$$

$$Var(Z_2) = (\frac{\delta_2}{1-\delta_1-\delta_2})(1+\frac{\delta_2}{1-\delta_1-\delta_2})+2\left(\overline{\alpha}\pi+\alpha\overline{\pi}\right)\left(\frac{\delta_2}{1-\delta_1-\delta_2}\right)^2,$$

and

$$Cov(Z_1,Z_2) = \left\{\frac{\delta_1\delta_2}{(1-\delta_1-\delta_2)^2}+4\pi\overline{\pi}(2\alpha-1)\left(\frac{\delta_1\delta_2}{(1-\delta_1-\delta_2)^2}\right)\right\}.$$

For the estimation of π and α, we may use the following alternatives:

1. The *MLE* estimators obtained as in Lemma 6 using the R function “constrOptim”.
2. The *MME* of (72)-(73).

In each of Figures 9 and 10, we give the scatter plot of a random sample of size $n = 100$ from the *BRST* of the *BGD* for selected values of α,π and $\underline{\lambda}$.

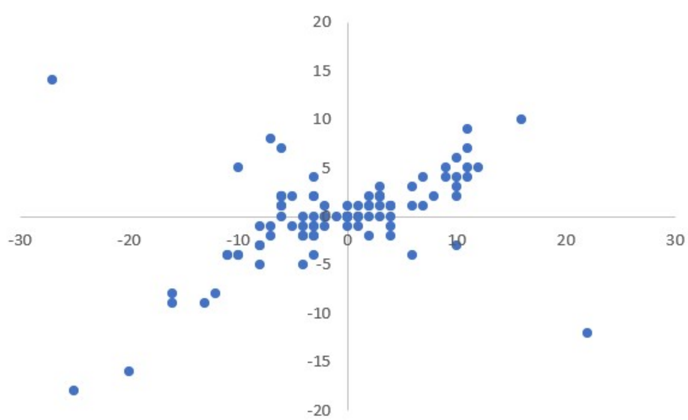

Figure 9: *BRST* of *BGD* with $\underline{\theta} = (0.6,0.3,0.1), \pi = 0.6, \alpha = 0.7, \rho = 0.4917$, and $n = 100$.

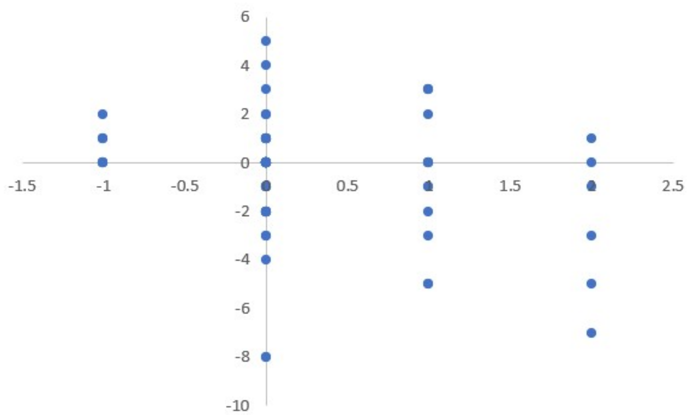

Figure 10: $BRST$ of BGD with $\underline{\theta} = (0.2, 0.48, 0.32), \pi = 0.7, \alpha = 0.5, \rho = -0.24661$, and $n = 100$.

5.3 Models based on the $BVBer(\beta)$ distribution

Assume that (U_1, U_2) has the $BVBer(\beta)$ distribution of (4). In this case, the $jpmf$ of Z_1 and Z_2 is as given in (74)-(77) after replacing $f(\cdot,\cdot)$ by the $jpmf$ of (9). The marginal $pmf's$ of Z_1 and Z_2 are given as in (78) after replacing $f_1(\cdot)$ by the pdf of $Geo\left(\frac{1-\delta_1-\delta_2}{1-\delta_2}\right)$ and $f_2(\cdot)$ by the pdf of $Geo\left(\frac{1-\delta_1-\delta_2}{1-\delta_1}\right)$. We can show that

$$E(Z_i) = 0, \qquad i = 1, 2,$$

$$Var(Z_i) = \frac{\delta_i}{1-\delta_1-\delta_2}\left(1 + \frac{2\delta_i}{1-\delta_1-\delta_2}\right), \qquad i = 1, 2,$$

and

$$Cov(Z_1, Z_2) = \frac{2(2\beta-1)\delta_1\delta_2}{(1-\delta_1-\delta_2)^2}.$$

For the estimation of β, we may use the following alternatives:

1. The MLE estimator of (83).

2. The MME of (84).

In each of Figures 11 and 12, we give the scatter plot of a random sample of size $n = 100$ from the $BRST$ of the BGD for selected values of β and $\underline{\theta}$.

6 Simulations

We have conducted 12 simulation studies to asses the performance of the MLE and the MME estimators of the model parameters. In each simulation, we used 10,000 realizations of samples of

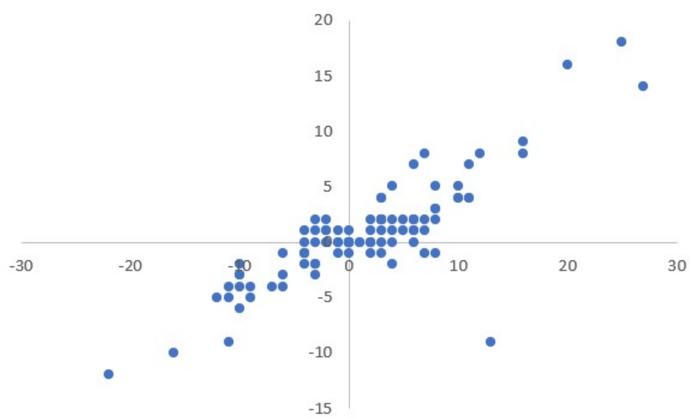

Figure 11: $BRST$ of BGD with $\underline{\theta} = (0.6, 0.3, 0.1), \beta = 0.8, \rho = 0.837$, and $n = 100$.

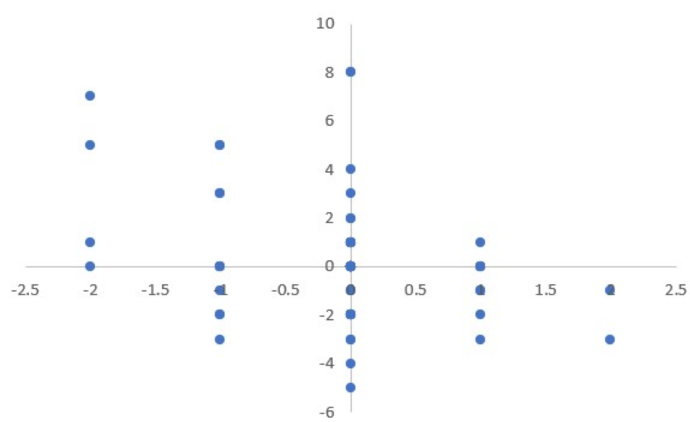

Figure 12: $BRST$ of BGD with $\underline{\theta} = (0.2, 0.48, 0.32), \beta = 0.4, \rho = -0.3335$, and $n = 100$.

Table 1: BRST of BPD using $BVBer(\underline{\beta})$.

	$\lambda_1 = 3$	$\lambda_2 = 2$	$\lambda_3 = 1$
MLE	2.998(0.199)	1.996(0.192)	1.002(0.182)
MME	3.005(0.228)	2.003(0.222)	0.995(0.214)
	$\beta_{11} = 0.2$	$\beta_{10} = 0.3$	$\beta_{01} = 0.35$
MLE	0.205(0.016)	0.283(0.021)	0.357(0.017)
MME	0.2(0.028)	0.306(0.033)	0.356(0.033)

Table 2: BRST of BPD using $BVBer(\pi, \alpha)$.

	$\lambda_1 = 3$	$\lambda_2 = 2$	$\lambda_3 = 1$	$\pi = 0.7$	$\alpha = 0.5$
MLE	2.998(0.199)	1.996(0.192)	1.002(0.182)	0.685(0.023)	0.5(0.021)
MME	3.005(0.228)	2.003(0.222)	0.995(0.214)	0.7(0.029)	0.501(0.086)

Table 3: BRST of BPD using $BVBer(\beta)$.

	$\lambda_1 = 3$	$\lambda_2 = 2$	$\lambda_3 = 1$	$\beta = 0.4$
MLE	2.998(0.199)	1.996(0.192)	1.002(0.182)	0.4(0.031)
MME	3.005(0.228)	2.003(0.222)	0.995(0.214)	0.4(0.041)

Table 4: BRST of BGD using $BVBer(\beta)$.

	$p_0 = 0.2$	$p_1 = 0.48$	$p_2 = 0.32$
MLE	0.2(0.013)	0.48(0.016)	0.321(0.015)
MME	0.2(0.013)	0.48(0.016)	0.321(0.015)
	$\beta_{11} = 0.2$	$\beta_{10} = 0.3$	$\beta_{01} = 0.35$
MLE	0.2(0.0111)	0.277(0.024)	0.35(0.011)
MME	0.201(0.056)	0.299(0.056)	0.349(0.059)

Table 5: BRST of BGD using $BVBer(\pi, \alpha)$.

	$p_0 = 0.2$	$p_1 = 0.48$	$p_2 = 0.32$	$\pi = 0.7$	$\alpha = 0.5$
MLE	0.2(0.013)	0.48(0.016)	0.321(0.015)	0.675(0.037)	0.502(0.026)
MME	0.2(0.013)	0.48(0.016)	0.321(0.015)	0.699(0.05)	0.5(0.018)

Table 6: BRST of BGD using $BVBer(\beta)$.

	$p_0 = 0.2$	$p_1 = 0.48$	$p_2 = 0.32$	$\beta = 0.4$
MLE	0.2(0.013)	0.48(0.016)	0.321(0.015)	0.4(0.051)
MME	0.2(0.013)	0.48(0.016)	0.321(0.015)	0.4(0.086)

Table 7: BRST of independent Poisson RV using $BVBer(\beta)$.

	$\lambda_1 = 3$	$\lambda_2 = 2$	$\beta_{11} = 0.2$	$\beta_{10} = 0.3$	$\beta_{01} = 0.35$
MLE	2.999(0.099)	2(0.082)	0.201(0.008)	0.286(0.016)	0.351(0.008)
MME	2.999(0.099)	2.(0.082)	0.201(0.03)	0.3(0.033)	0.349(0.034)

Table 8: BRST of independent Poisson RV using $BVBer(\pi, \alpha)$.

	$\lambda_1 = 3$	$\lambda_2 = 2$	$\pi = 0.7$	$\alpha = 0.5$
MLE	2.999(0.099)	2(0.082)	0.684(0.025)	0.502(0.017)
MME	2.999(0.099)	2.(0.082)	0.7(0.031)	0.502(0.092)

Table 9: BRST of independent Poisson RV using $BVBer(\beta)$.

	$\lambda_1 = 3$	$\lambda_2 = 2$	$\beta = 0.4$
MLE	2.999(0.099)	2(0.082)	0.4(0.031)
MME	2.999(0.099)	2.(0.082)	0.4(0.040)

Table 10: BRST of independent geometric RV using $BVBer(\underline{\beta})$.

	$\theta_1 = 0.5$	$\theta_2 = 0.6$	$\beta_{11} = 0.2$	$\beta_{10} = 0.3$	$\beta_{01} = 0.35$
MLE	0.501(0.02)	0.601(0.022)	0.199(0.003)	0.284(0.022)	0.35(0.005)
MME	0.501(0.02)	0.601(0.022)	0.236(0.064)	0.263(0.066)	0.32(0.067)

Table 11: BRST of independent geometric RV using $BVBer(\pi, \alpha)$.

	$\theta_1 = 0.5$	$\theta_2 = 0.6$	$\pi = 0.7$	$\alpha = 0.5$
MLE	0.501(0.02)	0.601(0.022)	0.683(0.033)	0.504(0.059)
MME	0.501(0.02)	0.601(0.022)	0.701(0.046)	0.5(0.01)

size 300 from the considered model. The mean (the standard deviation) of the $10,000$ estimators of each parameter are reported next in Tables 1 to 12.

The results of the Tables 1-12 suggest that both the ***MLE*** and the ***MME*** estimators perform well in all the considered models. However, for most of the models, the ***MLE*** estimators have smaller standard deviations than the corresponding ***MME*** estimators for all parameters.

7 Data analysis

The data of this example are based on the results of the 2019 UEFA Europa League. The 48 teams of this competition are divided into 12 groups of four teams each. Each team plays one home match and one away match against the other three teams of its group. For each team, we obtained one observation computed by taking the difference between a) the sum of scores of its three home matches and b) the sum of scores of its three away matches. For example, the observation of team Apoel of Group A (Apoel, Dudelange, Qarabağ and Sevilla) is obtained as follows. The sum of Apoel's three home scores $((3,4),(2,1)$, and $(1,0))$ is $(6,5)$ and the sum of Apoel's three away scores $((2,0),(2,2)$, and $(0,1))$ is $(4,3)$. Hence, the difference $((6,5)-(4,3))$ is $(2,2)$. The resulting bivariate data of 48 observations is as follows:

(0,-2)	(0,-2)	(0,4)	(0,-4)	(0,5)	(0,7)	(-1,0)	(1,1)	(1,-1)	(-1,-1)	(1,3)	(-1,-3)
(-1,-3)	(1,-4)	(1,-4)	(1,-5)	(-1,-6)	(2,0)	(2,-1)	(-2,1)	(2,2)	(2,-3)	(-2,-3)	(-2,-3)
(2,-4)	(-2,4)	(-2,-4)	(-3,-1)	(3,2)	(3,3)	(3,-3)	(4,-1)	(4,-2)	(4,-2)	(4,3)	(4,-4)
(-4,-5)	(5,-2)	(-5,2)	(6,-2)	(6,-2)	(6,4)	(2,-3)	(3,-5)	(-1,3)	(2,-2)	(3,-5)	(7,0).

To fit the above data we will explore the following bivariate BRST models:

1. P1 (BPD based on $BBer(\underline{\beta})$), P2 (BPD based on $BBer(\pi, \alpha)$) and P3 (BPD based on $BBer(\beta)$).

Table 12: BRST of independent geometric RV using $BVBer(\beta)$.

	$\theta_1 = 0.5$	$\theta_2 = 0.6$	$\beta = 0.4$
MLE	0.501(0.02)	0.601(0.022)	0.4(0.063)
MME	0.501(0.02)	0.601(0.022)	0.4(0.092)

2. P4 (independent Poisson RV based on $BBer(\underline{\beta})$), P5 (independent Poisson RV based on $BBer(\pi,\alpha)$) and P6 (independent Poisson RV based on $BBer(\beta)$).

3. P7 (BGD based on $BBer(\underline{\beta})$), P8 (BGD based on $BBer(\pi,\alpha)$) and P9 (BGD based on $BBer(\beta)$).

4. P10 (independent Geometric RV based on $BBer(\underline{\beta})$), P11 (independent Geometric RV based on $BBer(\pi,\alpha)$) and P12 (independent Geometric RV based on $BBer(\beta)$).

In all of the above models, we obtained the MLE. For models P1, P4, and P10, $\widehat{\beta}_{11} = 0.19800, \widehat{\beta}_{10} = 0.46751$, and $\widehat{\beta}_{01} = 0.113099$. For models P2, P5, and P11, $\widehat{\pi} = 0.659675$ and $\widehat{\alpha} = 0.418447$. For models P3, P6, and P12, $\widehat{\beta} = 0.421$. For models P1-P3, $\widehat{\lambda}_1 = 1.634857, \widehat{\lambda}_2 = 2.572357$, and $\widehat{\lambda}_3 = 0.7193102$. For models P4-P6, $\widehat{\lambda}_1 = 2.35$ and $\widehat{\lambda}_2 = 2.83$. For models P7-P9, $\widehat{\theta}_1 = 0.38, \widehat{\theta}_2 = 0.457$, and $\widehat{\theta}_3 = 0.162$. For models P10-P12, $\widehat{\theta}_1 = 0.298$ and $\widehat{\theta}_2 = 0.261$.

We divided Z^2 into the nine mutually exclusive areas corresponding to $n_0, n_{0,\mp}, n_{\mp,0}, n_{\pm,-}$, and $n_{\mp,+}$ of (32)-(36). The expected counts of each of these areas are computed using (37)-(45). For example, the computations for model P_1 are given next. Note that in this case,

$$f(0,0;\underline{\widehat{\lambda}}) = 0.0072517, \quad f_1(0;\underline{\widehat{\lambda}}) = 0.094973, \text{ and } f_2(0;\underline{\widehat{\lambda}}) = 0.037192. \tag{85}$$

Hence, by (37)-(45) and (85), we obtain the following Table 13 for Model P1.

Table 13: Observed and expected counts for Model P1.

P1	n_0	$n_{+,0}$	$n_{-,0}$	$n_{0,+}$	$n_{0,-}$	$n_{+,+}$	$n_{-,+}$	$n_{+,-}$	$n_{-,-}$
Observed	0	2	1	3	4	7	4	18	9
Expected	0.348	0.946	0.492	1.31	2.901	8.326	4.742	19.31	9.6266

For each of the 12 models, we computed the Chi-square test statistic and the corresponding P-value. The P-values for models P2, P3, and P6-P12 are all less than 0.05. The Chi-square test statistics and the corresponding P-values of the remaining models are as follows: P1 (5.104, 0.078), P4 (3.0386, 0.386), and P5 (7.8953, 0.096). It is clear from these results that model P4 provides the best fit for the considered data.

8 Conclusions

We extended the RST of Aly (2018) to produce bivariate integer-valued random vectors on Z^2. The proposed Bivariate RST (BRST) is also an extension of the family of bivariate discrete distributions on Z^2 of Chesneau et al. (2018). We studied in details our proposed family and considered, in particular, a number of new bivariate integer-valued distributions on Z^2. The proposed BRST can be applied to other bivariate nonnegative integer-valued random vectors to produce new families of bivariate integer-valued random vectors on Z^2.

As an illustration, we applied the proposed families to a real data set developed based on the results of the 2019 UEFA Europa League. One of our proposed models provided an excellent fit of this data.

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References

Aly, E.-E. (2018). A unified approach for developing Laplace-type distributions. *Journal of the Indian Society for Probability and Statistics*, **19**, 245–269.

Bulla, J., Chesneau, C. and Kachour, M. (2015). On the bivariate Skellam distribution. *Communications in Statistics-Theory and Methods*, **44**, 4552–4567.

Chesneau, C., Kachour, M. and Bakouch, H. S. (2018). A family of bivariate discrete distributions on Z^2 based on the Rademacher distribution. *In ProbStat Forum*, **11**, 53–66.

Genest, C. and Mesfioui, M. (2014). Bivariate extensions of Skellam's distribution. *Probability in the Engineering and Informational Sciences*, **28**, 401–417.

Hogg, R. V., McKean, J. W. and Craig, A. T. (2005). *Introduction to Mathematical Statistics, 6th ed.* Pearson Education, New Delhi.

Johnson, N. L., Kotz, S. and Balakrishnan, N. (1997). *Discrete multivariate distributions.* Wiley Series in Probability and Statistics, **165**, Wiley, New York.

Karlis, D. and Mamode Khan, N. (2023). Models for integer data. *Annual Review of Statistics and Its Application*, **10**, 297–323.

Karlis, D. and Ntzoufras, I. (2005). Bivariate Poisson and diagonal inflated bivariate Poisson regression models in R. *Journal of Statistical Software*, **14** (i10).

Kocherlakota, S. and Kocherlakota, K. (1992). *Bivariate Discrete Distributions.* Statistics: textbooks and monographs, **132**, Marcel Dekker, New York.

Krishna, H. and Pundir, P. S. (2009). A bivariate geometric distribution with applications to reliability. *Communications in Statistics–Theory and Methods*, **38**, 1079–1093.

Lai, C.D. (2006). *Constructions of Discrete Bivariate Distributions.* In Advances in distribution theory, order statistics, and inference, 29–58, Birkhäuser.

Odhah, O.H. (2013). Probability Distributions on the Class of Integer Numbers. Doctoral dissertation, King Saud University.

Omair M.A., Alomani, G. A. and Alzaid A. (2022). Bivariate distributions on Z^2. *Bulletin of the Malaysian Mathematical Sciences Society*, **45**, 425–444.

Omair, M, A, Alzaid A., Odhah, O. H. (2016). A trinomial difference distribution. *Revista Colombiana de Estadística,* **39**, 1–15.

Phatak, A. G. and Sreehari, M. (1981). Some characterizations of a bivariate geometric distribution. *Journal of Indian Statistical Association*, **19**, 141–146.

Sarabia Alegría, J. M. and Gómez Déniz, E. (2008). Construction of multivariate distributions: a review of some recent results. SORT, **32**, 3–36.

Shannon, C. E. (1951). Prediction and entropy of printed English. *Bell System Technical Journal*, **30**, 50–64.